On Supremal and Maximal Sets with Respect to Random Partial Orders

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Abstract

The paper deals with definition of supremal sets in a rather general framework where deterministic and random preference relations (preorders) and partial orders are defined by continuous multi-utility representations. It gives a short survey of the approach developed in [5], [6] with some new results on maximal sets.

Keywords: Preference Relation, Partial Order, Random cones, Transaction costs, European/American options, Hedging.

2000 MSC: 60G44, G11-G13.

1. Introduction

The classical notion of *essential supremum* plays an important role in the theory of frictionless markets serving to define the generalized Snell envelope of the payoff process, an important tool to characterize the set of super-replicating prices of European and American contingent claims, see [7] Contrarily to the frictionless financial markets, in models with proportional transaction costs, a portfolio process is vector-valued and its dynamic depends on a set-valued adapted process $(G_t)_{t\in[0,T]}$ in $\mathbf{R}^d$, $d \geq 1$, whose values are the solvency cones in physical units. In the discrete-time setting, a self-financing portfolio process $(\hat{V}_t)_{t=0,\cdots,T}$ has the increments $\hat{V}_t - \hat{V}_{t-1} \in -G_t$, $t \geq 1$, i.e. $\hat{V}_{t-1} \geq_{G_t} \hat{V}_t$ where $\geq_{G_t}$ denotes the random preorder (preference

relation) generated by the random cone G_t. In this context, it is reasonable to characterize the minimal vector-valued prices and minimal portfolio processes super-hedging contingent claims in the sense of random preorders defined by the solvency cones. Mathematically, it is rather natural to study questions on existence and properties of suitably defined supremal sets in a much more general framework of random preorders or partial orders. This study was initiated in our papers [5], [6] where we use systematically the description of preorders and partial orders in terms of continuous multi-utility representation. It seems that this approach is new even in the deterministic case. Specific and (pleasant) feature is that we do not require that the partial order generates a structure (i.e. any two elements admit maximum and minimum).

In the present article we give a short survey of the approach developed in [5], [6] with some new results on maximal sets. Moreover, we provide some applications to models of financial markets with transaction costs.

2. Supremum and maximum with respect to a preorder in a deterministic setting

2.1. Vocabulary

Let $\succeq$ be a *preorder* (=*preference relation*) in X, i.e. a binary relation between certain elements of a set X which is reflexive ($x \succeq x$) and transitive (if $x \succeq y$ and $y \succeq z$ then $x \succeq z$). The elements x and y are *equivalent* if $x \succeq y$ and $y \succeq x$; we write $x \sim y$ in this case. The preorder is called *partial order* if it is antisymmetric (if $x \succeq y$ and $y \succeq x$ then $x = y$). For the partial order the classes of equivalence are singletons.

The word "binary relation" means simply that we are given an indicator function $I_D : X \times X \to \{0, 1\}$ and the notation $x \succeq y$ is equivalent to the equality $I_D(x, y) = 1$, i.e. (x, y) belongs to the set D. For the preorder, the diagonal of the product space should be a subset of D and if the points (x, y), (y, z) are in D, then $(x, z) \in D$. For the partial order, (x, y), (y, x) are in D if and only if they belong to the diagonal. Let a set $D \subseteq X \times X$ define a preorder. Then its subset D' not containing the diagonal and such that for any points (x, y), (y, z) in D' the point $(x, z) \in D'$ defines a non-reflexive transitive relation denoted $\succ$.

We define the order intervals $[x, y] := \{z \in X : \ y \succeq z \succeq x\}$,

$$]-\infty, x] := \{z \in X : \ x \succeq z\}, \qquad [x, \infty[:= \{z \in X : \ z \succeq x\}$$

(the latter objects are called sometimes *lower* and *upper contour sets*). If Γ is a subset of X, the notation $\Gamma \succeq x$ means that $y \succeq x$ for all $y \in \Gamma$ while $\Gamma_1 \succeq \Gamma$ means that $x \succeq y$ for all $x \in \Gamma_1$ and $y \in \Gamma$;

$$[\Gamma, \infty[:= \bigcap_{x \in \Gamma} \{z \in X : \ z \succeq x\}$$

is the set of upper bounds of Γ, and so on. We shall use the notation $x \preceq z$ synonymous with $z \succeq x$.

A preorder $\succeq$ in a topological space X is *continuous* if its graph, i.e. the set $\{(x, y) : \ x \succeq y\}$, is closed.

2.2. Maximal and supremal sets

In the ample literature on preference theory and vector optimization one can find a number of definitions of maximal and supremal sets under various hypotheses on X. On the informal level, the maximal set of Γ is defined from the primary object, i.e. from the set Γ itself, while the supremal set is defined from the object, dual to Γ in an appropriate sense, namely, from the set $[\Gamma, \infty[$ of upper bounds of Γ. The difference between the two approaches is noticeable already in the case of the real line $\mathbf{R}$ with its usual total (linear) order. Indeed, any set $\Gamma \subset \mathbf{R}$ bounded from above has a supremum but may not have a maximum in the usual sense. But, the set being "improved" by passing to its closure, will have one, coinciding with the supremum.

With partial orders the situation is more complicated. Let us consider the partial order in $\mathbf{R}^2$ where $x \succeq y$ means that $x^i \geq y^i$ for $i = 1, 2$ (that is the partial order generated by $\mathbf{R}^2_+$). Let $\Gamma = \{(0, 0), \ (1, 0), \ (0, 1)\}$ and let $\mathbf{1} = (1, 1)$. Then $[\Gamma, \infty[= \mathbf{1} + \mathbf{R}^2_+$. This set has a minimal element with respect to the partial order, namely, $\mathbf{1}$, which is a good candidate to be considered as the supremum of Γ. The unpleasant feature is that it lays far from Γ. On the other hand, the set $\{(1, 0), \ (0, 1)\}$ looks, intuitively, as a good candidate for the maximum: it is a subset of Γ and for any its element x the intersection of Γ and $[x, \infty[$ is the singleton $\{x\}$.

Generalizing the above examples we arrive to the following notions.

Definition 2.1. *Let Γ be a non-empty subset of X and let $\succeq$ be a preorder. We denote by* $\operatorname{Sup} \Gamma$ *the* **largest** *subset $\hat{\Gamma}$ of X such that the following conditions hold:*

(a_0) $\hat{\Gamma} \subseteq [\Gamma, \infty[$;

(b_0) *if* $x \in [\Gamma, \infty[$, *then there is* $\hat{x} \in \hat{\Gamma}$ *such that* $\hat{x} \preceq x$;

(c_0) *if two elements* $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$ *are comparable, then they are equivalent.*

In the case of partial order the word "largest" in the above definition can be omitted: then the comparable elements of $\hat{\Gamma}$ coincide and it is not difficult to prove that the set with the listed properties is unique, see Lemma 3.3 in [5].

Note that the definition does not involve any additional structure on X. In contrast to this, to define maximal sets we assume that X is a topological space.

Definition 2.2. *Let* Γ *be a non-empty subset of a topological space* X *and let* $\succeq$ *be a preorder. We put*

$$\operatorname{Max}\Gamma = \{x \in \overline{\Gamma} : \ \overline{\Gamma} \cap [x, \infty[= [x, x]\}.$$

Definition 2.3. *Let* Γ *be a non-empty subset of a topological space* X *and let* $\succeq$ *be a preorder. We denote by* $\operatorname{Max}_1\Gamma$ *the largest subset* $\hat{\Gamma} \subseteq \overline{\Gamma}$ *(possibly empty) such that the following conditions hold:*

(α) *if* $x \in \overline{\Gamma}$, *then there is* $\hat{x} \in \hat{\Gamma}$ *such that* $\hat{x} \succeq x$;

(β) *if two elements* $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$ *are comparable, then they are equivalent.*

It is easy to understand that in the case of $\mathbf{R}$ our definition of supremum coincides with the classical one: $\operatorname{Sup}\Gamma = \{\sup\Gamma\}$ when a (non-empty) set Γ is bounded from above. Moreover, if Γ is closed and bounded from above, then $\operatorname{Max}\Gamma = \operatorname{Max}_1\Gamma = \{\max\Gamma\} = \operatorname{Sup}\Gamma$. For non-closed bounded $\Gamma \neq \emptyset$ the value $\max\Gamma$ in the classical sense may not exist while the sets $\operatorname{Max}\Gamma$ and $\operatorname{Max}_1\Gamma$ are well-defined and non-empty. In the case of our introductory example above where $\Gamma = \{(0,0),\ (1,0),\ (0,1)\}$, we have $\sup\Gamma = \mathbf{1}$ and $\max\Gamma = \operatorname{Max}_1\Gamma = \{(1,0),\ (0,1)\}$.

Remark 2.4. For a closed set Γ, the set $\operatorname{Max}\Gamma$ is just the set of maximal points of Γ and it plays an important role in multicriteria optimization. In the latter theory the simplest standard problem is the following: given a compact set Γ and a continuous function $\mathbf{u} : \mathbf{R}^n \to \mathbf{R}^n$, find $\operatorname{Max}\Gamma$ with respect to the preorder defined by the multi-utility representation $\{u^i,\ 1 \leq i \leq d\}$, formed by the component of the vector function $\mathbf{u}$. Recall that in this theory the set $\operatorname{Max}\mathbf{u}(\Gamma) \subset \mathbf{R}^n$ defined by the "natural" partial order in $\mathbf{R}^n$, i.e. generated by the cone $\mathbf{R}^n_+$, is called the Pareto frontier.

Though the definitions of $\operatorname{Max}\Gamma$ and $\operatorname{Max}_1\Gamma$ looks quite different, in the case where $\operatorname{Max}_1\Gamma \neq \emptyset$, both sets coincide.

Lemma 2.5. $\operatorname{Max}_1\Gamma \subseteq \operatorname{Max}\Gamma$.

Proof. Assume that $\operatorname{Max}_1\Gamma \neq \emptyset$ (otherwise the claim is trivial). Consider $\hat{x}_1 \in \operatorname{Max}_1\Gamma$ and $x \in \bar{\Gamma}$ such that $\hat{x}_1 \preceq x$. By (α), there exists $\hat{x}_2 \in \operatorname{Max}_1\Gamma$ such that $x \preceq \hat{x}_2$. Hence, by (β), $\hat{x}_1 \sim \hat{x}_2 \sim x$, i.e. $\hat{x}_1 \in \operatorname{Max}\Gamma$. $\square$

Lemma 2.6. *Let* $\operatorname{Max}_1\Gamma \neq \emptyset$. *Then* $\operatorname{Max}_1\Gamma = \operatorname{Max}\Gamma$.

Proof. By Lemma 2.5 above it is sufficient to check that the set $\operatorname{Max}\Gamma$ satisfies the properties (α) and (β) in Definition DefiMax1. Since $\operatorname{Max}_1\Gamma \neq \emptyset$ and $\operatorname{Max}_1\Gamma \subseteq \operatorname{Max}\Gamma$ the condition (α) is satisfied and it remains to observe that (β) automatically holds by definition of $\operatorname{Max}\Gamma$. $\square$

2.3. Existence results

The existence theorem below is a synthesis of several, more general, results from [5] and [6].

Theorem 2.7. *Let* $\succeq$ *be a continuous partial order in the Euclidean space* $\mathbf{R}^d$ *such that all order intervals* $[x, y]$, $y \succeq x$, *are compact. Let* Γ *be a non-empty subset bounded from above, i.e. the order interval* $[\Gamma, \infty[\neq \emptyset$. *Then* $\operatorname{Sup}\Gamma \neq \emptyset$ *and* $\operatorname{Max}\Gamma = \operatorname{Max}_1\Gamma \neq \emptyset$.

We recall some important facts on multi-utility representations of preorders on topological vector space X.

We say that a set $\mathcal{U}$ of real-valued functions defined on X is a multi-utility representation of the preorder $\succeq$ if for any $x, y \in X$,

$$x \succeq y \;\Leftrightarrow\; u(x) \geq u(y) \quad \forall u \in \mathcal{U}.$$

It is easy to see that any preorder admits a multi-utility representation given by the family of indicator functions $u(x) = I_{[x,\infty[}$, $x \in X$. Note that the terminology, taken from [3], does not coincide with with the standard one: properties usually associated with the utility functions are not required from the elements of $\mathcal{U}$.

The interest in multi-utility representations lays in the possibility to formulate, in terms of comprehensive properties of functions, assumptions on preorders and partial orders.

For example, if a preorder admits a continuous multi-utility representation (i.e. given by continuous functions), then, of course, this preorder is continuous. Under a suitable assumption the converse is also true: if X is a locally compact and σ-compact Hausdorff space, then a continuous preorder admits a continuous multi-utility representation, see [3].

As a corollary, we get that any continuous preorder on $\mathbf{R}^d$ (or, more generally, on any locally compact and σ-compact Hausdorff space) admits a countable continuous multi-utility representation, see [6].

Proposition 2.8. *Let X be a σ-compact metric space. Suppose that a family $\mathcal{U}$ of continuous functions defines a preorder on X. Then this preorder can be defined by a countable subfamily of $\mathcal{U}$.*

It is obvious that a preorder on $\mathbf{R}^d$ defined by a closed (convex) cone G ($x \succeq y$ means that $x - y \in G$) is continuous. Such a preorder is a partial order if and only if G is a proper cone, i.e. $G^0 := G \cap (-G) = \{0\}$.

Lemma 2.9. *Let $\succeq$ be a partial order defined by a closed proper cone G in $\mathbf{R}^d$. Than the order intervals $[x, y]$ are compact.*

Proof. Suppose that $z^n \in [x, y]$ and its Euclidean norm $|z^n| \to \infty$. Put $\tilde{z}^n = z^n/(1 + |z^n|)$, $\tilde{x}^n = x/(1 + |z^n|)$, and $\tilde{y}^n = y/(1 + |z^n|)$. Then $\tilde{x}^n \preceq \tilde{z}^n \preceq \tilde{y}^n$. Using the compactness of the unit ball in $\mathbf{R}^d$, we may assume that $\tilde{z}^n \to \tilde{z}$ such that $|\tilde{z}| = 1$. On the other hand, $\tilde{x}^n$, $\tilde{y}^n \to 0$. Hence, $0 \preceq \tilde{z} \preceq 0$, i.e. $\tilde{z} \in G \cap (-G) = \{0\}$ contrary to the assumption. $\square$

Thus, Theorem 2.7 implies as corollary the following result which seems to be well adapted to the needs of the theory of financial markets with proportional transaction costs:

Theorem 2.10. *Let $\succeq$ be a continuous partial order generated by a proper closed convex cone $G \subset \mathbf{R}^d$. Let Γ be a non-empty subset bounded from above. Then* $\operatorname{Sup}\Gamma \neq \emptyset$ *and* $\operatorname{Max}\Gamma = \operatorname{Max}_1\Gamma \neq \emptyset$.

It is rather natural to place the question on the existence of non-empty supremal sets in a more general context of a preorder on a topological space. Several results in this direction can be found in the paper [6] where the principal assumptions are: the preorder admits a countable multi-utility representation with lower semicontinuous functions and the order intervals in the quotient space $\tilde{X} = X/\sim$ (generated by the equivalence $x \sim y$) are compact. Of course, the quotient mapping q induces the relation between classes of

equivalence which is a partial order and the quotient space is equipped with the weakest topology such that q is continuous. But even for the preorder in $\mathbf{R}^d$, the quotient space is, in general, an abstract topological space and formulations of the corresponding results are too technical except the case when the preorder is generated by a cone. Also the topological assumptions we need to use our techniques (requiring the existence of countable representing family) are considered in the literature as too restrictive. In view of this in the present survey we concentrate ourselves on partial orders.

Theorem 2.11. *Let $\succeq$ be a partial order in a topological space X represented by a countable family $\mathcal{U}$ of lower semicontinuous functions and such that all order intervals $[x, y]$, $y \succeq x$, are compact. If a subset Γ is bounded from above, then* $\operatorname{Sup}\Gamma \neq \emptyset$ *and* $\operatorname{Max}\Gamma = \operatorname{Max}_1\Gamma \neq \emptyset$.

Recall that a function $u : X \to \mathbf{R}$ is called lower semicontinuous (l.s.c.) if for any point $x \in X$

$$\liminf_{x_\alpha \to x} u(x_\alpha) \geq u(x).^{1}$$

Equivalently, u is l.s.c. if all lower level sets $\{x \in X : \ u(x) \leq c\}$ are closed, see [1]. A function $g : \tilde{X} \to \mathbf{R}$ is l.s.c. if and only if $g \circ q : X \to \mathbf{R}$ is l.s.c. If a function $f : X \to \mathbf{R}$ is l.s.c. and constant on the classes of equivalences $[x]$, then the function $g : \tilde{X} \to \mathbf{R}$ with $g([x]) = f(x)$ is l.s.c.

We complete this section by an example showing that $\operatorname{Sup}\Gamma$ might not be closed even if Γ is closed.

Example Let $G \subseteq \mathbf{R}^2_+$ be defined as

$$G = \{(0, 0)\} \cup \{(x, y) \in \mathbf{R}^2_+ : \ x + y \geq 2\}.$$

As $G \cap (-G) = \{0\}$ and $G + G \subseteq G$, the relation $x \succeq y$ if $x - y \in G$ defines a partial order. Let us consider the set $\Gamma := \{(0, 0), (4, -1)\}$. Then,

$$[\Gamma, \infty) = \{(x, y) \in \mathbf{R}_+ : \ x + y \geq 5 \,\text{and}\, x \geq 4\}.$$

Indeed, $(x, y) \succeq \Gamma$ if and only if $(x, y) \in G$, i.e. $(x, y) \in \mathbf{R}^2_+$ and $x + y \geq 2$, and $(x - 4, y + 1) \in G$, i.e. $x \geq 4$ and $x - 4 + y + 1 \geq 2$. We deduce that

$$\operatorname{Sup}\Gamma = \{(x, y) \in \mathbf{R}_+ : \ 7 > x + y \geq 5 \,\text{and}\, x \geq 4\}. \tag{2.1}$$

[1] Recall that $(x_\alpha)_{\alpha \in I}$ designates a net, i.e. a sequence of elements in X indexed by an upward directed set I, such that for all open set $\mathcal{O}$ containing x, $(x_\alpha)_{\alpha \in I}$ eventually belongs to $\mathcal{O}$

Indeed, it suffices to observe that

$$\begin{aligned}(4,1)+G &= \{(4,1)\}\cup\{(x,y)\in\mathbf{R}_+:\ x+y\geq 7;\ y\geq 1\},\\ (5,0)+G &= \{(5,0)\}\cup\{(x,y)\in\mathbf{R}_+:\ x+y\geq 7;\ x\geq 5\},\end{aligned}$$

so that the points in $\mathbf{R}_+$ above the line $x+y\geq 7$ are greater that $(4,1)$ or $(5,0)$ with respect to G. If two points belong to the set given by (2.1), then they are not comparable. Observe that this set is not closed though G and Γ are closed.

Remark 2.12. For unbounded Γ it may happen that $\operatorname{Max}\Gamma\neq\operatorname{Max}_1\Gamma=\emptyset$. Indeed, let us consider in $\mathbf{R}^2$ the partial order generated by the closed cone $G=\mathbf{R}_+e_1$. For the set $\Gamma=\{e_2\}\cup G$ we have $\operatorname{Max}\Gamma=\{e_2\}$ while $\operatorname{Max}_1\Gamma=\emptyset$.

2.4. Relations with other concepts

In this subsection we discuss some concepts existing in the literature and very close to those introduced above.

We start with the notion of $\operatorname{supset}\Gamma$.

Definition 2.13. *Let Γ be a non-empty subset of X and let $\succeq$ be a order. Put*

$$\operatorname{supset}\Gamma:=\{z\in X:\ z\succeq\Gamma \text{ and } w=z\ \forall\, w\succeq\Gamma \text{ such that } z\succeq w\}.$$

It is easily seen that $\operatorname{supset}\Gamma$ satisfies the conditions (a_0) and (c_0) of Definition 2.1, i.e.

(a_0) $\operatorname{supset}\Gamma\subseteq[\Gamma,\infty[$;

(c_0) if two elements $\hat{x}_1,\hat{x}_2\in\operatorname{supset}\Gamma$ are comparable, then $x_1=x_2$.

Also, $\operatorname{Sup}\Gamma\subseteq\operatorname{supset}\Gamma$.

The definition of the supset was introduced and studied in papers [9], [10] [11], [12] under assumption that X is a vector space and the partial order is given by a proper cone G such that $G-G=X$. In these papers the authors introduced the property (denoted Condition (A)) of the space X requiring that for every subset Γ and every $a\in[\Gamma,\infty[$ there is a minimal element $x\in[\Gamma,\infty[$ such that $x\succeq a$, i.e. an element of $\operatorname{supset}\Gamma$. Thus, $\operatorname{supset}\Gamma$ satisfies all properties of Sup and, by the uniqueness of the latter, both sets coincide. Condition (A) is satisfied when the space X is finite-dimensional.

In the paper [12], again in the same setting, the optimal set of Γ is defined as the set of maximal elements of Γ. In the case where Γ is closed this definition coincides with $\operatorname{Max}\Gamma$.

3. Essential supremum in $L^0(X, \mathcal{F})$

In this section, we discuss the concept of the Essential Supremum for sets of random variables. In the scalar case, the traditional definition is obtained by lifting the linear order of the real line (i.e. the order generated by the cone (ray) $\mathbf{R}_+$) to the space L^0 of classes of equivalent random variables. The straightforward analog for the vector-valued random variables could be a procedure consisting in lifting the preorder or partial order in $\mathbf{R}^d$ given by a fixed cone. A slightly more sophisticated generalization is related with random partial orders given by random cones in $\mathbf{R}^d$, the situation, typical in models of financial markets with transaction costs. In view of the previous section, the it is natural to study the notion Esssup for the case when the preorder in $L^0(\mathbf{R}^d)$ is given by a countable random multi-utility representation. We consider the setting where the supremal set consists of $\mathcal{H}$-measurable random vectors, where $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{F}$. This additional feature seems to be new even in the scalar case where, usually, either $\mathcal{H} = \mathcal{F}$, or $\mathcal{H} = \{\emptyset, \Omega\}$, but we do not insist on this.

3.1. Essential supremum in a general setting

Let $(X, \mathcal{B}_X)$ be a separable metric space with its Borel σ-algebra and let $(\Omega, \mathcal{F}, P)$ be a probability space. As usual, $E\xi$ is the expectation of a real-valued random variable ξ. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. On the space $L^0(X, \mathcal{F})$ (of classes of equivalence) of X-valued random variables a preference relation is defined by a countable family $\mathcal{U} = \{u_j : \ j = 1, 2, ...\}$ of Carathéodory functions $u_j : \Omega \times X \to \mathbf{R}$, i.e. functions with the following properties:

(i) $u_j(., x) \in L^0(X, \mathcal{F})$ for every $x \in X$;
(ii) $u_j(\omega, .)$ is continuous for almost all $\omega \in \Omega$.

We recall that the important property of a Carathéodory function u on a separable metric space is that it is $\mathcal{F} \otimes \mathcal{B}_X$-measurable. Note that an order generated by a random cone can be generated by a countable family of linear Carathéodory functions, see the next subsection.

If $\gamma_1, \gamma_2 \in L^0(X, \mathcal{F})$, the relation $\gamma_2 \succeq \gamma_1$ means that $u_j(\gamma_2) \succeq u_j(\gamma_1)$ a.s. for all j. The just mentioned property ensures that the superpositions $u_j(\gamma_1)$, $u_2(\gamma_1)$ are random variables. The equivalence relation $\gamma_2 \sim \gamma_1$ has an obvious meaning.

We associate with an order interval $[\gamma_1, \gamma_2]$ in $L^0(X, \mathcal{F})$ its ω-sections, that is the order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$ in X corresponding to the partial orders represented by the families $\mathcal{U}(\omega) = \{u_j(\omega) : \ j = 1, 2, ...\}$.

Definition 3.1. *Let Γ be a subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}$-Esssup Γ the maximal subset $\hat{\Gamma}$ of $L^0(X, \mathcal{H})$ such that the following conditions hold:*

(a) $\hat{\Gamma} \succeq \Gamma$;

(b) if $\gamma \in L^0(X, \mathcal{H})$ and $\gamma \succeq \Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\gamma \succeq \hat{\gamma}$;

(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

An inspection of the proof of Theorem 3.7 in [5] (which deals with a partial order in $\mathbf{R}^d$) leads to the following statement [6]:

Theorem 3.2. *Let $\succeq$ be a preference relation in $L^0(X, \mathcal{F})$ represented by a countable family of Carathéodory functions. Let $\Gamma \neq \emptyset$ be such that $\bar{\gamma} \succeq \Gamma$ for some $\bar{\gamma} \in L^0(X, \mathcal{H})$. Suppose that for every $\gamma \in L^0([\Gamma, \infty[, \mathcal{H})$*

$$\Lambda(\gamma) := \operatorname{argmin}_{\zeta \in L^0([\Gamma, \gamma], \mathcal{H})} Eu(\zeta) \neq \emptyset, \tag{3.1}$$

where $u(\omega, z) = \sum_{j=1}^{\infty} 2^{-j} \arctan u_j(\omega, z)$. Then

$$\mathcal{H}\text{-Esssup}\, \Gamma = \cup_{\gamma \in L^0([\Gamma, \infty], \mathcal{H})} \Lambda(\gamma) \neq \emptyset. \tag{3.2}$$

Solving (3.1), we "minimize" the $\mathcal{H}$-measurable random variables ζ dominating Γ with respect to every "direction" u_j. It is easy to check that under (3.1) the set defined by the right-hand side of (3.2) satisfies the properties required of $\mathcal{H}$-Esssup Γ. The verification of the condition (3.1) is far from being trivial. At the moment we are able to do this only in the case where the ω-sections of the order intervals $[\gamma_1, \gamma_2]$ are compact, see Theorem 3.7 in [5]. It is not clear how to extend this theorem, for a general $\mathcal{H}$ to the case of preorders, even under the assumption of compactness of the order intervals in the quotient space (the difficulty is that the quotient mapping is only $\mathcal{F}$-measurable).

3.2. Essential supremum in $L^0(X)$ with respect to a random cone

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let X be a separable Hilbert space. Let $\omega \mapsto G(\omega) \subseteq X$ be a measurable set-valued mapping whose values are closed convex cones. The measurability means that

$$\text{graph}\, G := \{(\omega, x) \in \Omega \times X : \ x \in G(\omega)\} \in \mathcal{F} \otimes \mathcal{B}_X.$$

The positive dual $G^*(\omega)$ of $G(\omega)$ is defined as the set

$$G^*(\omega) := \{x \in X : \ xy \geq 0, \forall y \in G(\omega)\},$$

where xy is the scalar product generating the norm $||.||$ in X. Recall that a measurable mapping whose values are closed subsets admits a Castaing representation, i.e. there is a countable set of measurable selectors ξ_i of G such that $G(\omega) = \overline{\{\xi_i(\omega) : \ i \in \mathbf{N}\}}$ for all $\omega \in \Omega$. Thus,

$$\text{graph } G^* = \{(\omega, y) \in \Omega \times X : \ y\xi_i(\omega) \geq 0, \forall i \in \mathbf{N}\} \in \mathcal{F} \otimes \mathcal{B}_X,$$

hence, G^* is a measurable mapping and it admits a Castaing representation, i.e. there exists a countable set of $\mathcal{G}$-measurable selectors η_i of G^* such that $G^*(\omega) = \overline{\{\eta_i(\omega) : \ i \in \mathbf{N}\}}$ for all $\omega \in \Omega$.

Since $G = (G^*)^*$,

$$G(\omega) = \{(\omega, x) \in \Omega \times X : \ \eta_i(\omega)x \geq 0, \forall i \in \mathbf{N}\}. \tag{3.3}$$

Therefore, the relation $\gamma_2 - \gamma_1 \in G$ (a.s.) defines a preference relation $\gamma_2 \succeq \gamma_1$ in $L^0(X, \mathcal{F})$ and possesses a countable multi-utility representation given by the random linear functions $u_j(\omega, x) = \eta_j(\omega)x$ where η_j is a Castaing representation of G^*.

Notation. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$ and let $\Gamma \subseteq L^0(X, \mathcal{F})$. We shall use the notation $(\mathcal{H}, G)$-Esssup Γ instead of $\mathcal{H}$-Esssup Γ to indicate that the partial order is generated by the random cone G. In the following, we use the notation $G^0 := (-G) \cap G$.

We get the following [5]:

Theorem 3.3. *Let X be a separable Hilbert space and let $\succeq$ be a preference relation in $L^0(X, \mathcal{F})$ defined by a random cone G. Suppose that the subspaces $(G^0(\omega))^\perp$ are finite-dimensional a.s. Let $\Gamma \neq \emptyset$ be such that $\bar{\gamma} \succeq \Gamma$ for some $\bar{\gamma} \in L^0(X, \mathcal{F})$. Then $\mathcal{F}$-Esssup $\Gamma \neq \emptyset$.*

Remark 3.4. If we suppose that the ω-sections of $G \subseteq \mathbf{R}^d$ are proper, i.e. $G^0 = \{0\}$, then the order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$ are compact. Therefore, the set $(\mathcal{H}, G)$-Esssup Γ exists if Γ is bounded from above (i.e. if there exists $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$ such that $\bar{\gamma} - \Gamma \in G$.

4. Essential maximum in $L^0(X, \mathcal{F})$

4.1. Decomposability-based approach

Let us recall that the fastest proof of the existence of the element $\operatorname{esssup}\Gamma$ in the scalar case is assuming from the very beginning that the elements of Γ take values in the interval $[0, 1]$. It is sufficient to notice that one can replace Γ by the upward completion Γ^{up}, the smallest set containing Γ and closed with respect to operation $\vee$. Consider a sequence ξ_n on which $\sup_{\xi\in\Gamma}$ is attained, replace it by the monotone sequence $\xi^{(n)} = \xi_1 \vee ... \vee \xi_n$, and check that the limit of the latter is the required random variable. In the case where Γ^{up} is closed in L^0, it belongs to this set. It happens that this strategy of proof, appropriately modified, may work in the vector case and leads to a satisfactory definition of the maximal set. The approach presented in this section is developed in [5] and based on the notion of decomposability.

We start from some minor generalization of classical concepts, see, e.g. [14], [7].

Definition 4.1. *The set $\Gamma \subseteq L^0(X, \mathcal{F})$ is $\mathcal{H}$-decomposable if for any its elements γ_1, γ_2 and $A \in \mathcal{H}$ the random variable $\gamma_1 I_A + \gamma_2 I_{A^c}$ belongs to Γ.*

Definition 4.2. *We denote by $\operatorname{env}_{\mathcal{H}}\Gamma$ the smallest $\mathcal{H}$-decomposable subset of $L^0(X, \mathcal{F})$ containing Γ and by $\operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$ its closure in $L^0(X, \mathcal{F})$.*

The "interior" description of the $\mathcal{H}$-envelope of Γ is as follows:

Lemma 4.3. *The set $\operatorname{env}_{\mathcal{H}}\Gamma$ is formed by all random variables $\sum \gamma_i I_{A_i}$ where $\gamma_i \in \Gamma$ and $\{A_i\}$ is an arbitrary finite partition of Ω into $\mathcal{H}$-measurable subsets. Moreover, $\mathcal{H}$-$\operatorname{cl}\operatorname{env}\Gamma$ is $\mathcal{H}$-decomposable.*

We recall the two notions of Essential Maximum introduced in [6]:

Definition 4.4. *Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We put*

$$\mathcal{H}\text{-Essmax}\,\Gamma = \{\gamma \in \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma : \ \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma \cap [\gamma, \infty[= [\gamma, \gamma]\}.$$

Definition 4.5. *Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}$-$\operatorname{Essmax}_1\Gamma$ the largest subset $\hat{\Gamma} \subseteq \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$ such that the following conditions hold:*

(i) if $\gamma \in \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\hat{\gamma} \succeq \gamma$;

(ii) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

In the same way we define similar notions of $\mathcal{H}$-Essmin and $\mathcal{H}$-Essmin$_1$.

The proofs of the following two lemmata is exactly the same as of Lemma 2.5 and Lemma 2.6.

Lemma 4.6. $\mathcal{H}$-Essmax$_1\Gamma \subseteq \mathcal{H}$-Essmax Γ.

Lemma 4.7. *Let* $\mathcal{H}$-Essmax$_1\Gamma \neq \emptyset$. *Then* $\mathcal{H}$-Essmax$_1\Gamma = \mathcal{H}$-EssmaxΓ.

In [6], it is shown that $\mathcal{H}$-Essmax$_1\Gamma = \mathcal{H}$-Essmax $\Gamma \neq \emptyset$ as soon as the preference relation is a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. The same result is also claimed for preference relations when Γ is a set of measurable selectors of a closed random set.

Corollary 4.8. *Let* $\succeq$ *be a partial order in* $L^0(\mathbf{R}^d)$ *such that all order intervals* $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, *are compacts a.s. Let* $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ *be such that there is* $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$ *such that* $\bar{\gamma} \succeq \Gamma$. *Then*

$$\mathcal{H}\text{-Esssup}\,\Gamma = \text{Essmin}\, L^0([\Gamma, \infty), \mathcal{H}) = \text{Essmin}_1 L^0([\Gamma, \infty), \mathcal{H}) \neq \emptyset.$$

Proof. Observe that $L^0([\Gamma, \infty), \mathcal{H}) \succeq \tilde{\gamma}$ where $\tilde{\gamma} \in \Gamma$ is arbitrary. Moreover, $L^0([\Gamma, \infty), \mathcal{H})$ is closed and $\mathcal{H}$-decomposable. Therefore, by [6], the set $\text{Essmin}\, L^0([\Gamma, \infty), \mathcal{H}) = \text{Essmin}_1 L^0([\Gamma, \infty), \mathcal{H}) \neq \emptyset$ is nonempty and satisfies the required properties to be the unique set $\mathcal{H}$-EsssupΓ. □

4.2. Convexity-based approach

In this subsection we suggest a new notion of the maximal set for the case where X is a separable normed space. The most interesting case: $\mathbf{R}^d$ with a partial order defined by a closed cone. Of course, for the scalar case and a closed set Γ, this maximal set is reduced to the singleton, containing the maximal point of Γ.

We denote by $\text{conv}\,\Gamma$ the smallest convex subset of $L^0(X, \mathcal{F})$ containing Γ and by $\text{cl}\,\text{conv}\,\Gamma$ its closure in $L^0(X, \mathcal{F})$.

Definition 4.9. *Let* Γ *be a non-empty subset of* $L^0(X, \mathcal{F})$. *We put*

$$\text{Essmax}^c\,\Gamma = \{\gamma \in \text{cl}\,\text{conv}\,\Gamma : \ \text{cl}\,\text{conv}\,\Gamma \cap [\gamma, \infty[= [\gamma, \gamma]\}.$$

Remark 4.10. Suppose that $\Gamma \subseteq L^0(X, \mathcal{F})$ is both $\mathcal{H}$-decomposable, convex and closed where $\mathcal{H} \subseteq \mathcal{F}$ is a sub-σ-algebra. Then $\Gamma = \text{cl}\,\text{conv}\,\Gamma = \text{cl}\,\text{env}_{\mathcal{H}}\Gamma$ and, therefore, $\text{Essmax}^c\,\Gamma = \mathcal{H}$-Essmax Γ.

Definition 4.11. *Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$.*

We denote by $\mathrm{Essmax}_1^c \Gamma$ *the largest subset* $\hat{\Gamma} \subseteq \mathrm{cl}\,\mathrm{conv}\,\Gamma$ *such that the following conditions hold:*

(i) if $\gamma \in \mathrm{cl}\,\mathrm{conv}\,\Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\hat{\gamma} \succeq \gamma$;

(ii) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

As in the last section we have the following:

Lemma 4.12. $\mathcal{H}$-$\mathrm{Essmax}_1^c \Gamma \subseteq \mathcal{H}$-$\mathrm{Essmax}^c \Gamma$.

Lemma 4.13. *Let $\mathcal{H}$-$\mathrm{Essmax}_1^c \Gamma \neq \emptyset$. Then $\mathcal{H}$-$\mathrm{Essmax}_1^c \Gamma = \mathcal{H}$-$\mathrm{Essmax}^c \Gamma$.*

Proposition 4.14. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family of linear functions satisfying (i), (ii) and such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. Let Γ be a non-empty subset of $L^0(\mathbf{R}^d, \mathcal{F})$. Suppose that there exists $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F})$ such that $\bar{\gamma} \succeq \Gamma$. Then* $\mathrm{Essmax}^c\, \Gamma = \mathrm{Essmax}_1^c \Gamma \neq \emptyset$.

Proof. Note that the set $\mathrm{Essmax}^c\, \Gamma$ obviously satisfies (ii) and it remains only to check (i) and that $\mathrm{Essmax}^c\, \Gamma \neq \emptyset$. For $\gamma \in \mathrm{cl}\,\mathrm{conv}\,\Gamma$, we define random variables

$$\alpha_j(\gamma) := \alpha_j(\omega, \gamma) := 2^{-j}/(1 + |u_j(\gamma(\omega))| + |u_j(\bar{\gamma}(\omega))|).$$

Put

$$u(x, \gamma) := u(x, \omega, \gamma) := \sum_j \alpha_j(\gamma) u_j(\omega, x).$$

Then the mapping $\xi \mapsto u(\xi, \gamma)$ is well-defined for $\xi \in [\gamma, \bar{\gamma}]$ and for such an argument $u(\xi, \gamma)$ is a random variable with values in the interval $[-1, 1]$. Let

$$c := \sup_{\tilde{\gamma} \in \mathrm{cl}\,\mathrm{conv}\,\Gamma \cap L^0([\gamma, \infty), \mathcal{F})} Eu(\tilde{\gamma}).$$

Let $(\tilde{\gamma}_n)$ be a sequence on which the supremum in the above definition is attained. As the set $\mathrm{cl}\,\mathrm{conv}\,\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{H})$ is convex and $[\gamma, \bar{\gamma}]$ is compact a.s., we may assume without loss of generality (by applying Theorem 5.2.3 [7] on convergent subsequences) that the sequence of $\tilde{\gamma}_n$ converges a.s. to some $\tilde{\gamma}_\infty \in \mathrm{cl}\,\mathrm{conv}\,\Gamma \cap L^0([\gamma, \infty), \mathcal{F})$ such that $c := Eu(\tilde{\gamma}_\infty)$.

By definition of c, it is straightforward that $\tilde{\gamma}_\infty \in \mathrm{Essmax}\,\Gamma$ and the conclusion follows. $\square$

4.3. Comment on essential maximum of processes

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in\mathbf{R}_+}, P)$ be a stochastic basis satisfying the usual assumptions and let $X = (X_t)_{t\in\mathbf{R}_+}$ and $Y = (Y_t)_{t\in\mathbf{R}_+}$ be two real-valued measurable processes. Following Dellacherie [2], we say that the process Y is *essential majorant* of X if the set $\{X > Y\} = \{(\omega, t) : \ X_t(\omega) > Y_t(\omega)\}$ is negligeable, i.e. its projection on Ω has zero probability (the projection is measurable because the σ-algebra is complete). Let Γ be an arbitrary set of measurable processes. The measurable process Y is the *essential supremum* of Γ (notation: $Y = \text{ess.sup}\,\Gamma$) if Y is an essential majorant for every process from Γ and any other process Y' with the same property is an essential majorant of Y. Of course, in this definition the word "measurable" can be replaced by the words "optional", "predictable", etc. To get results one needs to impose certain assumptions on the regularity of trajectories. In view of financial applications, it is interesting to study the problem for the vector-valued processes. This is a problem for further studies.

5. Applications in finance

In this section, we recall two applications in finance given in [5], [6] for a financial model with proportional transaction costs.

In the model we are given a closed proper convex cone $K \subset \mathbf{R}^d$ whose interior contains $\mathbf{R}^d_+ \setminus \{0\}$ and a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t=0,...,T}, P)$ with a d-dimensional adapted process $S = (S_t)$ with strictly positive components.

Define the random diagonal operators

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S^1_t, ..., x^d/S^d_t), \qquad t = 0, ..., T,$$

and relate with them the random cones $\widehat{K}_t := \phi_t K$. We consider the set $\widehat{\mathcal{V}}$ of $\mathbf{R}^d$-valued adapted processes $\widehat{V}$ such that $\Delta\widehat{V}_t := \widehat{V}_t - \widehat{V}_{t-1} \in -\widehat{K}_t$ for all t and the set $\mathcal{V}$ whose elements are the processes V with $V_t = \phi_t^{-1}\widehat{V}_t$, $\widehat{V} \in \widehat{\mathcal{V}}$.

In the context of the theory of markets with proportional transaction costs, K is the solvency cone in a model with efficient friction corresponding to the description in terms of a numéraire, $\mathcal{V}$ is the set of value processes of self-financing portfolios. The notations with hat correspond to the description of the model in terms of "physical" units where the portfolio dynamics is much simpler because it does not depend on price movements. A typical example is the model of currency market defined via the matrix of transaction costs coefficients $\Lambda = (\lambda^{ij})$ with non-negative entries and $\lambda^{ii} = 0$. In

this case

$$K = \text{cone}\,\{(1+\lambda^{ij})e_i - e_j,\ e_i, 1 \le i,j \le d\}.$$

Another example is the commodity market where all transactions are payed from the money account. In this case

$$K = \text{cone}\,\{\gamma^{ij}e_1 + e_i,\ (1+\gamma^{1i})e_1 - e_i,\ (-1+\gamma^{j1})e_1 + e_j,\ e_i,\ 1 \le i,j \le d\}.$$

We assumed for simplicity that K is constant. In general, $K = (K_t)$ is an adapted random process whose values are convex closed proper cones, e.g., given by an adapted matrix-valued process $\Lambda = (\Lambda_t)$. But even in the constant case $\widehat{K} = (\widehat{K}_t)$ is a random cone-valued process. Note that one can use modeling involving only $\widehat{K}$ defined, e.g., by the bid-ask (adapted matrix-vaued) process but this is just a different parametrization leading to the same geometric structure.

5.1. Hedging of European options in a discrete-time model with transaction costs

In this model, the contingent claim is a d-dimensional random vector. We shall use the notation Y_T when the contingent claim is expressed in units of the numéraire and $\widehat{Y}_T$ when it is expressed in physical units. The relation is obvious: $\widehat{Y}_T = \phi_T Y_T$.

The value process $V \in \mathcal{V}$ is called *minimal* if $V_T = Y_T$ and any process $W \in \mathcal{V}$ such that $W_T = Y_T$ and $W_t \preceq_K V_t$ for all $t \le T$ coincides with V. The questions of interest are whether minimal portfolios do exist and how they can be found. We denote $\mathcal{V}^E_{min}$ the set of all minimal processes. The set $\widehat{\mathcal{V}}^E_{min}$ is defined in the obvious way. In the following, the notation $\widehat{W} \succeq_{\widehat{K}} \widehat{V}$ for two processes $\widehat{V}, \widehat{W}$ means that $\widehat{W}_t - \widehat{V}_t \in \widehat{K}_t$ for all $t = 0, \cdots, T$.

Proposition 5.1. *Suppose that $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \le T-1$, and suppose there exits a least one $\widehat{V} \in \widehat{\mathcal{V}}$ such that $\widehat{V}_T \ge_{\hat{K}_T} \hat{Y}_T$. Then $\widehat{\mathcal{V}}^E_{min} \ne \emptyset$ and $\widehat{\mathcal{V}}^E_{min}$ coincides with the set of solutions of backward inclusions*

$$\widehat{V}_t \in (\mathcal{F}_t, \widehat{K}_{t+1})\text{-Esssup}\,\{\widehat{V}_{t+1}\}, \quad t \le T-1, \quad \widehat{V}_T = \widehat{Y}_T. \tag{5.4}$$

Moreover, any $\widehat{W} \in \mathcal{V}$ with $\widehat{W}_T \succeq \widehat{Y}_T$ is such that $\widehat{W} \succeq_{\widehat{K}} \widehat{V}$ for some $\widehat{V} \in \mathcal{V}^E_{min}$.

The hypothesis $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \leq T-1$, of the above proposition is equivalent to the absence of arbitrage opportunities of the second kind (**NA2**), see [7], Th. 3.2.20. Note that it is always fulfilled when the price process S admits an equivalent martingale measure. Actually, we have the following equivalence below where $\mathcal{V}_t^E(Y_T)$ and $\mathcal{V}_{t,min}^E(Y_T)$ are defined similarly for the model starting at time t.

Theorem 5.2. **NA2** *holds if and only if for all European claim $\widehat{Y}_T$ such that $\widehat{\mathcal{V}}_t^E(\widehat{Y}_T) \neq \emptyset$, $\widehat{\mathcal{V}}_{t,min}^E(\widehat{Y}_T)$ coincides with the set of solutions of backward inclusions*

$$\widehat{V}_s \in (\mathcal{F}_s, \widehat{K}_{s+1})\text{-Esssup}\,\{\widehat{V}_{s+1}\}, \quad t \leq s \leq T-1, \quad \widehat{V}_T = \widehat{Y}_T. \tag{5.5}$$

Proof. The direct implication is immediate by virtue of Proposition 5.1. Reciprocally, we need to show that **NA2** holds if the backward inclusions above characterize the minimal super-hedging prices. To do so, suppose there exists a process $\widehat{W} \in \widehat{\mathcal{V}}_t^E(0) \neq \emptyset$. By assumption, there exists $\widehat{V} \in \widehat{\mathcal{V}}_{t,min}^E(0)$ such that $\widehat{W} \succeq \widehat{V}$. But $\widehat{\mathcal{V}}_{t,min}^E(0) = \{0\}$ hence $\widehat{W}_s \geq_{\widehat{K}_s} 0$ i.e. **NA2** holds. □

Definition 5.3. *We say that $\widehat{V}_0$ is a minimal super-hedging price of $\widehat{Y}_T$ if the property $\widehat{V}_0 \geq_{\widehat{K}_0} \widehat{W}_0$ for some super-hedging price $\widehat{W}_0$ of $\widehat{Y}_T$ implies $\widehat{V}_0 = \widehat{W}_0$, i.e. the set of minimal super-hedging price for $\widehat{Y}_T$ is defined as $\widehat{K}_0\text{-Min}\,\Gamma^E_{\widehat{Y}_T}$ (with respect to the partial order generated by $\widehat{K}_0$) where $\Gamma^E_{\widehat{Y}_T}$ is the set of all super-hedging prices of $\widehat{Y}_T$ expressed in physical units.*

Proposition 5.4. *Any minimal super-hedging price of $\widehat{Y}_T$ is the initial value of a minimal super-hedging portfolio process of $\widehat{Y}_T$.*

Proof. Consider a minimal super-hedging price of $\widehat{Y}_T$. This is the initial value $\widehat{W}_0$ of a portfolio process $\widehat{W}$ super-hedging the payoff $\widehat{Y}_T$. We know there exists a minimal super-hedging portfolio process $\widehat{V}$ of $\widehat{Y}_T$ such that $\widehat{W} \geq_{\widehat{K}} \widehat{V}$ hence $\widehat{W}_0 \geq_{\widehat{K}_0} \widehat{V}_0$ which implies that $\widehat{W}_0 = \widehat{V}_0$ since $\widehat{W}_0$ is minimal. Therefore, $\widehat{W}_0 = \widehat{V}_0$ is the initial endowment of the minimal super-hedging portfolio process $\widehat{V}$. □

Corollary 5.5. *Suppose that $\Gamma^E_{\widehat{Y}_T}$ is closed. Let $\widehat{\mathcal{V}}^{E,0}_{\min}(\widehat{Y}_T)$ be the set of all initial values of a minimal super-hedging portfolio process of $\widehat{Y}_T$. Then,*

$$\widehat{K}_0\text{-Min}\,\Gamma^E_{\widehat{Y}_T} = \widehat{K}_0\text{-Min}\,\widehat{\mathcal{V}}^{E,0}_{\min}(\widehat{Y}_T).$$

Observe that $\Gamma^E_{\widehat{Y}_T}$ is closed under robust absence of arbitrage opportunities [7] or equivalently when there exists a *strictly consistent price system*, i.e. a martingale $Z \in \operatorname{int} \hat{\mathrm{K}}^*$. The result above gives a constructive approach of the set $\Gamma^E_{\widehat{Y}_T}$ of all super-hedging prices of $\widehat{Y}_T$. Indeed, the set $\mathcal{V}^{E,0}_{\min}(\widehat{Y}_T)$ is obtained by the backward inclusions given in Proposition 5.1. Moreover, $\Gamma^E_{\widehat{Y}_T} = \widehat{K}_0\text{-Min}\,\Gamma^E_{\widehat{Y}_T} + \widehat{K}_0$.

5.2. Equivalent characterization of minimal super-hedging portfolio processes of European claims

In the following, we assume that $\widehat{Y}_T \geq_{\widehat{K}_T} -k\mathbf{1}$ for some $k \geq 0$ and Condition **NA2** holds. Let us define for $t = 0, \cdots, T$ the super-hedging prices of $\widehat{Y}_T$ at time t:

$$P_t(\widehat{Y}_T) := \{x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t) : \; x_t \in -A^T_t + \widehat{Y}_T\},$$

where

$$A^T_u := \sum_{r=u}^{T} L^0(-\widehat{G}_r, \mathcal{F}_r), \quad u = 0, \cdots, T.$$

As Condition **NA2** holds, the Robust No Arbitrage condition $\mathbf{NA}^r$ holds ([7]) and, we know that A^T_u is closed in probability for each $u \geq 0$ hence $P_t(\widehat{Y}_T)$ is also closed. Moreover, a dual characterization of $P_t(\widehat{Y}_T)$ is given in [7], i.e. $x_t \in P_t(\widehat{Y}_T)$ if an only if $EZ_t x_t \geq EZ_T\widehat{Y}_T$ for every consistent price system Z, i.e. a martingale Z such that $Z_t \in \widehat{K}^*_t$ for all $s = t, \cdots, T$. It follows that, if $\widehat{Y}_T \in L^0(\mathbf{R}^d, \mathcal{F}_u)$ where $u \geq t+1$, then

$$P_t(\widehat{Y}_T) := \{x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t) : \; x_t \in -A^u_t + \widehat{Y}_T\}.$$

Observe that $P_t(\widehat{Y}_T)$ is $\mathcal{F}_t$-decomposable and bounded from below. Indeed, any $x_t \in P_t(\widehat{Y}_T)$ is a super-hedging price for $\widehat{Y}_T$ hence

$$Z_t x_t \geq E(Z_T\widehat{Y}_T|\mathcal{F}_t) \geq -kZ_t\mathbf{1},$$

for any consistent price system Z (see [7]). Under **NA**2 or equivalently under Condition **B** (see [7], page 160), we deduce that $x_t \geq_{\widehat{K}_t} -k\mathbf{1}$, i.e. $P_t(\widehat{Y}_T) \geq_{\widehat{K}_t} -k\mathbf{1}$. By [6], it follows that

$$X_t := (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}\, P_t(\widehat{Y}_T) = (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}_1 P_t(\widehat{Y}_T) \neq \emptyset.$$

This is the set of all minimal prices at date t. At last, observe the tower rule $P_t(P_{t+1}(\widehat{Y}_T)) = P_t(\widehat{Y}_T)$ where

$$\begin{aligned} P_t(P_{t+1}(\widehat{Y}_T)) &:= \bigcup_{x_{t+1}\in P_{t+1}(\widehat{Y}_T)} P_t(x_{t+1}), \\ P_t(x_{t+1}) &:= \{x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t) : \ x_t \in -A_t^{t+1} + x_{t+1}\}. \end{aligned}$$

By induction, we deduce that $x_t \in P_t(x_{t+1})$ implies that $x_t \geq_{\widehat{K}_t} -k\mathbf{1}$ hence $(\mathcal{F}_t, \widehat{K}_t)$-Essmin $P_t(X_{t+1}) \neq \emptyset$ by [6].

Proposition 5.6. $(\mathcal{F}_t, \widehat{K}_t)$-Essmin $P_t(X_{t+1}) \subseteq X_t$.

Proof. Observe that

$$(\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}P_t(X_{t+1}) \subseteq P_t(X_{t+1}) \subseteq P_t(P_{t+1}(Y_T)) = P_t(Y_T).$$

Let us consider $\gamma_t \in (\mathcal{F}_t, \widehat{K}_t)$-Essmin $P_t(X_{t+1})$ and suppose that $\tilde{\gamma}_t \leq_{\widehat{K}_t} \gamma_t$ where $\tilde{\gamma}_t \in P_t(\widehat{Y}_T)$. By the tower rule, $\tilde{\gamma}_t \in P_t(\tilde{\gamma}_{t+1})$ where $\tilde{\gamma}_{t+1} \in P_{t+1}(\widehat{Y}_T)$. By definition of X_{t+1}, there exists $\bar{\gamma}_{t+1} \in X_{t+1}$ such that $\tilde{\gamma}_{t+1} \geq_{\widehat{K}_{t+1}} \bar{\gamma}_{t+1}$. It follows that $\tilde{\gamma}_t \in P_t(\bar{\gamma}_{t+1}) \subseteq P_t(X_{t+1})$.

Therefore, as $\gamma_t \in (\mathcal{F}_t, \widehat{K}_t)$-Essmin $P_t(X_{t+1})$, we have $\tilde{\gamma}_t = \gamma_t$. From there, we infer that $\gamma_t \in (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}P_t(\widehat{Y}_T) = X_t$. □

Theorem 5.7. *Suppose that* **NA**2 *holds and* $\widehat{Y}_T \geq_{\widehat{K}_T} -k\mathbf{1}$ *where* $k \in \mathbf{R}$. *The set of minimal portfolio processes super-replicating the European claim* $\widehat{Y}_T$ *is given by the solutions of backward inclusions:*

$$x_t \in (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}P_t(x_{t+1}), \quad x_T = \widehat{Y}_T.$$

Proof. Let $x_T \in X_T = \{\widehat{Y}_T\}$. Suppose that $x_{t+1} \in X_{t+1}$ for some $t \leq T-1$. Then, we choose any $x_t \in (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}P_t(x_{t+1}) \neq \emptyset$. Indeed, by proposition 5.6, $x_t \in X_t$ and $x_t \in P_t(x_{t+1})$ implies that

$$\Delta x_{t+1} \in -\widehat{K}_t - \widehat{K}_{t+1},$$

i.e. $x_u := x_0 + \sum_{t=1}^{u} \Delta x_t$, $u \leq T$, defines a portfolio process. Reciprocally, consider a minimal porfolio process v. Then $v_T = \widehat{Y}_T$ by definition, and $v_{T-1} \geq_{\widehat{K}_{T-1}} x_{T-1}$ for some $x_{T-1} \in X_{T-1}$ by definition of X_{T-1}. Moreover, $v_{T-2} \in P_{T-2}(v_{T-1})$ implies that $v_{T-2} \in P_{T-2}(x_{T-1})$. Therefore, there exists $x_{T-2} \in (\mathcal{F}_{T-2}, \widehat{K}_{T-2})\text{-Essmin}_1 P_{T-2}(x_{T-1}) \subseteq X_{T-2}$ such that $v_{T-2} \geq_{\widehat{K}_{T-2}} x_{T-2}$. We pursue the reasoning and, finally, we find a portfolio process x with $x_t \in X_t$ and $x_t \in (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}P_t(x_{t+1})$ such that $v \succeq x$. Since v is minimal, we have $x = v$ and the conclusion follows. □

5.3. Hedging of american options in a discrete-time model with transaction costs

Let $Y = (Y_t)$ be an American contingent claim expressed in units of the numéraire and let $\widehat{Y} = (\widehat{Y}_t)$ when it is expressed in physical units.

The value process $\widehat{V} \in \widehat{\mathcal{V}}$ is called *minimal* if $\widehat{V} \succeq_{\widehat{K}} \widehat{Y}$ and any process $\widehat{W} \in \widehat{\mathcal{V}}$ such that $\widehat{Y} \preceq_{\widehat{K}} \widehat{W} \preceq_{\widehat{K}} \widehat{V}$ coincides with $\widehat{V}$. The notation means that to compare values of the processes at time t one uses the partial order generated by the random cone $\widehat{K}_t$. We denote $\widehat{\mathcal{V}}^A_{min}$ the set of all minimal processes.

Proposition 5.8. *Suppose there is at least one $\hat{V} \in \widehat{\mathcal{V}}$ such that $\widehat{V} \succeq_{\widehat{K}} \widehat{Y}$. The set $\mathcal{V}^A_{min}$ is non-empty and coincides with the set of solutions of backward inclusions*

$$\widehat{V}_t \in (\mathcal{F}_t, \widehat{K}_t)\text{-Essmin}_1 L^0((\widehat{Y}_t+\widehat{K}_t)\cap(\widehat{V}_{t+1}+\widehat{K}_{t+1}), \mathcal{F}_t), \quad t \leq T-1, \quad \widehat{V}_T = \widehat{Y}_T. \tag{5.6}$$

Moreover, any $\widehat{W} \in \widehat{\mathcal{V}}$ such that $\widehat{W} \succeq_{\widehat{K}} \widehat{Y}$ is such that $\widehat{W} \succeq_{\widehat{K}} \widehat{V}$ for some $\widehat{V} \in \mathcal{V}^A_{min}$.

Adapting the notations used for European claims, we also get the following result:

Corollary 5.9. *Suppose that $\Gamma^A_{\widehat{Y}}$ is closed. Let $\widehat{\mathcal{V}}^{A,0}_{\min}(\widehat{Y})$ be the set of all initial values of a minimal super-hedging portfolio process of $\widehat{Y}$. Then,*

$$\widehat{K}_0\text{-Min}\,\Gamma^A_{\widehat{Y}_T} = \widehat{K}_0\text{-Min}\,\widehat{\mathcal{V}}^{A,0}_{\min}(\widehat{Y}).$$

It appears that $\Gamma^A_{\widehat{Y}}$ is also closed when we suppose the existence of a strictly consistent price system [7]. Observe that these results still hold for more general proper convex random cones. Moreover, it would be interesting to extend them to random solvency sets G which are not necessarily convex. To do so, it suffices that G defines a partial order such that the essential supremum of a family of random variables bounded from above with respect to G exists.

Our results are well adapted to the specific problem of super hedging a vector-valued contingent claim in a multivariate financial model defined by a random partial order since almost sure comparisons are required. It would also be interesting to consider a geometrical approach via random sets for applications in finance, see for instance [16] or [4] and [15] for risk measures. This is left for a future study.

6. Appendix

In the papers [9], [10], [11] and [12], where the preorder is defined by a deterministic cone, another definition of deterministic supremum of a set Γ, denoted by $\operatorname{supset}\Gamma$, is given. In the following, we compare $\operatorname{supset}\Gamma$ with $\sup\Gamma$ and show that they coincide when $\sup\Gamma \neq \emptyset$ in the case of an arbitrary poset X, i.e. a set endowed with a partial order $\succeq$.

Definition 6.1. *Let Γ be a subset of X. We denote by* $\operatorname{supset}\Gamma$ *the largest subset of X which satisfies the following conditions:*

- $\operatorname{supset}\Gamma \succeq \Gamma$,
- $z \in \operatorname{supset}\Gamma$ *and* $z \succeq \tilde{z} \succeq \Gamma$ *implies that* $z = \tilde{z}$.

Proposition 6.2. *Let Γ be a subset of X. We have:*

- $\sup\Gamma \subseteq \operatorname{supset}\Gamma$,
- $\sup\Gamma \neq \emptyset \Rightarrow \sup\Gamma = \operatorname{supset}\Gamma$.

Proof. In the case where $\sup\Gamma = \emptyset$, the inclusion $\sup\Gamma \subseteq \operatorname{supset}\Gamma$ is trivial. Otherwise, since $\sup\Gamma \succeq \Gamma$, it suffices to consider $\hat{z} \in \sup\Gamma$ such that $\hat{z} \succeq y \succeq \Gamma$ and to show that $\hat{z} = y$. Since we suppose that $\sup\Gamma \neq \emptyset$, there exists, by (b_0), $\hat{y} \in \sup\Gamma$ such that $y \succeq \hat{y}$ hence $\hat{z} \succeq \hat{y}$ and finally $\hat{z} = \hat{y} = y$ by (c_0). Reciprocally, since $\sup\Gamma \neq \emptyset$, for all $z \in \operatorname{supset}\Gamma$, there exists $\hat{z} \in \sup\Gamma$ such that $z \succeq \hat{z}$. By definition of $\operatorname{supset}\Gamma$, we have $z = \hat{z}$ and finally $z \in \sup\Gamma$, i.e. $\operatorname{supset}\Gamma \subseteq \sup\Gamma$. $\square$

In the papers [9], [10], [11] and [12], recall that the partial order $\succeq$ is generated by a proper cone G such that $X = G - G$. Moreover, a condition (A) is considered under which any subset Γ is such that $\operatorname{supset}\Gamma$ satisfies (b_0) provided that $\operatorname{supset}\Gamma \neq \emptyset$. By proposition 6.2, we deduce that a poset X such that $\sup\Gamma \neq \emptyset$ for all bounded from above subset of X satisfies (A) since, in this case, $\operatorname{supset}\Gamma = \sup\Gamma$ satisfies (b_0). As shown before, this is the case when the order intervals are compact, in particular when $X = \mathbf{R}^d$ and G is a proper cone even if $\mathbf{R}^d \neq G - G$.

We show a counterexample where $\sup\Gamma = \emptyset$ while $\operatorname{supset}\Gamma \neq \emptyset$:
Counterexample. Consider the mapping $x \mapsto u(x)$ on $X = \mathbf{R}_+$ whose graph is given in Picture 1. We suppose that u is continuous on R_+ except at $x = 1$ where it is left continuous. Moreover, $u(x) > 1$ for all $x > 1$.

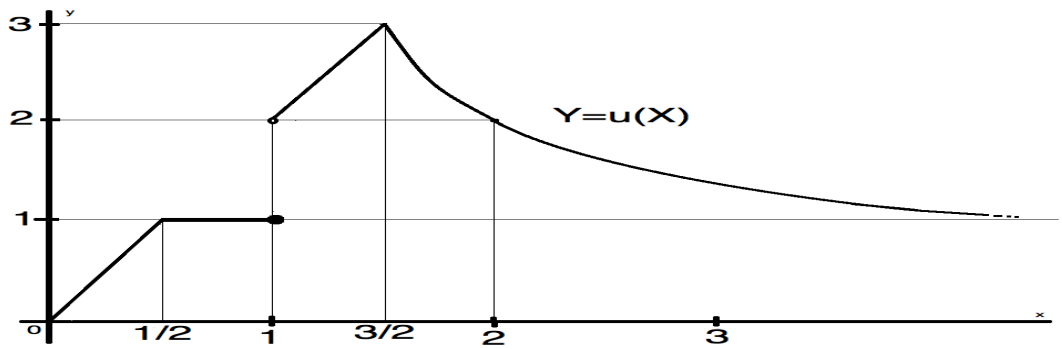

Figure 1: Graphic representation of $x \mapsto u(x)$.

Let $x \succeq y$ if, simultaneously, $u(x) > u(y)$ and $x > y$ when $x \neq y$. Clearly, for all $x \in \mathbf{R}_+$, $x \succeq x$, i.e. $\succeq$ is reflexive. Suppose that $x \succeq y$ and $y \succeq x$. In the case where $x \neq y$, then $x > y > x$ i.e. a contradiction. At last, suppose that $x \succeq y \succeq z$. In the case where $x = y$ or $y = z$, we have $x \succeq z$. Otherwise, $u(x) > u(y) > u(z)$ and $x > y > z$, i.e. $x \succeq z$. It follows that $\succeq$ is a partial order.

Let us consider Γ the set of all x such that $0 \leq x \leq 1$. Suppose that $x \succeq \Gamma$. If $x \in \Gamma$, then $x > y$ for all $y \in \Gamma \backslash \{x\}$ hence $x \geq 1$, i.e. $x = 1$ hence $1 \succeq 1/2$. It follows that $u(1) > u(1/2)$, which yields a contradiction. We deduce that $u(x) > u(y)$ and $x > y$ for all $y \in \Gamma$. Therefore, $x > 1$. Reciprocally, it is clear that $x > 1$ implies that $x \succeq \Gamma$, see Picture 1. We deduce that $[\Gamma, \infty) = \mathbf{R}_+ \backslash \Gamma$.

Let us show that each $x \geq 2$ belongs to $\operatorname{supset} \Gamma$. To do so, suppose that $x \succeq y \succeq \Gamma$, $y \in X$. In the case where $x \neq y$, $u(x) > u(y)$ where $1 < y < x$ which is in contradiction with the variations of u. Moreover, for all x such that $1 < x < 2$, there exists $y \succeq \Gamma$ such that $x < y$ and $u(y) < u(x)$, see Picture 1, i.e. such that $1 \preceq y \preceq x$ and $y \neq x$. It follows that $x \notin \operatorname{supset} \Gamma$ and finally $x \in \operatorname{supset} \Gamma$ if and only if $x \geq 2$.

At last, suppose that $\sup \Gamma \neq \emptyset$. Since $3/2 \succeq \Gamma$, there exists, by (b_0), $\hat{x} \in \sup \Gamma$ such that $3/2 \succeq \hat{x}$. Recall that by Proposition 6.2, $\hat{x} \in \operatorname{supset} \Gamma$ hence $\hat{x} \geq 2$. It follows that $3/2 \neq \hat{x}$ hence $3/2 > 2$, i.e. a contradiction. $\square$

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