Essential Supremum with Respect to a Random Partial Order

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Abstract

Inspired by the theory of financial markets with transaction costs, we study a concept of essential supremum in the framework where a random partial order in $\mathbf{R}^d$ is lifted to the space $L^0(\mathbf{R}^d)$ of d-dimensional random variables. In contrast to the classical definition, we define the essential supremum as a subset of random variables satisfying some natural properties. Applications of the introduced notion to a hedging problem under transaction costs and set-valued dynamic risk measures are given.

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1. Introduction

The aim of this paper is to study a seemingly new concept of essential supremum in the framework where a rather general (possibly, random) partial order in $\mathbf{R}^d$ is lifted to the space $L^0(\mathbf{R}^d, \mathcal{F})$ of d-dimensional random variables. In contrast to the classical definition of the esssup in $L^0 = L^0(\mathbf{R}, \mathcal{F})$ as a random variable, we define, for $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ and a sub-σ-algebra $\mathcal{H} \subseteq \mathcal{F}$, the essential supremum, $\mathcal{H}$-$\operatorname{Esssup}\Gamma$, as a set of $\mathcal{H}$-measurable random variables satisfying some natural properties. In the classical case, where the partial order in L^0 is generated by the linear order of the real-line, this set is a singleton, consisting from the usual esssup when $\mathcal{H} = \mathcal{F}$ and vraimax when $\mathcal{H}$ is trivial. The importance of considering intermediate σ-algebras is obvious because of applications of such a concept to dynamical models arising in mathematical finance.

Our interest to such objects as set-valued essential supremum/infimum originates from an attempt to give a description of the minimal portfolios dominating a contingent claim in the hedging problem in the presence of proportional transaction costs.

To explain the motivation and the necessity of this study we recall, first, some basic facts about the classical hedging problem for the discrete-time model of frictionless market which can be formulated as follows. We are given a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t=0,\dots,T}, P)$ with a d-dimensional adapted process $S = (S_t)$ whose first component is equal identically to unit and a scalar random variable ξ or a scalar adapted processes $A = (A_t)$. In financial context S models the price process of d traded securities (the first one is the numéraire) while ξ and A stand for pay-offs of a European or American contingent claim, respectively.

The hedging problem is to find the set $\Gamma \in \mathbf{R}$ of points x for those there exist d-dimensional processes $H \in \mathcal{P}$ such that $x + H \cdot S_T \geq \xi$ (for the European claim) or $x + H \cdot S \geq A$ (for the American claim). Here $\mathcal{P}$ denotes the space of predictable processes, i.e. such that H_t is $\mathcal{F}_{t-1}$-measurable and $H \cdot S_T := \sum_{r \leq T} H_r \Delta S_r$ where $\Delta S_r := S_r - S_{r-1}$. The process $x + H \cdot S$ gives the dynamics of values of self-financing portfolio containing at time t the vector $H_t^2, ..., H_t^d$ of units of the risky assets chosen at the previous date. The values of H^1 are irrelevant (because $\Delta S^1 = 0$) and can be chosen arbitrary. Obviously, the set Γ, if non-empty, is a semi-infinite interval.

The hedging problem admits a very simple solution which was suggested, and in a great generality, by Dmitri Kramkov, [12]. In the case of the arbitrage-free market, assuming an appropriate integrability of ξ, we have that $\Gamma = [\underline{x}, \infty[$ where $\underline{x} = \sup_{Q \in \mathcal{Q}} E_Q \xi$ where $\mathcal{Q}$ is the set of equivalent martingale measures. To get this result one needs two theorems. The first one asserts that the processes defined, respectively, as

$$X_t := \text{esssup}_{Q \in \mathcal{Q}} E_Q(\xi | \mathcal{F}_t) \qquad \text{or} \qquad X_t := \text{esssup}_{Q \in \mathcal{Q},\, \tau \in \mathcal{T}_t} E_Q(A_\tau | \mathcal{F}_t),$$

where $\mathcal{T}_t$ is the set of stopping times τ with $\tau \geq t$, are Q-martingale with respect to every $Q \in \mathcal{Q}$. These processes are called generalized Snell envelopes[1]. The optional decomposition theorem says that $X = \underline{x} + H \cdot S - B$ where B is an increasing process with $B_0 = 0$. Thus, we have the required domination property of $\underline{x} + H \cdot S$ and, hence, the inclusion $\Gamma \supseteq [\underline{x}, \infty[$. The opposite inclusion is obvious. Note that these arguments not only solves the problem "in principle" but also give, e.g., in the case of multinomial models, a way how to compute the minimal value $\underline{x}$ of the initial capitals and how to find the corresponding hedging strategy H — its values are Lagrange multipliers removing constraints in simple linear programming problems, see [7].

The model above is fairly standard and mathematically transparent. However, one should be aware that the given formulation hides some important issues which happen to be crucial if one wants to consider models with friction. Historically, the contingent claims or option were baskets of assets to be delivered. E.g., the European-type call option with a strike K can be interpreted as the contract to deliver the basket which is represented by the $\mathcal{F}_T$-measurable vector $\widehat{C}_T$ with $\widehat{C}^1_T = -KI_{\{S_T \geq K\}})$ units of money and $\widehat{C}^2_T = I_{\{S_T \geq K\}}$ units of stock. In general, the nominal (monetary) value of assets to be delivered is $\xi = \widehat{C}_T S_T$. Thus, the above description of the hedging set means that the capital x allows the option seller to hedge (superreplicate) the contingent claim expressed in physical units if and only if

$$x \geq \sup_{Q \in \mathcal{Q}} E_Q \widehat{C}_T S_T = \sup_{Z \in \mathcal{Z}} E \widehat{C}_T Z_T,$$

where $\mathcal{Z}$ denotes the set of $\mathbf{R}^d$-valued martingales Z of the form $Z_t = \rho_t S_t$ and $\rho = (\rho_t)$ runs the set of density processes of equivalent martingale measures, that is, $\rho_t = E(dQ/dP|\mathcal{F}_t)$.

In the theory of markets with proportional transaction costs a model can be given by an adapted (polyhedral) cone-valued process $\widehat{K} = (\widehat{K}_t)$. The value processes are d-dimensional adapted processes and contingent claims are d-dimensional random variables or processes, see the book [10] where the "hat" notations are used to express assets in physical units as opposed to the "countability" in monetary terms, i.e. of units of numéraire. Hedging a European-type contingent claim $\widehat{C} \in L(\mathbf{R}^d, \mathcal{F}_T)$ means to find a self-financing

[1]In the particular case of the complete market when $\mathcal{Q}$ is a singleton the second process is the classical Snell envelop with respect to Q, i.e. the minimal Q-supermartingale dominating the process A.

portfolio with the value process $\widehat{V} = (\widehat{V}_t)$ whose terminal value dominates the claim in the sense that the difference $\widehat{V}_T - \widehat{C}$ belongs to the random solvency cone $\widehat{K}_T$. In other words, $\widehat{V}_T(\omega)$ dominates $\widehat{C}(\omega)$ (a.s.) in the sense of the preference relation generated by the cone $\widehat{K}_T(\omega)$. The solvency cone is the fundamental notion of the theory giving a geometric description of the vectors of investor's positions that can be converted (paying the transaction costs) into vectors with non-negative components. The self-financing condition (with free disposal) means simply that the increments of the value process are non-negative in the sense of partial orderings, i.e. $\Delta\widehat{V}_t \in L^0(-\widehat{K}_t, \mathcal{F}_t)$ for $t = 0, 1, ..., T$. In general, the solvency cones $\widehat{K}_t$ depend on time t as well as on the state of the nature ω — even in the case of constant transaction costs when the solvency cone K in the monetary representation, i.e. in terms of the numéraire, is constant.

The hedging theorem for the European contingent claim $\widehat{C}$ has a form very similar to that given above for the frictionless market. Namely, assuming an appropriate no-arbitrage condition and integrability of $\widehat{C}$ we have that the set Γ of initial capitals permitting hedging is the set of points $v \in \mathbf{R}^d$ such that

$$vZ_0 \geq E\widehat{C}_T Z_T \qquad \forall\, Z \in \mathcal{Z},$$

where $\mathcal{Z}$ denotes the set of consistent price systems, i.e. martingales Z evolving in the (positive) duals of the solvency cones, i.e. with $Z_t \in L^0(\widehat{K}_t^*, \mathcal{F}_t)$. In the case of frictionless markets with traded numéraire this set $\mathcal{Z}$ coincides with that introduced above[2]. Note that the hedging theorem for American options is not a straightforward generalization of the hedging theorem for frictionless case: it has a more complicated structure and involves the so-called coherent price systems.

In contrast to the frictionless case, the arguments used in the available proofs of the latter theorem rely upon the convex duality and do not use order considerations hidden in the concept of Snell envelope. This leads to the natural and intriguing question whether one can define similar notions in the context of stochastic set-valued dynamics and associated random preference relations constituting the mathematical foundations of the theory of

[2]These forms of the hedging theorem reconciles mathematical finance and mathematical economics if one believes that the difference between them is in systematic use of martingale measures for the former and prices for the latter: martingales measures simply provide stochastic deflators to get consistent price systems from asset quotes.

markets with transaction costs. It happens that even such an elementary concept as the essential supremum is not adapted to random vectors, at least, to our knowledge of the literature. The present study was planned to fill this lacuna. It was restricted initially to the random partial ordering defined by proper polyhedral cones corresponding to models with so-called efficient friction. Very soon it became clear that it is rather natural to put the question in a much more general abstract framework. This logic lead our research beyond the scope of mathematical finance and opened perspectives for potential applications in mathematical economics, vector and set-valued optimization, risk theory etc.

It is important to note that in multi-asset models with transaction costs the duals to ordering cones have, in general, more generators than the number of assets. This implies that the corresponding partial orders do not generate lattices, i.e. even for a pair of elements their supremum or infimum might not exist. In this perspective, the assumption that the partially ordered set is a lattice, frequent in the literature, is too restrictive. Apparent advantage of our approach is that for any Γ bounded from above the set $\operatorname{Esssup}\Gamma$ exists and is non-empty for any continuous partial order.

Returning back to the hedging problem under transaction costs it is worthy to recall that finding the set of hedging initial capitals and hedging strategies are problems of great importance. There is a growing literature on these issues, in particular, on numerical methods and the modern trend seems to be the approach based on the set-valued dynamics, see [14] and the references therein. In the cited paper Löhne and Rudloff suggested a numerically efficient method of calculating the hedging sets by backward recurrence and, afterwards, the hedging strategies by forward recurrence using in the latter procedure also optimization. In the present paper we make an attempt to contribute to this tendency by looking for the minimal hedging portfolios.

For a fixed contingent claim $\widehat{C}$ the value (portfolio) process $\widehat{V}$ is called *minimal* if at the terminal date $\widehat{V}_T = \widehat{C}$ and any value process $\widehat{W}$ terminating at $\widehat{C}$ and dominated by $\widehat{V}$ (i.e. such that $\widehat{V}_t - \widehat{W}_t \in \widehat{K}_t$ for all t) coincides with $\widehat{V}$. The problem of interest is whether the minimal portfolios do exist and how they can be found. We provide a description of the set of minimal portfolios as the solution of (backward) recursive inclusions involving Esssup. In this context questions when Esssup does exist and when it is a singleton are very natural.

Recall that in the case of real numbers the supremum of any subset $\Gamma \subset \mathbf{R}$

can be defined either as the minimum of its upper bounds or as the maximum of its closure $\bar{\Gamma}$. In the case of linear ordering lifted to the space $L^0 = L^0(\mathbf{R}, \mathcal{F}_T)$ of scalar random variables the essential supremum of a set $\Gamma \subseteq L^0$ also can be defined in two ways: either as the minimal point of the set of upper bounds of Γ or, alternatively, as maximal point of the upward completion of Γ (that is the smallest set containing Γ closed under convergence in probability and stable under the operation $\wedge$).

In the present paper we study the properties of the set obtained by an extension of the first approach to the case of partial order. To keep the presentation readable we do not work here at full generality: our goal is to fix ideas and built a platform for further studies.

In [11] we discuss extensions to the situation of random preference relations (preorders) given on a Hausdorff topological space and under weaker assumptions. The interest to such generalizations is easy to explain. Indeed, some classical models of mathematical finance suggest infinite and even uncountable sets of securities (e.g., Vasicek and HJM models of term structure of interest rates in the theory of bond markets, Dupire model of derivative market etc.). Their analogs, taking into account transaction costs, involve cone-valued processes in infinite-dimensional spaces, see very recent works [3] and [2]. Also our approach to find extensions to random preorders is based on the passage to the partial order in the quotient space and the latter, in general, is not a nice one. This gives an extra motivation to go beyond $\mathbf{R}^d$. We believe that these issues are important but lay apart of the main line and must be considered separately.

We also relay to [11], and this more important, the development of the second idea mentioned above. Namely, we provide in the companion paper a study of properties of sets of maximal points of a suitable completion of Γ at the higher level of generality. Its main object is a concept which is different from $\operatorname{Esssup} \Gamma$ and can be viewed as an analog of the Pareto frontier. The notion happens to be useful in a description of minimal hedging portfolios in the hedging problem of American options under transaction cost. We hope that the readers will benefit from splitting of the material into two papers with similar structure. The present one is intended for the initial acquaintance with problematics (we avoid discussions of delicate topological aspects) but we tried to make them both sufficiently self-contained.

The main technical hypothesis we use in this paper is the existence of countable continuous multi-utility representation (in the terminology of [5]). For $\mathbf{R}^d$ this means that the considered partial orders are continuous, see [11], Remark 2.9.

The structure of the paper is the following. In Section 2 we consider a purely deterministic setting. We give the definition of $\operatorname{Sup}\Gamma$ for $\Gamma \subseteq \mathbf{R}^d$ and prove a theorem giving a sufficient condition ensuring that it does exist and (as) a non-empty set. In the case, where the partial order is generated by a convex closed proper cone this condition is simply the boundedness of Γ from above in the sense of partial order. We do not insist on any novelty at this section but the comparison with the literature shows that our approach and the related techniques are different from those we could find, see, e.g. [4] and [18] for numerous definitions of supremum-like objects. In Section 3 we define $\operatorname{Esssup}\Gamma$ for $\Gamma \subseteq L^0(\mathbf{R}^d)$ and establish its numerous properties. In particular, we are interested in conditions when it is a singleton. In Section 4 we consider a bit more specific model where the random partial order is given by a random cone. In Section 5 we give an application to the hedging problem for European contingent claims under transaction costs providing a system of backward inclusion to calculate minimal hedging portfolios. In concluding Section 6 we give an example of construction of dynamic set-valued risk measure using the notion of Esssup.

2. Supremum with Respect to a Partial Order in $\mathbf{R}^d$

2.1. Basic Concepts

We start with some basic concepts and notations restricting ourselves to the Euclidean space $\mathbf{R}^d$.

Let $\succeq$ be a *partial order* in $\mathbf{R}^d$, i.e. a binary relation between certain its elements, which is reflexive ($x \succeq x$), transitive (if $x \succeq y$ and $y \succeq z$ then $x \succeq z$) and antisymmetric (if $x \succeq y$ and $y \succeq x$ then $x = y$).

Define an order interval $[x, y] := \{z \in \mathbf{R}^d : \ y \succeq z \succeq x\}$ and extend naturally the notation by putting

$$]-\infty, x] := \{z \in \mathbf{R}^d : \ x \succeq z\}, \qquad [x, \infty[:= \{z \in \mathbf{R}^d : \ z \succeq x\}.$$

The notation $\Gamma \succeq x$, where Γ is a set, means that $y \succeq x$ for all $y \in \Gamma$. In the same spirit: $\Gamma_1 \succeq \Gamma$ means that $x \succeq y$ for all $x \in \Gamma_1$ and $y \in \Gamma$; $[\Gamma, \infty[:= \cap_{x\in\Gamma}[x, \infty)$ etc. Sometimes we shall use the notation $x \preceq z$ instead of $z \succeq x$.

A partial order is *upper semi-continuous* (respectively, *lower semi-continuous*) if $[x, \infty[$ (respectively, $]-\infty, x]$) is closed for any $x \in \mathbf{R}^d$ and *semi-continuous* if it is both upper and lower semi-continuous. Finally, it is called *continuous* if its graph $\{(x,y) : \ y \succeq x\}$ is a closed subset of $\mathbf{R}^d \times \mathbf{R}^d$.

We say that a set $\mathcal{U}$ of real-valued functions defined on $\mathbf{R}^d$ *represents* the partial order $\succeq$ if for any $x, y \in \mathbf{R}^d$,

$$x \succeq y \ \Leftrightarrow \ u(x) \geq u(y) \quad \forall u \in \mathcal{U}.$$

This set $\mathcal{U}$ is called multi-utility representation of the partial order. If its elements are continuous functions, we say that $\mathcal{U}$ is a *continuous multi-utility representation* of the partial order.

Clearly, any partial order can be represented by the family of indicator functions $\mathcal{U} := \{I_{[x,\infty[}, \ x \in \mathbf{R}^d\}$.

The following statement follows from a more general result due to Evren and Ok: any continuous partial order admits a continuous multi-utility representation (see [5], Th. 1).

Note that an arbitrary family $\mathcal{U}$ defines a partial order if the equalities $u(x) = u(y)$ for all $u \in \mathcal{U}$ imply that $x = y$.

The object of our main interest is given by the following

Definition 2.1. *Let* Γ *be a non-empty subset of* $\mathbf{R}^d$ *and let* $\succeq$ *be a partial order. We denote by* $\operatorname{Sup} \Gamma$ *a subset* $\hat{\Gamma}$ *of* $\mathbf{R}^d$ *such that the following conditions hold:*

(a_0) $\hat{\Gamma} \succeq \Gamma$*;*

(b_0) *if* $x \succeq \Gamma$*, then there is* $\hat{x} \in \hat{\Gamma}$ *such that* $x \succeq \hat{x}$*;*

(c_0) *if* $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$*, then* $\hat{x}_1 \succeq \hat{x}_2$ *implies* $\hat{x}_1 = \hat{x}_2$*.*

Remark 2.2. Such a set $\hat{\Gamma}$, if exists, is necessarily unique as shown in Lemma 3.3. Note that our definition is "assumption-free" and can be apply to any $\Gamma \neq \emptyset$ and allows us to formulate the question whether it does exist and is non-empty. To compare it with concepts in the literature, namely, with those in [4] let $x \succ y$ mean that $x \succeq y$ and $x \neq y$. Define, for a non-empty set A, the subset

$$\min A := \{y \in A : \text{ the relation } z \succ y \text{ hold only if } z \notin A\}.$$

We say that A has the submission property if for any $y \in A$ there exists $y_0 \in \min A$ such that $y \succeq y_0$ (thus, this property assumes that $\min A \neq \emptyset$). It is easy to see that if the set $[\Gamma, \infty[$ satisfies the submission property, then $\operatorname{Sup} \Gamma = \min [\Gamma, \infty[$.

The following lemma follows immediately from the continuity assumption.

Lemma 2.3. *Let $\succeq$ be a partial order represented by a family of continuous functions. Let (x_n) and (y_n) be two sequences of $\mathbf{R}^d$ such that $y_n \succeq x_n$ for all n. Suppose that these sequences converge, respectively, to x_∞ and y_∞. Then*

$$\bigcap_n [x_n, y_n] \subseteq [x_\infty, y_\infty].$$

Theorem 2.4. *Let $\succeq$ be a partial order represented by a countable family of continuous functions and such that all order intervals $[x, y]$, $y \succeq x$, are compacts. If the subset Γ is such that $\bar{x} \succeq \Gamma$ for some $\bar{x}$, then $\operatorname{Sup}\Gamma \neq \emptyset$. Moreover, if Γ is totally ordered, then $\operatorname{Sup}\Gamma$ is a singleton formed by a limit point of Γ.*

Proof. Let $\mathcal{U} = \{u_j\}_{j\geq 1}$ be a representing family and fix $x_0 \in \Gamma$. Without loss of generality we may assume that $|u_j| \leq 1$. We define the function $u(.) = \sum 2^{-j} u_j(.)$. Observe that this function is continuous. Put $a(x) := \inf_{y \in [\Gamma, x]} u(y)$. Since $[\Gamma, x] \subseteq [x_0, x]$, the value $a(x) \geq u(x_0)$. Let $y_n \in [\Gamma, x]$ be a sequence such that $u(y_n) \to a(x)$. Since $[\Gamma, x]$, being an intersection of compacts, is also a compact, we may assume, passing to a subsequence, that $y_n \to y_\infty$. In virtue of continuity, $a(x) = u(y_\infty)$.

Define $\Lambda(x)$ as the set of all $y_\infty \in [\Gamma, x]$ with $u(y_\infty) = a(x)$ and put $\hat{\Gamma} := \bigcup_{x \succeq \Gamma} \Lambda(x)$. By above the properties (a_0) and (b_0) hold. Let $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$, $\hat{x}_1 \succeq \hat{x}_2$. There is $x_1 \succeq \Gamma$ such that $\hat{x}_1 \in \Lambda(x_1)$, i.e. $u(\hat{x}_1) = a(x_1)$. By monotonicity, $u(\hat{x}_1) \geq u(\hat{x}_2)$. But $\hat{x}_2 \in [\Gamma, \hat{x}_1] \subseteq [\Gamma, x_1]$ and, therefore, $u(\hat{x}_2) \geq a(x_1)$, i.e. $u(\hat{x}_1) = u(\hat{x}_2)$. Since $u_j(\hat{x}_1) \geq u_j(\hat{x}_2)$ for all j we have necessarily that $u_j(\hat{x}_1) = u_j(\hat{x}_2)$ for all j. Therefore, $\hat{x}_1 = \hat{x}_2$ and (c_0) holds.

Finally, suppose that Γ is totally ordered. Define $b := \sup_{x \in \Gamma} u(x)$. There exists $x_n \in \Gamma$ such that $u(x_n) \to b$. Since Γ is totally ordered, we can assume without loss of generality that the sequence (x_n) is increasing and satisfy $x_n \in [x_0, \bar{x}]$. Arguing as previously, we get that $x_n \to x_\infty$ and $b = u(x_\infty)$. Let $y \in \Gamma$. We have two possibilities. If $y \in [x_{n'}, \bar{x}]$, for some subsequence, then we obtain that $y \in [x_\infty, \bar{x}]$ by virtue of Lemma 2.3. It follows that $u(y) = u(x_\infty)$ hence $y = x_\infty$. If $y \in \Gamma$ is such that $y \preceq x_{n'}$ for some infinite subsequence $(x_{n'})$ then $y \preceq x_\infty$. We conclude that $\operatorname{Sup}\Gamma = \{x_\infty\}$. $\square$

Remark 2.5. It is easily seen that the result holds (with the same proof) not only for $\mathbf{R}^d$ but for any topological space such that the order intervals are sequentially compact.

Proposition 2.6. *Suppose that the partial order is given by a countable family* $\mathcal{U} = \{u_j\}_{j\ge 1}$ *of continuous homogeneous functions. Then all the order intervals are compacts.*

Proof. Let us consider an arbitrary order interval $[x, z]$, $z \succeq x$, and let $y_n \in [x, z]$ be such that $|y_n| \to \infty$. Passing to a subsequence we may assume that a sequence $\tilde{y}_n := y_n/|y_n|$ converges to a point $\tilde{y}_\infty$ with $|\tilde{y}_\infty| = 1$. Due to the homogeneity, $x/|y_n| \succeq \tilde{y}_n \succeq z/|y_n|$ and, therefore,

$$u_j(x/|y_n|) \ge u_j(\tilde{y}_n) \ge u_j(z/|y_n|), \quad j \ge 1.$$

Taking the limit, we obtain that $u_j(\tilde{y}_\infty) = 0$ for all j, i.e. $\tilde{y}_\infty = 0$. A contradiction. So, the order interval $[x, z]$ is bounded, hence, compact. □

2.2. Partial Order in $\mathbf{R}^d$ *Defined by a Cone*

In this paper oriented towards financial applications we are interested mainly by the partial order defined by a closed proper convex cone $G \subseteq \mathbf{R}^d$. In this case, the relation $x \succeq 0$ means that $x \in G$, and $y \succeq x$ means that $y - x \succeq 0$, i.e. $y \in x + G$. Obviously, it is *homogeneous*: $y \succeq v$ implies that $\lambda y \succeq \lambda x$ for any $\lambda \ge 0$. Also if $y \succeq x$, $v \succeq u$, then $x + v \succeq y + u$. The order intervals $[x, \infty[= x + G$ and $] - \infty, x] = x - G$ are closed. Hence, the partial order is semicontinuous. In fact, it is continuous since its graph $\{(x, y) :\ y - x \in G\}$ is a closed subset of $\mathbf{R}^d \times \mathbf{R}^d$.

By the classical separation theorem the cone G is the intersection of the family of closed half-spaces $L = \{x \in \mathbf{R}^d :\ lx \ge 0\}$ containing G. Its complement G^c is the union of the open half-spaces L^c. In $\mathbf{R}^d$ any covering of an open set contains a countable subcovering. Hence, there exists a countable family of vectors l_j such that $G = \cap_j \{x \in \mathbf{R}^d :\ l_j x \ge 0\}$. It follows that a countable family of linear functions $u_j(x) = l_j x$ represents the partial order defined by G. So, the partial order defined by a closed proper convex cone $G \subseteq \mathbf{R}^d$ can be generated by a countable family of linear functions. Clearly, the converse is true.

For the partial order given by a cone G the properties defining the set $\hat{\Gamma} = \operatorname{Sup}\Gamma$ can be reformulated in geometric terms as follows:

(a_0') $\hat{\Gamma} - \Gamma \subseteq G$;

(b_0') if $x - \Gamma \subseteq G$, then there is $\hat{x} \in \hat{\Gamma}$ such that $x - \hat{x} \in G$;

(c_0') if $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$, then $\hat{x}_1 - \hat{x}_2 \notin G \setminus \{0\}$.

Notation. To distinguish partial orders generated by various cones (a typical situation in financial applications) we shall use sometimes the notation $\succeq_G$.

As corollary of Theorem 2.4 we have the following result:

Theorem 2.7. *Let $\succeq$ be the partial order generated by a closed proper convex cone $G \subseteq \mathbf{R}^d$. If $\Gamma \subseteq \mathbf{R}^d$ is such that $\bar{x} \succeq \Gamma$ (i.e. $\bar{x} - \Gamma \subseteq G$) for some $\bar{x} \in \mathbf{R}^d$, then* $\operatorname{Sup} \Gamma \neq \emptyset$.

Remark 2.8. Any subset $G \subset \mathbf{R}^d$ with $G + G \subseteq G$ and $G \cap (-G) = \{0\}$ allows us to define a partial order by putting $x \succeq y$ if $x - y \in G$. Supremum of sets for such partial orders may have rather exotic features. For example, let $d = 2$ and

$$G := \{(0,0)\} \cup \{(x,y) \in \mathbf{Z}^2_+ : \; x + y \geq 2\}.$$

Under the corresponding partial order, for the set $\Gamma := \{(0,0),(4,-1)\}$ consisting from two points we have $\operatorname{Sup} \Gamma = \{(4,1);(4,2);(5,0);(5,1);(6,0)\}$.

Remark 2.9. Let us consider the triangle Γ generated by the points $(0,0)$, $(1,0)$, and $(0,1)$ in $\mathbf{R}^2$ where the partial order is generated by $\mathbf{R}^2_+$. In our definition $\operatorname{Sup} \Gamma = \{(1,1)\}$, i.e. it contains the minimum of upper bounds. It lays outside of Γ. On the other hand, the segment with extremities at $(1,0)$ and $(0,1)$ consisting of maximal points of Γ looks also as a good candidate for a supremum. In the companion paper we analyze in details objects of this type in the stochastic setting and provide an application to hedging of American optons.

Remark 2.10. In the literature one can find also other definitions of supremum for partial order. E.g., in the book by Löhne [13] there is a definition adapted to the needs of the vector optimization theory. For the case of $\mathbf{R}^d$ with the partial order given by a convex cone $G \neq \mathbf{R}^d$ with non-empty interior it can be described as follows. First, it is defined the *lower closure of A* as the set

$$\operatorname{Cl}_{-} A := \{x \in \mathbf{R}^d : \; x - \operatorname{int} G \subseteq A - \operatorname{int} G\}$$

and the *set of weak maximal points* of A

$$\operatorname{wMax} A := \{x \in A : \; (x + \operatorname{int} G) \cap A = \emptyset\}.$$

Finally, $\operatorname{Sup}_{\mathrm{w}} A := \operatorname{wMax} \operatorname{Cl}_{-} A$. For the cone $G = \mathbf{R}^d_+$ and $A = -\mathbf{R}^d$ we have that $\operatorname{Cl}_{-} A = A$ and $\operatorname{Sup}_{\mathrm{w}} A = \operatorname{wMax} A = -\partial \mathbf{R}^d_+$. According to our definition $\operatorname{Sup} A = \{0\}$.

3. Essential Supremum in $L^0(\mathbf{R}^d)$

3.1. Setting

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. We consider in the space $L^0(\mathbf{R}^d, \mathcal{F})$ of d-dimensional random variables a partial order defined by a countable family $\mathcal{U} = \{u_j : \ j = 1, 2, ...\}$ of functions $u_j : \Omega \times \mathbf{R}^d \to \mathbf{R}$ with the following properties:

(i) $u_j(., x) \in L^0(\mathbf{R}, \mathcal{F})$ for every $x \in \mathbf{R}^d$;

(ii) $u_j(\omega, .)$ is continuous for almost all $\omega \in \Omega$.

Namely, for elements $\gamma_1, \gamma_2 \in L^0(\mathbf{R}^d, \mathcal{F})$, the relation $\gamma_2 \succeq \gamma_1$ means that $u_j(\gamma_2) \geq u_j(\gamma_1)$ (a.s.) for all j.

Definition 3.1. *Let Γ be a subset of $L^0(\mathbf{R}^d, \mathcal{F})$. We denote by $\mathcal{H}$-*$\mathrm{Esssup}\,\Gamma$ *a subset $\hat{\Gamma}$ of $L^0(\mathbf{R}^d, \mathcal{H})$ such that the following conditions hold:*

(a) $\hat{\Gamma} \succeq \Gamma$;

(b) if $\gamma \in L^0(\mathbf{R}^d, \mathcal{H})$ and $\gamma \succeq \Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\gamma \succeq \hat{\gamma}$;

(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 = \hat{\gamma}_2$.

If $\mathcal{H}$-$\mathrm{Esssup}\,\Gamma$ is a singleton, we denote by $\mathcal{H}$-$\mathrm{esssup}\,\Gamma$ its unique element. Since in this section $\mathcal{H}$ is fixed, we shall omit this symbol and write simply $\mathrm{Esssup}\,\Gamma$ and $\mathrm{esssup}\,\Gamma$. When needed, we note $\mathrm{Esssup}^{\mathcal{U}}\Gamma$ where $\mathcal{U}$ is the family of functions representing the partial order.

We define $\mathrm{Essinf}^{\mathcal{U}}\Gamma := \mathrm{Esssup}^{-\mathcal{U}}(\Gamma)$ and $\mathrm{essinf}^{\mathcal{U}}\Gamma := \mathrm{esssup}^{-\mathcal{U}}(\Gamma)$.

Note that for every ω (except a null set) the countable family of functions $\{u_j(\omega, .)\}$ defines a partial order in $\mathbf{R}^d$. In the sequel we associate with an order interval $[\gamma_1, \gamma_2]$ in $L^0(\mathbf{R}^d, \mathcal{F})$ the order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$ in $\mathbf{R}^d$ corresponding to these families.

Remark 3.2. *Let $d = 1$ and $\succeq$ is the usual total order on the real line. Then $\mathcal{F}$-*$\mathrm{Esssup}\,\Gamma = \{\mathcal{F}\text{-}\mathrm{esssup}\,\Gamma\}$ *where $\mathcal{F}$-*$\mathrm{esssup}\,\Gamma$ *is the classical essential supremum of Γ. Also in the scalar case, if $\Gamma = \{\xi\}$ and the σ-algebra $\mathcal{H}$ is trivial, then $\mathcal{H}$-*$\mathrm{Esssup}\,\Gamma = \{\mathrm{vraimax}\,\xi\}$.

3.2. Elementary Properties

In this subsection, we consider a partial order $\succeq$ in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family of functions satisfying (i), (ii).

Lemma 3.3. *The set* $\operatorname{Esssup}\Gamma$ *is uniquely defined.*

Proof. Let us consider two subsets $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ satisfying (a)–(c). Consider $\hat{\gamma}_1 \in \hat{\Gamma}_1$. Since $\hat{\gamma}_1$ dominates the set Γ, there exists $\hat{\gamma}_2 \in \hat{\Gamma}_2$ with $\hat{\gamma}_1 \succeq \hat{\gamma}_2$. Similarly, there exists $\hat{\gamma}_1' \in \hat{\Gamma}_1$ satisfying $\hat{\gamma}_2 \succeq \hat{\gamma}_1'$. Then, $\hat{\gamma}_1 \succeq \hat{\gamma}_1'$. But (c) implies that $\hat{\gamma}_1 = \hat{\gamma}_1'$. Hence, $\hat{\gamma}_1 = \hat{\gamma}_1' = \hat{\gamma}_2$, i.e. $\hat{\gamma}_1 \in \hat{\Gamma}_2$. So, $\hat{\Gamma}_1 \subseteq \hat{\Gamma}_2$ and, by symmetry, $\hat{\Gamma}_1 = \hat{\Gamma}_2$. $\Box$

Lemma 3.4. *The set* $\operatorname{Esssup}\Gamma$ *is decomposable, i.e. for any* $\hat{\gamma}_1, \hat{\gamma}_2 \in \operatorname{Esssup}\Gamma$ *and* $B \in \mathcal{H}$*, we have* $\hat{\gamma}_1 I_B + \hat{\gamma}_2 I_{B^c} \in \operatorname{Esssup}\Gamma$.

Proof. Let us consider the set $\tilde{\Gamma} := \operatorname{Esssup}\Gamma \cup \{\hat{\gamma}_1 I_B + \hat{\gamma}_2 I_{B^c}\}$. This set satisfies (a)–(c). By the previous lemma, $\tilde{\Gamma} = \operatorname{Esssup}\Gamma$. $\Box$

Lemma 3.5. *Let* $\succeq$ *be a partial order in* $L^0(\mathbf{R}^d, \mathcal{F})$ *represented by a countable family of linear functions (in* x*-variable) satisfying (i), (ii). If* $\operatorname{Esssup}\Gamma$ *is neither an empty set nor a singleton, then* $\operatorname{Esssup}\Gamma$ *is infinite.*

Proof. Suppose that $\operatorname{Esssup}\Gamma = \{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_m\}$ where $\hat{\gamma}_i \neq \hat{\gamma}_j$ if $i \neq j$. The linearity ensures that the random variable $(1/k)\hat{\gamma}_1 + (1 - 1/k)\hat{\gamma}_2$ dominates Γ for each $k \in \mathbf{N}$. The property (b) implies the existence of an integer $N_k \in \{1, \cdots, m\}$ such that $(1/k)\hat{\gamma}_1 + (1-1/k)\hat{\gamma}_2 \succeq \hat{\gamma}_{N_k}$. Using again the linearity and the property (c) we deduce that the index N_k is not equal to 1 or 2 (otherwise, $\hat{\gamma}_1 = \hat{\gamma}_2$). Hence, there exists $j \in \{3, \cdots, m\}$ and an infinite subsequence (k') such that $N_{k'} = j$ for all k'. By letting k' tend to infinity, we infer that $\hat{\gamma}_2 \succeq \hat{\gamma}_j$. therefore, by (c), we have $\hat{\gamma}_2 = \hat{\gamma}_j$. A contradiction. $\Box$

Example. The linearity assumption above is important. Indeed, let us consider the partial order in $\mathbf{R}^2$ given by the family $\mathcal{U} = \{u_1, u_2, u_3\}$, where $u_1(x) = ax$, $u_2(x) = bx$ with $a = (1/\sqrt{2}, 1/\sqrt{2})$, $b = (0, 1)$, and $u_3(x) = |x|$. Let $\Gamma = \{0, p\}$ where $p = (-1/\sqrt{2}, -1/\sqrt{2})$. Then $x \succeq \Gamma$ if and only if $ax \geq 0$, $bx \geq 0$, and $|x| \geq 1$. That is $x \in G$ and $|x| \geq 1$, where G is the cone $\{x : \; ax \geq 0, \; bx \geq 0\}$. One can easily check that $\operatorname{Sup}\Gamma = \{A, B\}$ where $A = (-1/\sqrt{2}, 1/\sqrt{2})$, $B = (1, 0)$, see Figure 1.

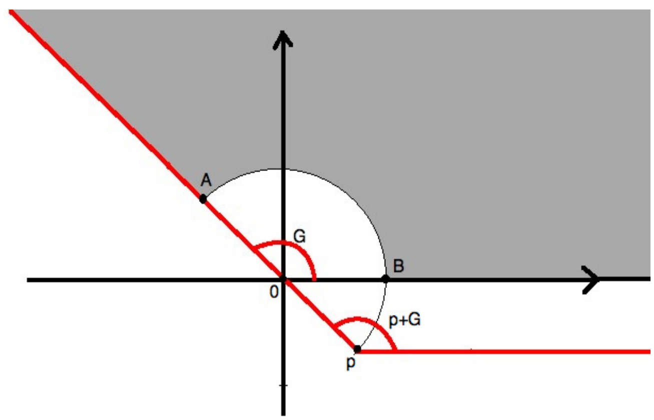

Figure 1: The grey-coloured domain corresponds to the set of all points dominating $\Gamma = \{0, p\}$, hence, $\mathrm{Sup}\,\Gamma = \{A, B\}$.

Lemma 3.6. *Assume that the essential supremum of any finite subset of $L^0(\mathbf{R}^d, \mathcal{F})$ is a singleton. For $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$, let the upward completion be defined as*

$$\Gamma^{\mathrm{up}} := \{\mathrm{esssup}\,\{\gamma_{j_1}, \cdots, \gamma_{j_n}\} : \gamma_{j_k} \in \Gamma,\, n \in \mathbf{N}\}\,. \tag{3.1}$$

Then

$$\mathrm{Esssup}\,\Gamma = \mathrm{Esssup}\,\Gamma^{\mathrm{up}}.$$

Proof. Put $\hat{\Gamma} := \mathrm{Esssup}\,\Gamma^{\mathrm{up}}$. Since $\mathrm{Esssup}\,\Gamma^{\mathrm{up}} \succeq \mathrm{esssup}\,\{\gamma\} \succeq \gamma$ whatever is $\gamma \in \Gamma$, the set $\hat{\Gamma}$ satisfies the property (a). Let $\gamma \in L^0(\mathbf{R}^d, \mathcal{H})$ be such that $\gamma \succeq \Gamma$. In particular, γ dominates the elements $\gamma_{j_k} \in \Gamma$, $1 \leq k \leq n$. By the assumption the essential supremum of the latter is a singleton. Therefore, $\gamma \succeq \Gamma^{\mathrm{up}}$. Thus, there is an element $\hat{\gamma} \in \mathrm{Esssup}\,\Gamma^{\mathrm{up}}$ such that $\gamma \succeq \hat{\gamma}$. That is $\hat{\Gamma}$ satisfies the property (b). Finally, let $\hat{\gamma}_1, \hat{\gamma}_2 \in \mathrm{Esssup}\,\Gamma^{\mathrm{up}}$ be such that $\hat{\gamma}_1 \succeq \hat{\gamma}_2$. By definition of Esssup we have that $\hat{\gamma}_1$ coincides with $\hat{\gamma}_2$ and (c) also holds. $\square$

3.3. Existence

Theorem 3.7. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d,\mathcal{F})$ represented by a countable family of functions satisfying (i), (ii) and such that all order intervals $[\gamma_1(\omega),\gamma_2(\omega)]$, $\gamma_2\succeq\gamma_1$, are compacts a.s. If a non-empty subset Γ is such that $\bar\gamma\succeq\Gamma$ for some $\bar\gamma\in L^0(\mathbf{R}^d,\mathcal{H})$, then $\mathcal{H}$-*Esssup $\Gamma\neq\emptyset$.

Proof. Let $\mathcal{U}=\{u_j\}_{j\geq 1}$ be a representing family. Without loss of generality we may assume that $|u_j|\leq 1$. Put

$$u(\omega,z):=\sum_{j\geq 1}2^{-j}u_j(\omega,z).$$

The function $z\mapsto u(\omega,z)$ is continuous and its absolute value is bounded by unit. Fix arbitrary $\gamma_0\in\Gamma$. Take $\gamma\in L^0(\mathbf{R}^d,\mathcal{H})$ such that $\gamma\succeq\Gamma$. For any $\zeta\in L^0([\Gamma,\gamma],\mathcal{H})$ the mapping $\omega\mapsto u(\omega,\zeta(\omega))$ is an $\mathcal{F}$-measurable random variable taking values in the interval $[-1,1]$. Put

$$a(\gamma):=\inf_{\zeta\in L^0([\Gamma,\gamma],\mathcal{H})}Eu(\zeta)$$

and consider a sequence $\zeta_n\in L^0([\Gamma,\gamma],\mathcal{H})$ such that $a(\gamma)=\lim_n Eu(\zeta_n)$.

Without loss of generality we may assume that the sequence of random variables $u(\zeta_n)$ is such that the conditional expectations $E(u(\zeta_n)|\mathcal{H})$ are decreasing. Indeed, we can replace the sequence ζ_n by the sequence ζ'_n by putting $\zeta'_1=\gamma$, and defining recursively the random variables

$$\zeta'_n:=\zeta'_{n-1}I_{\{E(u(\zeta'_{n-1})|\mathcal{H})\leq E(u(\zeta_n)|\mathcal{H})\}}+\zeta_nI_{\{E(u(\zeta'_{n-1})|\mathcal{H})>E(u(\zeta_n)|\mathcal{H})\}},\quad n\geq 2.$$

Due to the assumption of the theorem, the order intervals $[\gamma_0(\omega),\gamma(\omega)]$ are compact (a.s.). It follows that $\sup_n|\zeta_n|<\infty$ a.s. By virtue of the lemma on converging subsequences (Lemma 2.1.2 [10]) there exists a strictly increasing sequence of $\mathcal{H}$-measurable integer-valued random variables τ_k such that the sequence ζ_{τ_k} converges a.s. to some ζ such that $\Gamma\preceq\zeta\preceq\gamma$. The monotonicity implies that

$$E(u(\zeta_{\tau_k})|\mathcal{H})=\sum_{m\geq k}E(u(\zeta_m)|\mathcal{H})I_{\{\tau_k=m\}}\leq E(u(\zeta_k)|\mathcal{H}).$$

It follows that

$$Eu(\zeta_{\tau_k})\leq Eu(\zeta_k).$$

Using the continuity of $u(\omega,.)$ and the Lebesgue theorem on dominated convergence we have:

$$Eu(\zeta)=E\lim_k u(\zeta_{\tau_k})\leq \lim_k Eu(\zeta_k)=a(\gamma).$$

Thus,

$$a(\gamma)=Eu(\zeta). \tag{3.2}$$

We denote by $\Lambda(\gamma)$ the set of all random variables $\zeta\in L^0([\Gamma,\gamma],\mathcal{H})$ verifying (3.2) and define the set

$$\widehat{\Gamma}:=\bigcup_{\gamma\in L^0([\Gamma,\infty[,\mathcal{H})}\Lambda(\gamma). \tag{3.3}$$

It remains to show that this set satisfies (a)–(c). Obviously, $\widehat{\Gamma}\succeq\Gamma$, i.e. (a) holds. If $\zeta\in L^0([\Gamma,\infty[,\mathcal{H})$, then $\zeta\succeq\Lambda(\zeta)$ by construction, i.e. (b) holds. At last, consider $\widehat{\zeta}_1$, $\widehat{\zeta}_2\in\widehat{\Gamma}$ with $\widehat{\zeta}_1\in\Lambda(\gamma_1)$ and $\widehat{\zeta}_2\in\Lambda(\gamma_2)$, $\gamma_1,\gamma_2\in[\Gamma,\infty[$, such that $\widehat{\zeta}_1\succeq\widehat{\zeta}_2$. Suppose that $\widehat{\zeta}_1\neq\widehat{\zeta}_2$ and, hence, there is i for which $u_i(\widehat{\zeta}_1)-u_i(\widehat{\zeta}_2)\geq 0$ and the inequality is strict on a non-null set. It follows that there exists a non-null set $B\in\mathcal{H}$ on which

$$E(u_i(\widehat{\zeta}_1)-u_i(\widehat{\zeta}_2)|\mathcal{H})>0.$$

Observe that for all j

$$u_j(\widehat{\zeta}_2I_B+\widehat{\zeta}_1I_{B^c})=u_j(\widehat{\zeta}_2)I_B+u_j(\widehat{\zeta}_1)I_{B^c}$$

and

$$E(u_j(\widehat{\zeta}_2I_B+\widehat{\zeta}_1I_{B^c})|\mathcal{H})=E(u_j(\widehat{\zeta}_2)|\mathcal{H})I_B+E(u_j(\widehat{\zeta}_1)|\mathcal{H})I_{B^c}$$

for all j. It follows that

$$a(\gamma_1)=Eu(\widehat{\zeta}_1)>Eu(\widehat{\zeta}_2I_B+\widehat{\zeta}_1I_{B^c}) \tag{3.4}$$

where $\widehat{\zeta}_2I_B+\widehat{\zeta}_1I_{B^c}\in[\Gamma,\widehat{\zeta}_1]\subseteq[\Gamma,\gamma_1]$. This is a contradiction. Hence, (c) also holds. $\Box$

Lemma 3.8. *Let $\succeq$ be a partial order represented by a countable family of functions satisfying (i), (ii) and such that all order intervals $[\gamma_1(\omega),\gamma_2(\omega)]$, $\gamma_2\succeq\gamma_1$, are compacts a.s. Suppose that for any $\hat{\gamma}_1,\hat{\gamma}_2\in L^0(\mathbf{R}^d,\mathcal{F})$, the set* $\operatorname{Essinf}\{\hat{\gamma}_1,\hat{\gamma}_2\}$ *is either singleton or empty. Then, for any $\Gamma\subseteq L^0(\mathbf{R}^d,\mathcal{F})$ the set* $\operatorname{Esssup}\Gamma$ *is either singleton or empty.*

Proof. Suppose that some $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ is such that $\mathrm{Esssup}\,\Gamma$ contains two different points $\hat{\gamma}_1$ and $\hat{\gamma}_2$. By the existence Theorem 3.7 above the set $\mathrm{Essinf}\,\{\hat{\gamma}_1, \hat{\gamma}_2\}$ is non-empty and, by the hypothesis, it is a singleton formed by some element γ^*. If $\gamma \in \Gamma$, then $\gamma \preceq \{\hat{\gamma}_1, \hat{\gamma}_2\}$ and, therefore, $\gamma \preceq \gamma^*$, i.e. $\gamma^* \succeq \Gamma$. Hence, there exists $\hat{\gamma} \in \mathrm{Esssup}\,\Gamma$ such that $\gamma^* \succeq \hat{\gamma}$. Therefore, $\{\hat{\gamma}_1, \hat{\gamma}_2\} \succeq \hat{\gamma}$ implying that $\hat{\gamma}_1 = \hat{\gamma}$ and $\hat{\gamma}_2 = \hat{\gamma}$. A contradiction. □

3.4. More Properties of Esssup

To relate our result with the classical concept of essential supremum of a set of scalar random variables, it is more convenient to consider the space $L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{F})$. The natural partial order in this case can be given by a single function, e.g., $u(x) = x$ or any increasing strictly monotone function; the choice $u(x) = \arctan x$ is convenient since the latter is bounded.

Lemma 3.9. *Let $\Gamma \neq \emptyset$ be a subset of $L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{F})$. Then $\mathcal{H}$-*Esssup Γ *is a singleton. In particular, $\mathcal{H}$-*Esssup Γ=$\{\mathcal{H}$-esssup $\Gamma\}$.

Proof. Working with $u(x) = \arctan x$ we observe that the arguments of the previous theorem (with the random variable $\bar{\gamma}$ identically equal to infinity) require no changes. It is easy to see that $a(\gamma)$ does not depend on $\gamma \succeq \Gamma$. Finally, there exists only one element $\zeta \in L^0([\Gamma, \infty], \mathcal{H})$ such that (3.2) holds (otherwise we could diminish the value of the right-hand side by $\zeta \wedge \zeta'$). □

Remark 3.10. *We have $\mathcal{H}$-*esssup $\Gamma \succeq \mathcal{F}$-esssup Γ.

Proposition 3.11. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d)$ represented by a countable family $\mathcal{U}$ of functions satisfying (i), (ii). Let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{H})$ be a totally ordered subset such that there exists $\overline{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$ with $\Gamma \preceq \overline{\gamma}$. Then* Esssup Γ *is a singleton* esssup Γ*. Moreover, there is a strictly increasing sequence of $\mathcal{H}$-measurable integer-valued random variables τ_n such that (γ_{τ_n}) converges increasingly to* esssup Γ *a.s.*

Proof. Put $b := \sup_{\gamma \in \Gamma} Eu(\gamma)$ where u is defined above. Take $\gamma_n \in \Gamma$ such that $Eu(\gamma_n) \uparrow b$. The set Γ being totally ordered, we may assume without loss of generality that (γ_n) is order increasing. Since $\gamma_n \in [\gamma_1, \overline{\gamma}]$, we infer that $\liminf_n |\gamma_n| < \infty$. In virtue of Lemma 2.1.2 [10] there is a strictly increasing sequence of $\mathcal{H}$-measurable integer-valued random variables τ_n such that (γ_{τ_n}) converges a.s. to some $\hat{\gamma} \preceq \overline{\gamma}$. Recall that

$$\gamma_{\tau_n} = \sum_{m \geq n} \gamma_m I_{\{\tau_n = m\}} \in L^0(\mathbf{R}^d, \mathcal{H}).$$

Since both sequences (γ_n) and (τ_n) are increasing, we deduce that the sequence (γ_{τ_n}) is also increasing. As $\tau_n \geq n$, we get that $\gamma_{\tau_m} \succeq \gamma_n$ if $m \geq n$ implying that $\hat{\gamma} \succeq \gamma_n$ for every n and, also $b = Eu(\hat{\gamma})$. It follows that $\operatorname{Esssup} \Gamma = \{\hat{\gamma}\}$. Indeed, if $\gamma \succeq \Gamma$, then $\gamma \succeq \{\gamma_{\tau_n} : \ n \in \mathbf{N}\}$ and, taking the limit, we get that $\gamma \succeq \hat{\gamma}$, i.e. the singleton $\{\hat{\gamma}\}$ satisfies (a)–(c). □

3.5. Properties of Esssup *for Homogeneous Generating Functions*

In this subsection we shall work assuming that the functions defining the partial order are linear (in x variable). For a set $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ and $\lambda \in L^0(\mathbf{R}_+, \mathcal{H})$, we define the set $\lambda\Gamma := \{\lambda\gamma : \ \gamma \in \Gamma\}$.

Lemma 3.12. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family $\mathcal{U}$ of homogeneous functions satisfying (i), (ii). Let a set $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ be such that* $\operatorname{Esssup} \Gamma \neq \emptyset$. *If $\lambda \in L^0(\mathbf{R}_+, \mathcal{H})$, then*

$$\operatorname{Esssup} (\lambda\Gamma) = \lambda \operatorname{Esssup} \Gamma.$$

Proof. Let $\widehat{\Gamma}_\lambda := \lambda \operatorname{Esssup} \Gamma$. Since multiplication on elements of $L^0(\mathbf{R}_+, \mathcal{H})$ preserves the order, we have $\widehat{\Gamma}_\lambda \succeq \lambda\Gamma$. Let $\gamma \in L^0(\mathbf{R}^d, \mathcal{H})$ and $\gamma \succeq \lambda\Gamma$. Take an arbitrary element $\tilde{\gamma}_0 \in \operatorname{Esssup} \Gamma$. Then

$$\gamma_1 := \lambda^{-1}\gamma I_{\{\lambda \neq 0\}} + \tilde{\gamma}_0 I_{\{\lambda = 0\}} \succeq \Gamma$$

and, in virtue of the property (b) for Γ, there is $\hat{\gamma} \in \operatorname{Esssup} \Gamma$ such that $\gamma_1 \succeq \hat{\gamma}$. It follows that $\gamma \succeq \lambda\hat{\gamma} \in \widehat{\Gamma}_\lambda$. Finally, let $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}_\lambda$ be such that $\hat{\gamma}_1 \succeq \hat{\gamma}_2$. By definition of $\hat{\Gamma}_\lambda$ we have that $\hat{\gamma}_i = \lambda\tilde{\gamma}_i$, where $\tilde{\gamma}_i \in \operatorname{Esssup} \Gamma$, $i = 1, 2$. Also,

$$\bar{\gamma}_i := \tilde{\gamma}_i I_{\{\lambda \neq 0\}} + \tilde{\gamma}_0 I_{\{\lambda = 0\}} \in \operatorname{Esssup} \Gamma.$$

and $\bar{\gamma}_1 \succeq \bar{\gamma}_2$. The property (c) for Γ implies that $\bar{\gamma}_1 = \bar{\gamma}_2$. Hence, $\hat{\gamma}_1 = \hat{\gamma}_2$. Therefore, the set Γ_λ satisfies all the conditions defining $\operatorname{Esssup} (\lambda\Gamma)$. □

We introduce the following condition:

(iii) There is $\gamma^0 \in L^0(\mathbf{R}^d, \mathcal{H})$ such that $\{\zeta \in L^0(\mathbf{R}^d, \mathcal{F}) : \ |\zeta| \leq 1\} \preceq \gamma^0$.

Proposition 3.13. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family $\mathcal{U}$ of homogeneous functions satisfying (i), (ii), (iii). Let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ and $|\Gamma| := \{|\gamma| : \ \gamma \in \Gamma\}$. Suppose that $\xi := \operatorname{esssup} |\Gamma| < \infty$ a.s. Then $\operatorname{Esssup} \Gamma$ is not empty.*

Proof. Consider $\tilde{\Gamma} := (1+\xi)^{-1}\Gamma$. By (iii), $\tilde{\Gamma} \preceq \gamma^0$ where $\gamma^0 \in L^0(\mathbf{R}^d, \mathcal{H})$. Applying Theorem 3.7, we deduce that $\operatorname{Esssup} \widetilde{\Gamma}$ is not empty. It follows that $\operatorname{Esssup} \Gamma$ is not empty and is given by $\operatorname{Esssup} \Gamma = (1+\xi) \operatorname{Esssup} \widetilde{\Gamma}$. This identity follows from Lemma 3.12. $\square$

It is easily seen that the above results hold also when all functions of the representing family are (positive) homogeneous of order $\gamma \neq 0$.

Lemma 3.14. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family $\mathcal{U}$ of linear functions satisfying (i), (ii), and (iii) and let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$. Then*

$$\mathcal{F}\text{-esssup}\, |\Gamma| < \infty \quad \Leftrightarrow \quad \text{there are } \gamma_1, \gamma_2 \in L^0(\mathbf{R}^d, \mathcal{F}) \text{ such that } \gamma_1 \preceq \Gamma \preceq \gamma_2.$$

Proof. ($\Rightarrow$) By the above proposition there are $\gamma_1 \in \mathcal{F}\text{-Essinf}\, \Gamma \neq \emptyset$ and $\gamma_2 \in \mathcal{F}\text{-Esssup}\, \Gamma \neq \emptyset$ with the needed property.

($\Leftarrow$) Suppose that the set $B := \{\mathcal{F}\text{-esssup}\, |\Gamma| = \infty\}$ is non-null. Take a sequence $\gamma^n \in \Gamma$ such that $|\gamma^n| \uparrow \mathcal{F}\text{-esssup}\, |\Gamma|$. Let us define the random variables $\widetilde{\gamma}^n := \gamma^n(|\gamma^n|+1)^{-1}$, $\widetilde{\gamma}_1^n := \gamma_1(|\gamma^n|+1)^{-1}$, and $\widetilde{\gamma}_2^n := \gamma_2(|\gamma^n|+1)^{-1}$. Using Lemma 2.1.2 [10] we may assume that $\widetilde{\gamma}_n \to \widetilde{\gamma}$ with $|\widetilde{\gamma}| = 1$ on B. On the other hand, in virtue of (ii), we have that $\widetilde{\gamma}_1^n \preceq \widetilde{\gamma}_n \preceq \widetilde{\gamma}_2^n$ for all n. It follows that $\widetilde{\gamma} = 0$ on the set B. A contradiction. $\square$

4. Essential Supremum in $L^0(\mathrm{R}^d)$ with Respect to a Random Cone

4.1. Setting

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\omega \mapsto G(\omega) \subseteq \mathbf{R}^d$ be a measurable set-valued mapping whose values are **closed** convex cones. The measurability is understood as the measurability of the graph, i.e. we assume that

$$\operatorname{graph} G := \{(\omega, x) \in \Omega \times \mathbf{R}^d : \ x \in G(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d).$$

The positive dual G^* of G is defined as the mapping whose values are closed convex cones

$$G^*(\omega) := \{x \in \mathbf{R}^d : \ xy \geq 0, \, \forall y \in G(\omega)\},$$

where xy is the scalar product. Note that $0 \in L^0(G, \mathcal{F}) \neq \emptyset$.

The fundamental fact of the theory of set-valued analysis is that any measurable mapping whose values are closed subsets admits a Castaing representation (see, e.g. [15]). In our case this means that there exists a countable set of measurable selectors ξ_i of G such that $G(\omega) = \overline{\{\xi_i(\omega) : \ i \in \mathbf{N}\}}$ for all $\omega \in \Omega$. Thus,

$$\text{graph } G^* = \{(\omega, y) \in \Omega \times \mathbf{R}^d : \ y\xi_i(\omega) \geq 0, \, \forall i \in \mathbf{N}\} \in \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d),$$

i.e. G^* is a measurable mapping and admits a Castaing representation, i.e. there exists a countable set of measurable selectors η_i of G^* such that we have $G^*(\omega) = \overline{\{\eta_i(\omega) : \ i \in \mathbf{N}\}}$ for all $\omega \in \Omega$. Since $G = (G^*)^*$,

$$G(\omega) = \{(\omega, x) \in \Omega \times \mathbf{R}^d : \ \eta_i(\omega)x \geq 0, \, \forall i \in \mathbf{N}\}. \tag{4.1}$$

From now on we suppose that the values of G are proper cones, i.e. $G \cap (-G) = \{0\}$ a.s. (or, equivalently, $\operatorname{int} G^* \neq \emptyset$). In the terminology of mathematical finance this property is called the *efficient friction condition.*

Under the adopted hypothesis the relation $\gamma_2 - \gamma_1 \in G$ a.s. defines a partial order $\gamma_2 \succeq \gamma_1$ in $L^0(\mathbf{R}^d, \mathcal{F})$. Moreover, the countable family of functions $u_j(\omega, x) = \eta_j(\omega)x$ where η_j is a Castaing representation of G^*, represents the partial order defined by G. So, the above theory can be applied.

Notation. Let $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{F}$ and let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$. We shall use sometimes the notation $(\mathcal{H}, G)$-Esssup Γ instead of $\mathcal{H}$-Esssup Γ to indicate that partial order is generated by the random cone G.

Note that in some cases we may dispose an additional information about the measurability of G.

Of course, if the partial order is given by a countable family of functions $(\omega, x) \mapsto \eta_i(\omega)x$, then we can generate it by a random cone G given by (4.1). It is worth noting that in applications the partial order is usually given by a random cone rather than by a representing family.

Since in the considered case the order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$ are compacts, the set $(\mathcal{H}, G)$-Esssup Γ exists if Γ is bounded from above with respect to the partial order (i.e. there is $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$ such that $\bar{\gamma} - \Gamma \in G$).

4.2. Properties

In the following the partial order $\succeq$ is defined by the random cone G.

Lemma 4.1. *Suppose that the representation (4.1) is given by d elements $\eta_i \in L^0(G^*, \mathcal{H})$ such that the vectors $\eta_i(\omega)$ form a basis in $\mathbf{R}^d$ for almost all ω and $G^* = \operatorname{cone}\{\eta_i,\ 1 \le i \le d\}$. Let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ be such that $(\mathcal{H}, G)$-$\operatorname{Esssup}\Gamma$ is a singleton $\{\bar{\gamma}\}$. Then $\eta_i\bar{\gamma} = (\mathcal{H}, \mathbf{R}_+)$-$\operatorname{Esssup}(\eta_i\Gamma)$ for all $i \le d$.*

Proof. Let us consider $\eta := \eta_i$ for some $i = 1, \cdots, d$. Since $\bar{\gamma} \succeq \Gamma$, we have the inequalities $\eta\bar{\gamma} \ge \eta\gamma$ for all $\gamma \in \Gamma$. Therefore, $\eta\bar{\gamma} \ge (\mathcal{H}, \mathbf{R}_+)$-$\operatorname{Esssup}(\eta\Gamma)$. Suppose that the inequality above is strict on a non-null set. Without loss of generality we may assume that $\eta_1 = \eta$. Let us consider a $\mathcal{H}$-measurable random vector ξ with $|\xi| = 1$ such that $\xi(\omega)$ is orthogonal to the linear subspace generated by $\eta_i(\omega)$, $i = 2, ..., d$, for almost all ω. We may always assume that $\eta\xi \ge 0$. We can find a non-zero $\mathcal{H}$-measurable random variable $\alpha \ge 0$ such that

$$\eta(\bar{\gamma} - \alpha\xi) \ge (\mathcal{H}, \mathbf{R}_+)\text{-}\operatorname{Esssup}(\eta\Gamma).$$

It follows that $\bar{\gamma} - \alpha\xi \succeq \Gamma$ and, hence $\bar{\gamma} - \alpha\xi \succeq \bar{\gamma}$. Thus, we have that $\xi(\omega) \in G(\omega) \cap (-G(\omega))$ on the non-null set where $\alpha \neq 0$, i.e. $\xi(\omega) = 0$ on this set. This is a contradiction. $\square$

Remark 4.2. Recall that a closed cone in $\mathbf{R}^d$ generates a lattice structure (i.e. a preorder with respect to which for any two elements have infimum and supremum) if and only if the dual cone has d independent generators, see [16] and the correction of the statement in [1]. Thus, the hypotheses of the above lemma means that $L^0(\mathbf{R}^d, \mathcal{F})$ is a lattice.

Corollary 4.3. *Assume that there are d elements $\eta_i \in L^0(G^*, \mathcal{H})$ such that $G^* = \operatorname{cone}\{\eta_i,\ 1 \le i \le d\}$ where the vectors $\eta_i(\omega)$ form a basis in $\mathbf{R}^d$ a.s. Suppose that $(\mathcal{H}, G)$-Esssup of any finite subset of $L^0(\mathbf{R}^d, \mathcal{F})$ is a singleton. Let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ and let $(\mathcal{H}, G)$-$\operatorname{Esssup}\Gamma = \{\hat{\gamma}\}$. Then for any $\eta \in L^0(\mathbf{R}^d, \mathcal{H})$ there exists a sequence γ_n from the set Γ^{up} (defined by (3.1)) such that*

$$\eta\hat{\gamma} = \lim_n \eta\gamma_n.$$

In particular, if $\Gamma = \Gamma^{\mathrm{up}}$, then $\eta\hat{\gamma} \le (\mathcal{H}, \mathbf{R}_+)$-$\operatorname{esssup}(\eta\Gamma)$.

Proof. By virtue of Lemma 3.6, we may assume without loss of generality that $\Gamma = \Gamma^{\mathrm{up}}$, i.e. Γ is directed upwards. Since the vectors $\eta_j(\omega)$ form a basis in $\mathbf{R}^d$, $\eta = \sum_{j\le d} \alpha_j \eta_j$ where $\alpha_j \in L^0(\mathbf{R},\mathcal{H})$. By virtue of the above lemma,

$$\eta\hat{\gamma} = \sum_{j\le d} \alpha_j (\mathcal{H},\mathbf{R}_+)\text{-esssup}\,(\eta_j\Gamma).$$

For each j, the family $\eta_j\Gamma$ is directed upwards. Thus, there are sequences $\gamma_n^j \in \Gamma$ such that $\eta_j\gamma_n^j \uparrow (\mathcal{H},\mathbf{R}_+)\text{-esssup}\,(\eta_j\Gamma)$ a.s. Replacing the sequences (γ_n^j) by the sequence $\gamma_n := (\mathcal{H},G)\text{-esssup}\,\{\gamma_n^j:\ j\le d\} \in \Gamma$, we obtain that $(\mathcal{H},\mathbf{R}_+)\text{-esssup}\,(\eta_j\Gamma) = \lim_n \eta_j\gamma_n$. The statement follows from here immediately. □

4.3. Polyhedral Ordering Cones G with Linearly Independent Generators

Proposition 4.4. *Let* $G = \text{cone}\,\{\xi^i,\ i=1,\cdots,N\}$ *where* $\xi^i \in L^0(\mathbf{R}^d,\mathcal{F})$ *and, for every* ω, *the vectors* $\xi^i(\omega)$, $i=1,\cdots,N$, *are linearly independent (so,* $N\le d$*). Let a non-empty set* $\Gamma \subseteq L^0(\mathbf{R}^d,\mathcal{F})$ *be such that* $\Gamma \preceq \bar{\gamma}$ *for some* $\bar{\gamma} \in L^0(\mathbf{R}^d,\mathcal{F})$. *Then* $\mathcal{F}\text{-Esssup}\,\Gamma$ *is a singleton.*

Proof. Without loss of generality we may assume that $|\xi^i| = 1$. Let us consider the $\mathcal{F}$-measurable random linear subspace $G-G$. Any $\gamma \in L^0(G-G,\mathcal{F})$ admits a unique representation $\gamma = \sum_{i=1}^N \alpha^i(\gamma)\xi^i$ where the coefficients $\alpha^i \in L^0(\mathbf{R},\mathcal{F})$; they are all non-negative if and only if $\gamma \in L^0(G,\mathcal{F})$. Invariance under a shift on a fixed random vector allows us to reduce the problem to the case where $0\in\Gamma$. Since $\bar{\gamma}-\Gamma \subset L^0(G,\mathcal{F})$, the "bound" $\bar{\gamma} \in L^0(G,\mathcal{F})$ and $\Gamma \subset L^0(G-G,\mathcal{F})$. Then

$$\widehat{\alpha}^i := (\mathbf{R}_+,\mathcal{F})\text{-esssup}\,\{\alpha^i(\gamma),\ \gamma\in\Gamma\} \le \alpha^i(\bar{\gamma}) < \infty$$

(this is nothing but the classical essential supremum). It is easy to check that $\mathcal{F}\text{-Esssup}\,\Gamma = \{\widehat{\gamma}\}$ where $\widehat{\gamma} = \sum_{i=1}^N \widehat{\alpha}^i\xi^i$. □

Corollary 4.5. *Under the assumptions of the above proposition on G, if a non-empty set* $\Gamma \subseteq L^0(\mathbf{R}^d,\mathcal{F})$ *is such that* $(\mathbf{R}_+,\mathcal{F})\text{-esssup}\,|\Gamma| < \infty$, *then* $\mathcal{F}\text{-Esssup}\,\Gamma$ *is a singleton.*

5. Hedging of European Options in a Discrete-Time Model with Transaction Costs

In the model we are given a closed proper convex cone $K \subset \mathbf{R}^d$ whose interior contains $\mathbf{R}^d_+ \setminus \{0\}$ and a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t=0,...,T}, P)$ with a d-dimensional adapted process $S = (S_t)$ with strictly positive components.

Define the random diagonal operators

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S^1_t, ..., x^d/S^d_t), \qquad t = 0, ..., T,$$

and relate with them the random cones $\widehat{K}_t := \phi_t K$. We consider the set $\widehat{\mathcal{V}}$ of $\mathbf{R}^d$-valued adapted processes $\widehat{V}$ such that $\Delta \widehat{V}_t := \widehat{V}_t - \widehat{V}_{t-1} \in -\widehat{K}_t$ for all t and the set $\mathcal{V}$ whose elements are the processes V with $V_t = \phi_t^{-1} \widehat{V}_t$, $\widehat{V} \in \widehat{\mathcal{V}}$.

In the context of the theory of markets with transaction costs, K is the solvency cone in a model with efficient friction corresponding to the description in terms of a numéraire, $\mathcal{V}$ is the set of value processes of self-financing portfolios. The notations with hat correspond to the description of the model in terms of "physical" units where the portfolio dynamics is much simpler because it does not depend on price movements. A typical example is the model of currency market defined via the matrix of transaction costs coefficients $\Lambda = (\lambda^{ij})$ with non-negative entries and $\lambda^{ii} = 0$. In this case

$$K = \text{cone}\,\{(1 + \lambda^{ij})e_i - e_j,\ e_i, 1 \le i, j \le d\}.$$

Another example is the commodity market where all transactions are payed from the money account. In this case

$$K = \text{cone}\,\{\gamma^{ij} e_1 + e_i,\ (1 + \gamma^{1i})e_1 - e_i,\ (-1 + \gamma^{j1})e_1 + e_j,\ e_i,\ 1 \le i, j \le d\}.$$

We assumed for simplicity that K is constant. In general, $K = (K_t)$ is an adapted random process whose values are convex closed proper cones, e.g., given by an adapted matrix-valued process $\Lambda = (\Lambda_t)$. But even in the constant case $\widehat{K} = (\widehat{K}_t)$ is a random cone-valued process. Note that one can use modeling involving only $\widehat{K}$ defined, e.g., by the bid-ask (adapted matrix-vaued) process but this is just a different parametrization leading to the same geometric structure.

In this model the contingent claim is a d-dimensional random vector. We shall use the notation Y_T when the contingent claim is expressed in units of the numéraire and $\widehat{Y}_T$ when it is expressed in physical units. The relation is obvious: $\widehat{Y}_T = \phi_T Y_T$.

The value process $V \in \mathcal{V}$ is called *minimal* if $V_T = Y_T$ and any process $W \in \mathcal{V}$ such that $W_T = Y_T$ and $W_t \preceq_K V_t$ for all $t \le T$ coincides with V.

The questions of interest are whether minimal portfolios do exist and how they can be found. We denote $\mathcal{V}_{min}$ the set of all minimal processes. The set $\widehat{\mathcal{V}}_{min}$ is defined in the obvious way.

Proposition 5.1. *Suppose that $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \le T-1$, and suppose there exits a least one $\widehat{V} \in \widehat{\mathcal{V}}$ such that $\widehat{V}_T \ge_{\hat{K}_T} \hat{Y}_T$. Then $\widehat{\mathcal{V}}_{min} \neq \emptyset$ and $\widehat{\mathcal{V}}_{min}$ coincides with the set of solutions of backward inclusions*

$$\widehat{V}_t \in (\mathcal{F}_t, \widehat{K}_{t+1})\text{-Esssup}\,\{\widehat{V}_{t+1}\}, \quad t \le T-1, \quad \widehat{V}_T = \widehat{Y}_T. \tag{5.1}$$

Moreover, any $W \in \mathcal{V}$ with $W_T \succeq Y_T$ is such that $W \succeq_K V$ for some $V \in \mathcal{V}_{min}$.

Proof. Let $\widehat{W} \in \widehat{\mathcal{V}}$ be such that $\widehat{W}_T \succeq_{\widehat{K}_T} \widehat{Y}_T$. Since $\Delta\widehat{W}_T \in -\widehat{K}_T$, we have $\widehat{W}_{T-1} \succeq_{\widehat{K}_T} \widehat{W}_T$. By definition of $(\mathcal{F}_{T-1}, \widehat{K}_T)$-Esssup and Theorem 3.7, we obtain that $\widehat{W}_{T-1} \succeq_{\widehat{K}_T} \widehat{V}_{T-1}$ for some $\widehat{V}_{T-1} \in (\mathcal{F}_{T-1}, \widehat{K}_T)$-Esssup $\{\widehat{Y}_T\} \neq \emptyset$. Therefore, by the hypothesis, $\widehat{W}_{T-1} \succeq_{\widehat{K}_{T-1}} \widehat{V}_{T-1}$. Continuing the backward induction, we obtain that $\widehat{W}_t \succeq_{\widehat{K}_t} \widehat{V}_t$ where $\widehat{V}_t$ satisfies (5.1). We deduce that any portfolio $\widehat{W} \in \widehat{\mathcal{V}}_{min}$ satisfy (5.1). The same backward induction allows us to conclude that any $\widehat{V} \in \widehat{\mathcal{V}}$ which satisfies (5.1) is minimal. □

Remark 5.2. The hypothesis $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \le T-1$, of the above proposition is equivalent to the absence of arbitrage opportunities of the second kind, see [10], Th. 3.2.20. Note that it is always fulfilled when the price process S admits an equivalent martingale measure. This hypothesis is essential for the claimed property. Indeed, with $T = 1$, suppose that the inclusion $L^0(\widehat{K}_1, \mathcal{F}_0) \subseteq L^0(\widehat{K}_0, \mathcal{F}_0)$ does not hold. In this case, we may find $\hat{W}_0 \in L^0(\widehat{K}_1, \mathcal{F}_0)$ such that $\hat{W}_0 \notin \widehat{K}_0$. With the payoff $\hat{Y}_T = 0$, we easily obtain that the only minimal value process which satisfies (5.1) is given by $\hat{V}_1 = \hat{V}_0 = 0$. If the assertion of Proposition 5.1 holds, the value process ($\hat{W}_0$, $\hat{W}_1 = 0$) should satisfy $\hat{W}_0 \succeq \hat{V}_0$, i.e. $\hat{W}_0 \in \hat{K}_0$ and we get a contradiction.

In the above result we do not use the specificity of the model and it can be extended without any changes to an "abstract case" where $\widehat{K} = (\widehat{K}_t)$ is replaced by an adapted cone-valued process $G = (G_t)$.

Remark 5.3. In general, the essential supremum in the formula (5.1) is not a singleton. Take for instance the simple case where $\Omega := \{\omega_1, \omega_2\}$ and

$T = 1$. Consider $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_1$ is the σ-algebra of all subsets of Ω. Suppose that $\widehat{K}_0 = \widehat{K}_1(\omega_1)$ and $\widehat{K}_1(\omega_2) \neq \widehat{K}_0$. As illustrated in the picture below, if $\widehat{V}_1 = AI_{\{\omega_1\}} + BI_{\{\omega_2\}}$, then $(\mathcal{F}_0, \widehat{K}_1)$-Esssup $\{\widehat{V}_1\} = \Lambda$ where $\Lambda := (A + \widehat{K}_0) \cap (B + \widehat{K}_1(\omega_2))$. Indeed, the set of all deterministic points $V_0 \geq_{\widehat{K}_1} \widehat{V}_1$ is Λ and neither of them can be "reduced" in the direction of $\widehat{K}_1$ since $\widehat{K}_1(\omega_2) \cap \widehat{K}_1(\omega_1) = \{0\}$.

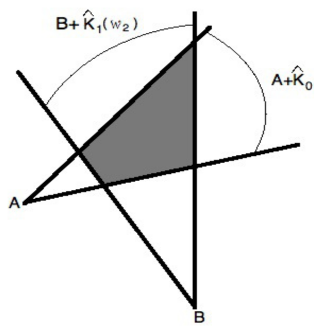

Figure 2: The grey-coloured domain corresponds to the set $\Lambda := (\mathcal{F}_0, \widehat{K}_1)$-Esssup $\{\widehat{V}_1\}$ where $\widehat{V}_1 = AI_{\{\omega_1\}} + BI_{\{\omega_2\}}$.

6. Set-Valued Dynamic Risk Measures

One of interesting recent new ideas in the theory of financial markets with transaction costs is the introduction of set-valued risk measures, see [9], [8], [6]. In the following illustrative examples we show that the notion Esssup can be used to construct such risk measures in dynamic setting.

Let us consider the model described in the previous section assuming that transaction costs coefficients are constant and the efficient friction condition holds. We denote by $\succeq$ the order in $L^0(\mathbf{R}^d, \mathcal{F}_T)$ defined by the solvency cone K.

We denote by D the set of $X \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ such that $X \succeq -kS_t$, $t \leq T$, for some $k \geq 0$.

Let X be a vector-valued random variable $X \in L^0(\mathbf{R}^d, \mathcal{F}_T)$. It models the value at date T of some multi-asset financial portfolio, the investment of a company at date T. A dynamic set-valued measure of risk $\rho = (\rho_t)$ is a family of set-valued mappings defined on D. The mapping ρ_t associates with $X \in D$ a non-empty set $\rho_t(X) \subset L^0(\mathbf{R}^d, \mathcal{F}_t)$, interpreted as positions ξ, when added to X, make the total position $X + \xi$ acceptable by the regulator/supervisor at date t. The acceptable positions are given by the solvency cone K, i.e. they are such that $\rho_t(X) + X \succeq 0$ or $\rho_t(X) + X \subseteq K$.

The natural requirements on such a risk measure are the following properties that can be described as follows:

(r_0) $\rho_t(0) \succeq 0$;.

(r_1) If $X \succeq 0$ and $X \in D$, then there exists a process (ξ_t) such that $\xi_t \in \rho_t(X)$ and $\xi_t \preceq 0$ for all $t \leq T$.

(r_2) for any $(\xi, \eta) \in \rho_t(X) \times \rho_t(Y)$ there is $\zeta \in \rho_t(X + Y)$ such that $\zeta \preceq \xi + \eta$.

(r_3) If $\lambda \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ and $X \in D$, then $\rho_t(\lambda X) = \lambda \rho_t(X)$.

(r_4) If $X \in D$ and $a \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, then $\rho_t(X + a) = \rho_t(X) - a$.

Proposition 6.1. *The mapping $\rho_t(X) := \mathcal{F}_t\text{-Esssup}\,\{-X\}$ defines a dynamic set-valued risk measure on D.*

Proof. In virtue of Theorem 3.7 the set $\mathcal{F}_t$-Esssup $\{-X\}$ is non-empty (though not necessary a singleton). Properties (r_0), (r_1), and (r_4) are obvious, (R_3) follows from Lemma 3.12. Finally, if $\xi, \in \rho_t(X)$ and $\eta \in \rho_t(Y)$, that is $\xi + X \in K$ and $\eta + Y \in K$, then $\xi + \eta + X + Y \in K$ a.s. In other words, $\xi + \eta \leq -(X + Y)$. According to the property (b) in Definition 3.1, there is $\zeta \preceq \xi + \eta$ in $\mathcal{F}_t$-Esssup $\{-(X + Y)\}$. Thus, (r_2) also holds. □

The definition of set-valued dynamic risk measures can be extended in an natural way to be applied to sets of random vectors. Let Γ be a non-empty subset of D. It can be interpreted as the set of potential investors position at time T. A dynamic set-valued measure of risk $\mathcal{R} = (\mathcal{R}_t)$ is a family of set-valued mappings defined on such subsets of D and associating with Γ a non-empty set $\mathcal{R}_t(\Gamma) \subset L^0(\mathbf{R}^d, \mathcal{F}_t)$. The interpretation: the total positions from $\Gamma + \mathcal{R}_t(\Gamma)$ are acceptable by the regulator/supervisor at date t, i.e. they are elements, i.e. they are such that $\rho_t(X) + X \succeq 0$.

For a dynamic set-valued measure defined on sets of random vectors the following properties should be fulfilled:

(R_0) $\mathcal{R}_t(\{0\}) \succeq 0$;.

(R_1) If $\Gamma \succeq 0$ and $\Gamma \subseteq D$, then there exists a process (ξ_t) such that $\xi_t \in \mathcal{R}_t(X)$ and $\xi_t \preceq 0$ for all $t \leq T$.

(R_2) for any $(\xi, \eta) \in \rho_t(\Gamma_1) \times \rho_t(\Gamma_2)$ there is $\zeta \in \mathcal{R}_t(\Gamma_1 + \Gamma_2)$ such that $\zeta \preceq \xi + \eta$.

(R_3) If $\lambda \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ and $\Gamma \in D$, then $\mathcal{R}_t(\lambda\Gamma) = \lambda\mathcal{R}_t(X)$.

(R_4) If $\Gamma \subseteq D$ and $a \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, then $\mathcal{R}_t(X + a) = \mathcal{R}_t(X) - a$.

As above we have:

Proposition 6.2. *The mapping* $\mathcal{R}_t(\Gamma) := \mathcal{F}_t\text{-Esssup}\,\{-\Gamma\}$ *defines a dynamic set-valued risk measure on subsets of* D*.*

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