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Consistent Price Systems and Arbitrage Opportunities of the Second Kind in Models with Transaction Costs

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Abstract In contrast with the classical models of frictionless financial markets, market models with proportional transaction costs, even satisfying usual no-arbitrage properties, may admit arbitrage opportunities of the second kind. This means that there are self-financing portfolios with initial endowments laying outside the solvency region but ending inside. Such a phenomenon was discovered by M. Rásonyi in the discrete-time framework. In this note we consider a rather abstract continuous-time setting and prove necessary and sufficient conditions for the property which we call No Free Lunch of the 2nd Kind, **NFL2**. We provide a number of equivalent conditions elucidating, in particular, the financial meaning of the property **B** which appeared as an indispensable "technical" hypothesis in previous papers on hedging (super-replication) of contingent claims under transaction costs. We show that it is equivalent to another condition on the "richness" of the set of consistent price systems, close to the condition **PCE** introduced by Rásonyi. In the last section we deduce the Rásonyi theorem from our general result using specific features of discrete-time models.

Keywords Transaction costs · Arbitrage · No Free Lunch · Consistent price systems · Set-valued processes · Martingales.

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1 Introduction

In the recent paper [28] M. Rásonyi discovered that financial market models with proportional transaction costs, even being arbitrage-free in the usual sense, nevertheless, may admit portfolios ending up, for sure, in the solvency region despite their initial values lay outside one. Working in the discrete-time framework and assuming the efficient friction he established some necessary and sufficient conditions for the absence of these arbitrage opportunities of the second kind[1] (shortly: **NA2**-property). One of such conditions is the existence of a strictly consistent price system (a martingale evolving in the interiors of dual of solvency cones) starting from an arbitrary initial value in the interior. The aim of the present note is to extend Rásonyi's results to the continuous-time setting.

To explain the financial motivation of the problem discussed here we recall some concepts and facts from the arbitrage theory following the book [21], Chs. 2 and 3. This theory formalizes the concept of arbitrage, that is the existence of portfolio strategies allowing for non-risky profits. One can imagine two kinds of arbitrage opportunities: 1) a portfolio starting from the zero initial value and ending up with a positive non-zero value; 2) a portfolio starting from a strictly negative value (i.e., the investor enters the market with a debt) and ending up with a positive value. In the common discrete-time model of frictionless financial market with the price process $S=(S^1,...,S^d)$ the arbitrage opportunities of the 2nd kind are not interesting and rarely mentioned. The reason is that the inequality $-a+H\cdot S_T\geq 0$, where $a>0$, implies that $H\cdot S_T\geq a$, i.e. the strategy H realizing the arbitrage opportunity of the 2nd kind realizes arbitrage opportunity of the 1st kind (and a good one!). Thus, the conventional **NA**-property, usually required from a market model and excluding the arbitrage opportunities of the 1st kind, automatically excludes the arbitrage opportunities of the 2nd kind.

Supposing that the first traded asset is the *numéraire*, i.e. $S^1=1$ and slightly abusing the financial terminology, we reformulate the classical criterion of absence of arbitrage as follows:

The **NA***-property holds if and only if there is a stochastic deflator, i.e. a strictly positive martingale ρ such that the process $Z:=\rho S$ is a martingale.*

The processes Z can be interpreted as "correct" or "fair" prices of financial securities allowing to compare the today value of securities and their expected future value. Usually, the stochastic deflators are normalized to have unit initial value and in this case they are just the densities of equivalent martingale measures involved in the "standard" formulation of the criterion and playing the fundamental role in the whole theory of financial markets.

Let us turn to the simplest discrete-time model of financial market with proportional transaction costs. The investor portfolio is now vector-valued and its evolution, in units of the *numéraire*, is given by the following controlled difference equation:

$$\Delta V_t=\operatorname{diag} V_{t-1}\Delta R_t+\Delta B_t,\qquad V_{-1}=v,$$

where $\Delta R^i_t=\Delta S^i_t/S^i_{t-1}$, $i\leq d$, is the relative price increment of the ith security, $\Delta B_t\in L^0(-K_t,\mathcal{F}_t)$ is the control, and $\operatorname{diag} x$ denotes the diagonal operator generated by the vector x. In other words, the investor action ΔB_t is an $\mathcal{F}_t$-measurable random

[1] M. Rásonyi calls this property **NGV**, No Sure Gain in Liquidation Value, (**NSP**, No Sure Profits in the preliminary version of his paper). We prefer a terminology consistent with earlier works on the large financial markets where a similar phenomenon was observed.

variable taking values in the cone $-K_t$. In the model where one can exchange any asset to any other with losses (see [21], Section 3.1.1), the solvency cones are defined by the matrices of transaction costs coefficients $\Lambda_t = (\lambda_t^{ij})$:

$$K_t = \text{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\}. \tag{1.1}$$

In the theory, as in practice, the coefficients $\lambda_t^{ij} \ge 0$ are adapted random processes. The above dynamics naturally falls into a scope of linear difference equations with controls constraint to be taken from random cones.

One can express the portfolio dynamics also in "physical units". It is much simpler. Assuming that $S_{-1} = S_0 = (1, ..., 1)$ and introducing the diagonal operators

$$\phi_t := (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d),$$

we have:

$$\Delta\widehat{V}_t = \widehat{\Delta B}_t, \qquad \widehat{V}_{-1} = v,$$

where $\widehat{V}_t = \phi_t V_t$, $\widehat{\Delta B}_t \in L^0(-\widehat{K}_t, \mathcal{F}_t)$, $\widehat{K}_t = \phi_t K_t$. Note that, in contrast with K_t, the cones $\widehat{K}_t$ are always random, even in the model with constant transaction costs. So, $(\widehat{K}_t)$ is an adapted cone-valued process. We shall consider also the adapted cone-valued process $(\widehat{K}_t^*)$ with $\widehat{K}_t^*(\omega)$ defined as the (positive) dual cone of $\widehat{K}_t(\omega)$. Though in financial models the cones $\widehat{K}_t(\omega)$ are polyhedral, for the control theory this looks too restrictive and the question about possible extensions to a "general" model, with $(\widehat{K}_t)$ replaced by an arbitrary adapted cone-valued process (G_t), arises naturally. The reader should be informed that for this "general" model a few results are available, e.g., until recently it was not known whether the principal theorems of [19] and [20] remain true for it. Only in the recent preprint [26] criterion for $\mathbf{NA}^r$-property was extended to an arbitrary cone-valued processes.

For models with transaction costs one can consider various types of arbitrage opportunities of the 1st kind with corresponding no-arbitrage properties. E.g., the weak no-arbitrage property ($\mathbf{NA}^w$) of the market, the most natural one, means that the intersection of the set of terminal values of portfolio processes $\widehat{A}_0^T = -\sum_{t=0}^T L^0(\widehat{K}_t, \mathcal{F}_t)$ with $L^0(\mathrm{R}_+^d, \mathcal{F}_T)$ is a singleton containing only the random variable identically equal to zero. For the model with a finite number of states of the nature the following criterion is well-known, [22]:

The $\mathbf{NA}^w$-property holds if and only if there is Z belonging to the set $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ of martingales taking values in the random cones $\widehat{K}_t^* \setminus \{0\}$.

Remarkably, this assertion holds for arbitrary Ω for two-asset model, [9], but fails to be true for models with a larger number of assets, [30]. In the latter paper it was shown that the condition $\mathcal{M}_0^T(\mathrm{ri}\,\widehat{K}^*) \ne \emptyset$, i.e. the existence of a martingale evolving in relative interiors of the dual cones, admits, for any number of assets, an equivalent characterization as the *robust no-arbitrage property* $\mathbf{NA}^r$ expressing the fact that $\mathbf{NA}^w$-property holds also for smaller transaction costs.

An inspection of results obtained for discrete-time models shows that the elements of $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ and $\mathcal{M}_0^T(\mathrm{ri}\,\widehat{K}^*)$ referred to as *consistent price systems* and *strictly consistent price systems*, respectively, play a fundamental role in the theory: they are direct generalizations of the stochastic deflators defined above because in the absence of transaction costs the random cones $\widehat{K}_t^*$ are reduced to the random rays R_+S_t. Note that the condition $\mathcal{M}_0^T(\mathrm{ri}\,\widehat{K}^*) \ne \emptyset$ ensures the closedness of A_0^T, but $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\}) \ne \emptyset$

does not — even for two-asset model (examples by Rásonyi and Grigoriev, see [27], [9], and [21], Section 3.2.3).

In contrast with the theory of frictionless market, no-arbitrage properties of the 1st kind, even the robust no-arbitrage property, do not eliminate the existence of an arbitrage opportunity of the 2nd kind. The latter is defined as the value process $(V_t)_{t=s,\dots,T}$ such that $V_s \notin G_s$ (a.s.) but $V_T \in G_T$. The Rásonyi theorem claims that for the "general" model with *efficient friction* (i.e. when all cones G_t are proper) and $\mathrm{R}^d_+ \subseteq G_t$ the absence of arbitrage opportunities of the 2nd kind is equivalent to the "richness" of the set of strictly consistent price systems formally expressed as the following condition "Prices Consistently Extendable":

PCE For any s and $\eta \in L^1(\mathrm{int}\, G^*_s, \mathcal{F}_s)$, there is $Z \in \mathcal{M}^T_s(\mathrm{int}\, G^*)$ such that $Z_s = \eta$.

Of course, such an important result immediately leads to a question about its counterpart for continuous-time models. As it is well-known, the corresponding theory, even for the frictionless markets, is much more complicated and involves delicate topological properties and specific admissibility restrictions on portfolio processes. The stochastic deflators (density processes of local martingale measures) remain the fundamental objects but the existence of one is no more equivalent to the **NA**-property but to the **NFL**-property ("No Free Lunch") introduced by Kreps. The latter involves the $\sigma\{L^\infty, L^1\}$-closure $\bar{A}^w$ of the set A of bounded hedgeable contingent claims. In general, elements of this closure can not be characterized as limits of weakly* convergent sequences of elements of A and a financial interpretation of the **NFL**-property is not satisfactory (though strongly enrooted in the financial literature, see, e.g. [13] and references therein). For some particular models it is known that **NFL**-property is equivalent to the **NFLVR**-property which definition uses the norm-closure of $\bar{A}$ and which admits a transparent financial interpretation, [6], [14]. It is worth to emphasize that results of such type are not easy to obtain and satisfactory analogues for markets with friction are not known to the date, see [10], [11], [4] for recent progress in this direction.

Though difficult, the theory of continuous trading under transaction costs is rapidly growing. To avoid the transformation of this introduction to a general survey of the field, we mention only a few relevant publications. First, we attract the reader's attention to the recent articles [2], [7], [1] making clear that in the case where the prices have jumps it is more natural to model the value processes as làdlàg and not càdlàg or càglàd as in the early papers (this was already observed in works on optimal control but in a rather implicit way). Second, we give references to the recent papers studying the question when the set $\mathcal{M}^T_0(\mathrm{int}\, \widehat{K}^*)$ is non-empty or, more generally, when a martingale selector of a set-valued process does exist: [11], [29].

One of the difficulties of continuous-time setting is due to the fact that even for constant transaction costs the cone-valued processes $(\widehat{K}_t)$ and $(\widehat{K}^*_t)$ may have jumps. To get satisfactory hedging (super-replication) theorems for European and American options one needs to impose certain regularity properties of these processes and their generators as well as to use rather sophisticated definitions of the value processes. Moreover, some extra properties on the structure of the set of consistent price systems seems to be unavoidable. One of such properties is the condition **B** which requires that the set of consistent price systems should be rich enough to test the evolution of the portfolio in the solvency region: the inequalities $V_t Z_t \geq 0$ for all Z have to imply that $V_t \in \widehat{K}_t$ (it was tacitly assumed in [24], appeared explicitly in [2] and used also in [21]). Another delicate property is the admissibility of strategies. It happens that the boundedness from below of value processes expressed in terms of the *numéraire*, a common assumption in the theory of frictionless markets, is not a good one. It is replaced by the boundedness from below of portfolio process with accounting in the "physical units". Since portfolios are vector-valued, one uses a partial ordering induced by solvency cones. This requires also an appropriate modification of the notion of Fatou-convergence.

The aim of the present note is to relate **B** with the condition **MCPS** ("Many Consistent Price Systems") close to **PCE**. The difference with the latter is that the prices should be extended to consistent (but not necessarily strictly consistent) price systems. Being inspired by Ràsonyi's work, we give an equivalent characterization of **B** in terms of a certain no-arbitrage property of the second kind involving weak* closure in the same line as was suggested by Kreps in his seminal work. In our study we follow the ideology of Kreps' **NFL**, the No Free Lunch condition. The question whether one can use in our context the norm-closure remains open.

The preliminary version of the paper used the framework developed in the papers [24], [2] and [7] and the results obtained there in. Unfortunately, this approach happened to be not adequate to the problems discussed here because it requires a lengthy repetition of rather "technical" definitions and hypotheses on the structure of cone-valued processes and portfolios.

That is why we opted to work here in a very general "abstract" mathematical setting using only a few comprehensive hypotheses. These hypotheses are fulfilled for the basic models in continuous as well as in discrete time. The chosen approach allows us not to enter in the discussion of specific models but, by providing necessary references, to arrive quickly to the essence of our note. We introduce the notion No Free Lunch of the Second Kind, **NFL2**, named in an obvious allusion with a concept **NAA2** that have been studied in the theory of large financial markets, [15], [16]. We establish several necessary and sufficient conditions for this property (Theorem 2.2). Our main conclusion is that the condition **B**, which appeared in all previous studies on the superreplication problems as a technical one, is equivalent to a financially meaningful condition, namely, to the absence of asymptotic arbitrage opportunities of the second kind. Under the assumption that the set of hedgeable claims is Fatou closed (this property always holds in the basic models of financial markets with transaction costs) we prove that **B** holds if and only if the condition **MCPS** is fulfilled.

2 Main Result

Let $\big(\Omega, \mathbf{F}=\mathcal{F}, (\mathcal{F}_t)_{t\le T}, P\big)$ be a continuous-time stochastic basis verifying the usual conditions. We are given a pair of set-valued adapted processes $G=(G_t)_{t\in[0,T]}$ and $G^*=(G^*_t)_{t\in[0,T]}$ whose values are closed cones in R^d in duality, i.e.

$$G^*_t(\omega)=\{y\in \mathrm{R}^d:\ yx\ge 0\ \forall x\in G_t(\omega)\}.$$

"Adapted" means that the graphs

$$\Big\{(\omega,x)\in\Omega\times\mathrm{R}^d:\ x\in G_t(\omega)\Big\}$$

are $\mathcal{F}_t\otimes\mathcal{B}(\mathrm{R}^d)$-measurable.

We assume that all cones G_t are proper, i.e. $G_t \cap (-G_t) = \{0\}$ or, equivalently, $\operatorname{int} G_t^* \neq \emptyset$. In financial context this means that the *efficient friction* condition (EF) is fulfilled, i.e. the market does not admit loops of transaction-free exchanges. We assume also that G_t dominates R_+^d, i.e. $G_t^* \backslash \{0\} \subset \operatorname{int} \mathrm{R}_+^d$.

In a more specific financial setting (see [24], [2], [7], [1], [27], [10]), the cones G_t are the solvency cones $\widehat{K}_t$ provided that the portfolio positions are expressed in physical units.

For each $s \in]0,T]$ we are given a convex cone $\mathcal{Y}_s^T$ of optional R^d-valued processes $Y = (Y_t)_{t \in [s,T]}$ with $Y_s = 0$.

It is assumed that $\mathcal{Y}_s^T$ is stable under multiplication by the positive bounded $\mathcal{F}_s$-measurable random variables, i.e. by the elements of the set $L_+^\infty(\mathcal{F}_s) = L^\infty(\mathrm{R}_+, \mathcal{F}_s)$. Moreover, if sets $A^n \in \mathcal{F}_s$ form a countable partition of Ω and $Y^n \in \mathcal{Y}_s^T$, then $\sum_n Y^n I_{A^n} \in \mathcal{Y}_s^T$.

The following notations will be used in the sequel:

- for d-dimensional processes Y and Y' the relation $Y \geq_G Y'$ means that the difference $Y - Y'$ evolves in G, that is $Y_t - Y_t' \in G_t$ a.s. for every t;
- $\mathbf{1}$ stands for a vector $(1, ..., 1) \in \mathrm{R}_+^d$;
- $\mathcal{Y}_{s,b}^T$ denotes the subset of $\mathcal{Y}_s^T$ formed by the processes Y dominated from below in the sense of partial ordering generated by G, i.e. such that there is a constant κ such that the process $Y + \kappa \mathbf{1}$ evolves in G;
- $\mathcal{Y}_{s,b}^T(T)$ is the set of random variables Y_T where $Y \in \mathcal{Y}_{s,b}^T$;
- $\mathcal{A}_{s,b}^T(T) = (\mathcal{Y}_{s,b}^T(T) - L^0(G_T, \mathcal{F}_T)) \cap L^\infty(\mathrm{R}^d, \mathcal{F}_T)$ and $\overline{\mathcal{A}_{s,b}^T(T)}^w$ is the closure of this set in $\sigma\{L^\infty, L^1\}$;
- $\mathcal{M}_s^T(G^*)$ is the set of all d-dimensional martingales $Z = (Z_t)_{t \in [s,T]}$ evolving in G^*, i.e. such that $Z_t \in L^0(G_t^*, \mathcal{F}_t)$.

Throughout the note we assume the following **standing hypotheses** on the sets $\mathcal{Y}_{s,b}^T$:

$\mathbf{S}_1$ $E\xi Z_T \leq 0$ for all $\xi \in \mathcal{Y}_{s,b}^T(T)$, $Z \in \mathcal{M}_s^T(G^*)$, $s \in [0,T[$.

$\mathbf{S}_2$ $\cup_{t \geq s} L^\infty(-G_t, \mathcal{F}_t) \subseteq \mathcal{Y}_{s,b}^T(T)$ for each $s \in [0,T]$.

The hypotheses $\mathbf{S}_1$ and $\mathbf{S}_2$ adopted in this note allows us to avoid the annoying repetitions and do not provide the full description of continuous-time models with transaction costs.

It is important to know only that these conditions are fulfilled for the known models, see [24], [2], [7]. Recall that in these financial models $\mathbf{S}_1$ holds because if one calculates the current portfolio value using a price system (that is a process from $\mathcal{M}_s^T(G^*)$) the resulting scalar process is a supermartingale. The following example illustrates this.

Example. Let consider the simplest continuous-time analogue of the model described in the introduction where the solvency cones K_t, $t \in [0,T]$, are given by the formula (1.1) (this model sometimes referred to as the model of a currency market). Now $G_t = \widehat{K}_t = \phi_t K_t$. Define the elements of $\mathcal{Y}_s^T$ as adapted processes Y of bounded variation with $Y_s = 0$ and such that $\dot{Y}_t \in -\widehat{K}_t$ where $\dot{Y} = Y/d||Y||$ and $||Y||$ is the total variation of Y. Take $Z \in \mathcal{M}_s^T(\widehat{K}^*)$. By the product formula we have that

$$ZY = Y_- \cdot Z + Z\dot{Y} \cdot ||Y||,$$

where the first integral in the right-hand is a local martingale while the second is a decreasing process. In the case where $Y \in \mathcal{Y}_{s,b}^T$, that is $Y_t + \kappa \mathbf{1} \in \widehat{K}_t$ for some $\kappa > 0$, the process ZY dominates the martingale $-\kappa Z\mathbf{1}$. Hence, it is a supermartingale and $EZ_TY_T \leq 0$. Thus, the hypothesis $\mathbf{S}_1$ is fulfilled. The hypothesis $\mathbf{S}_2$ holds since $\widehat{K}$ contains $\mathbf{R}^d_+$.

It is easily seen that in the above arguments only the duality of cones was used. They also can be extended to more sophisticated definitions of value processes as in [2], [7].

The hypothesis $\mathbf{S}_2$ expresses the fact that an investor has a right to take any position less advantageous than zero and keeps it until the end of the planning horizon. It is fulfilled in all financial models.

Now we introduce other properties of interest: No Free Lunch, No Free Lunch of the 2nd Kind, Many Consistent Price Systems.

NFL $\overline{\mathcal{A}^T_{s,b}(T)}^w \cap L^\infty(\mathrm{R}^d_+, \mathcal{F}_T) = \{0\}$ for each $s \in [0,T[$.

NFL2 For each $s \in [0,T[$ and $\xi \in L^\infty(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + \overline{\mathcal{A}^T_{s,b}(T)}^w) \cap L^\infty(\mathrm{R}^d_+, \mathcal{F}_T) \neq \emptyset$$

only if $\xi \in L^\infty(G_s, \mathcal{F}_s)$.

MCPS For any $\eta \in L^1(\mathrm{int}\, G^*_s, \mathcal{F}_s)$, there is $Z \in \mathcal{M}^T_s(G^* \setminus \{0\})$ with $Z_s = \eta$.

Finally, we recall one more condition:

B If ξ is an $\mathcal{F}_s$-measurable R^d-valued random variable such that $Z_s\xi \geq 0$ for any $Z \in \mathcal{M}^T_s(G^*)$, then $\xi \in G_s$ (a.s.).

The following assertion is a version of FTAP for the considered setting:

Proposition 2.1 **NFL** $\Leftrightarrow \mathcal{M}^T_0(G^*\backslash\{0\}) \neq \emptyset$.

Proof. ($\Leftarrow$) Let $Z \in \mathcal{M}^T_0(G^*\backslash\{0\})$. Then the components of Z_T are strictly positive and $EZ_T\xi > 0$ for all $\xi \in L^\infty(\mathrm{R}^d_+, \mathcal{F}_T)$ except $\xi = 0$. On the other hand, $E\xi Z_T \leq 0$ for all $\xi \in \mathcal{Y}^T_{s,b}(T)$ and so for all $\xi \in \overline{\mathcal{A}^T_{s,b}(T)}^w$.

($\Rightarrow$) The Kreps–Yan theorem on separation of closed cones in $L^\infty(\mathrm{R}^d, \mathcal{F}_T)$ implies the existence of $\eta \in L^1(\mathrm{int}\,\mathrm{R}^d_+, \mathcal{F}_T)$ such that $E\xi\eta \leq 0$ for every $\xi \in \overline{\mathcal{A}^T_{s,b}(T)}^w$, hence, by virtue of the hypothesis $\mathbf{S}_2$, for all $\xi \in L^\infty(-G_t, \mathcal{F}_t)$. Let us consider the martingale $Z_t = E(\eta|\mathcal{F}_t)$, $t \geq s$, with strictly positive components. Since $EZ_t\xi = E\xi\eta \geq 0$, $t \geq s$, for every $\xi \in L^\infty(G_t, \mathcal{F}_t)$, it follows that $Z_t \in L^1(G^*_t, \mathcal{F}_t)$ and, therefore, $Z \in \mathcal{M}^T_s(G^*\backslash\{0\})$. $\square$

Now we formulate the main result of this note where the equivalence of two central terms in the chain is a corollary of the above proposition.

Theorem 2.2 *The following relations hold:*

$$\mathbf{MCPS} \Rightarrow \{\mathbf{B},\ \mathcal{M}^T_0(G^*\backslash\{0\}) \neq \emptyset\} \Leftrightarrow \{\mathbf{B},\ \mathbf{NFL}\} \Rightarrow \mathbf{B} \Leftrightarrow \mathbf{NFL2}.$$

If, moreover, the sets $\mathcal{Y}^T_{s,b}(T)$ are Fatou-closed for any $s \in [0,T[$, then the first two conditions are equivalent.

In the above formulation the Fatou-closedness means that the set $\mathcal{Y}^T_{s,b}(T)$ contains the limit of any a.s. convergent sequence of its elements provided that the latter is bounded from below in the sense of partial ordering induced by G_T. In the presence of the condition **B** this property is fulfilled for the continuous-time financial models considered in [24], [2] (Th.14), [21] (Lemmas 3.6.6, 3.6.16). Establishing the Fatou-closedness of $\mathcal{Y}^T_{s,b}(T)$ is the most difficult part of proofs of hedging theorems relying upon continuity properties of the cone-valued processes G. Discussion of the latter is beyond the scope of the present study.

The role of the condition **B** in the theory of financial markets merits to be discussed in detail. The concept of the Fatou convergence (and related definitions) was widely used already in the analysis of frictionless markets, see [6] and references therein. It was related with the definition of an admissible strategy as that for which the (scalar) value process is bounded from below by a constant. The first attempt to extend it in a straightforward way to the simplest market model with constant proportional transaction costs was done in the paper [17] where the admissible portfolio processes, expressed in terms of the *numéraire*, were bounded from below in the sense of partial ordering induced by K by a constant vector. Such a straightforward definition happens to be not satisfactory: the hedging theorem in [17] covers only the case of bounded price processes because its proof requires the buy-and-hold strategies. In the next paper [18] it was suggested to consider as admissible the strategies whose portfolio processes in physical values are bounded from below by a constant in the sense of partial orderings induced by (random) cones $\widehat{K}_t$. This concept of *numéraire*-free admissibility is commonly accepted now though other forms are also discussed in the literature, see recent studies [2], [1], [8], [11], [3], [4]. Retrospectively, it was observed that a similar concept was introduced in the theory of frictionless markets by C. Sin in his thesis [32]. He discovered that it leads to the existence of equivalent martingale measure (hence, to strictly consistent price systems in the terminology adopted here) and not to just a local martingale related with the traditional definition of admissibility, see Ch. VII in [31] for a detailed discussion.

In the concluding section of this note we discuss in detail the discrete-time model which can be formally imbedded into considered general framework but possesses a number of specific features. We consider various no-arbitrage properties having transparent financial interpretations. In particular, we prove that for the discrete-time model all five properties in the formulation of Theorem 2.2 are equivalent without additional assumptions. Our results in this section can be considered as complementary to those of M. Rásonyi, [28]. His theorem establishes the equivalence of the condition **NA2** (which is weaker than **NFL2**) and **PCE** (which is stronger than **MCPS** and could be called **MSCPS** with the extra **S** for “Strictly”). Thus, all these conditions are equivalent.

3 Proof of the Main Result

MCPS $\Rightarrow \{\mathbf{B},\ \mathcal{M}^T_0(G^*\setminus\{0\}) \neq \emptyset\}$.

Let ξ be $\mathcal{F}_s$-measurable random variable such that $Z_s\xi \geq 0$ for any martingale $Z \in \mathcal{M}^T_s(G^*)$. Since **MCPS** holds, we have that $\eta\xi \geq 0$ for all $\eta \in L^1(\operatorname{int} G^*_s, \mathcal{F}_s)$, hence for all $\eta \in L^0(G^*_s, \mathcal{F}_s)$. This implies that $\xi \in G_s$ (a.s.) and the condition **B** holds. Since $\operatorname{int} G^*_s$ of the $\mathcal{F}_s$-measurable mapping G^*_s is also $\mathcal{F}_s$-measurable, it admits an $\mathcal{F}_s$-measurable selector which serves, by **MCPS**, as starting value of a martingale from $\mathcal{M}^T_s(G^*\setminus\{0\})$. □

B $\Rightarrow$ **NFL2**.

Let $\xi \in L^\infty(\mathrm{R}^d, \mathcal{F}_s)$ and let $V \in \overline{\mathcal{A}^T_{s,b}(T)}^w$ be such that $\xi + V \in L^\infty(G_T, \mathcal{F}_T)$. For any $Z \in \mathcal{M}^T_s(G^*)$ and $\Gamma \in \mathcal{F}_s$ the process $ZI_\Gamma \in \mathcal{M}^T_s(G^*)$ and we have:

$$0 \le EZ_T(\xi + V)I_\Gamma = EZ_s\xi I_\Gamma + EZ_T I_\Gamma V \le EZ_s\xi I_\Gamma$$

because $EZ_T I_\Gamma V \le 0$ due to the hypothesis $\mathbf{S}_1$. Thus, $EZ_s\xi I_\Gamma \ge 0$ for every $\Gamma \in \mathcal{F}_s$, i.e. $Z_s\xi \ge 0$. By virtue of **B** the random variable $\xi \in L^\infty(G_s, \mathcal{F}_s)$ and we conclude. $\Box$

NFL2 $\Rightarrow$ **B**.

Let $\zeta \in L^\infty(\mathrm{R}^d_+)$. We define the convex set

$$\Gamma_\zeta := \left\{ x \in \mathrm{R}^d : \ \zeta - x \in \overline{\mathcal{A}^T_{s,b}(T)}^w \right\}$$

and the closed convex set

$$D_\zeta := \{x \in \mathrm{R}^d : \ EZ_s x \ge EZ_T\zeta \ \ \forall Z \in \mathcal{M}^T_s(G^*)\}.$$

Lemma 3.1 $\Gamma_\zeta = D_\zeta$.

Proof. The argument is standard but we sketch it for the sake of completeness. The inclusion $\Gamma_\zeta \subseteq D_\zeta$ is obvious. For the converse, let us consider a point $x \in D_\zeta$ such that $\zeta - x \notin \overline{\mathcal{A}^T_{s,b}(T)}^w$. Using the Hahn–Banach theorem, we separate $\zeta - x$ and $\overline{\mathcal{A}^T_{s,b}(T)}^w$ by a hyperspace given by some $\eta \in L^1(\mathrm{R}^d)$ and define the martingale $Z^\zeta_t = E(\eta|\mathcal{F}_t)$ for which $EZ^\zeta_T\xi \le 0$ for all ξ from $\mathcal{A}^T_{s,b}(T)$. By our hypothesis $\mathbf{S}_2$ the latter set is rich enough to ensure that $Z^\zeta \in \mathcal{M}^T_s(G^*)$. The point $\zeta - x$ lays in the interior of the complementary subspace, i.e. the inequality $EZ^\zeta_T(\zeta - x) > 0$ holds. This contradicts to the definition of D_ζ. Thus, $\Gamma_\zeta = D_\zeta$. $\Box$

Suppose that $\xi \in L^\infty(\mathrm{R}^d, \mathcal{F}_s)$ is such that $Z_s\xi \ge 0$ for any $Z \in \mathcal{M}^T_s(G^*)$. It follows that $0 \in D_{-\xi}$ and, by the above lemma, $-\xi \in \overline{\mathcal{A}^T_{s,b}(T)}^w$. The last property means that

$$0 = \xi - \xi \in (\xi + \overline{\mathcal{A}^T_{s,b}(T)}^w) \cap L^\infty(\mathrm{R}^d_+).$$

In virtue of the condition **NFL2**, this may happen only if $\xi \in G_s$ a.s. So, the condition **B** is fulfilled. $\Box$

To finish the proof it remains to establish the following implication:

$\{$**B**, **NFL**$\}$ $\Rightarrow$ **MCPS**

Now we are working assuming that the sets $\mathcal{Y}^T_{s,b}(T)$ are Fatou-closed. The idea of the proof is to replace the cone G_s by a larger cone $\tilde{G}_s$, dual to $\mathrm{R}_+\eta$, check that for the extended model the set of hedgeable contingent claims $\widetilde{A}^T_{s,b}(T)$ is weakly* closed (the Fatou-closedness intervenes here, Lemmas 3.4 and 3.5) and the **NFL**-property is fulfilled (Lemma 3.6). To carry out this plan, we need of sequences of consistent price systems whose initial values converge to η in L^1 while the terminal values converge to an element of $L^1(G^*_T \setminus \{0\})$. The existence of such sequences is established in Lemmas 3.2 and 3.3.

Lemma 3.2 *Assume that* **B** *and* **NFL** *hold. Then for any* $\eta \in L^1(\mathrm{int}\, G^*_s, \mathcal{F}_s)$ *there exists a sequence* $Z^n \in \mathcal{M}^T_s(G^* \backslash \{0\})$ *such that* $Z^n_s \to \eta$ *in* L^1.

Proof. Suppose that $\eta \in L^1(\operatorname{int} G_s^*, \mathcal{F}_s)$ does not belong to the set $\bar{M}_s$, the closure in L^1 of the convex cone $M_s := \left\{ Z_s : Z \in \mathcal{M}_s^T(G^*\backslash\{0\}) \right\}$. By the Hahn–Banach theorem, there exists $\xi \in L^\infty(G_s, \mathcal{F}_s)$ such that

$$\sup_{Z \in \mathcal{M}_s^T(G^*\backslash\{0\})} EZ_s\xi < E\eta\xi.$$

Since the set $\mathcal{M}_s^T(G^*\backslash\{0\})$ is a cone, the left-hand side of the above inequality is zero.

We take a martingale $\tilde{Z} \in \mathcal{M}_s^T(G^*\backslash\{0\})$ existing by virtue of Proposition 2.1. For any $Z \in \mathcal{M}_s^T(G^*)$, $\Gamma \in \mathcal{F}_s$ and $k > 0$ the process $ZI_\Gamma + k^{-1}\tilde{Z}$ belongs to $\mathcal{M}_s^T(G^*\backslash\{0\})$ and $E(Z_s I_\Gamma + k^{-1}\tilde{Z}_s)\xi \le 0$. We deduce from here that $Z_s\xi \le 0$ for every $Z \in \mathcal{M}_s^T(G^*)$. The condition **B** implies that $\xi \in -G_s$ a.s. leading to a contradiction since $E\eta\xi > 0$. Hence, $\eta \in \bar{M}_s$, i.e. there exists a sequence $Z^n \in \mathcal{M}_s^T(G^*\backslash\{0\})$ such that $Z_s^n \to \eta$ in L^1. $\square$

Since the components of Z^n in the above are positive, the expectations of components of the vector Z_T^n coincide with the expectation of components of Z_s^n. It follows that the sequence Z_T^n is bounded in L^1 and the Komlós theorem can be applied. Replacing the original sequence by a sequence of Césaro means from the latter theorem we obtain a sequence in $\mathcal{M}_s^T(G^*\backslash\{0\})$ which terminal values converge a.s. to a random variable $Z_T \in L^1(G_T^*)$. The following lemma shows that we could do better.

Lemma 3.3 *Assume that* **B** *and* **NFL** *hold. Then for any* $\eta \in L^1(\operatorname{int} G_s^*, \mathcal{F}_s)$ *there exists a sequence* $Z^n \in \mathcal{M}_s^T(G^*\backslash\{0\})$ *such that* $Z_s^n \to \eta$ *in* L^1 *and* $Z_T^n \to Z_T$ *a.s. where* $Z_T \in L^1(G_T^* \setminus \{0\})$.

Proof. Let $\eta \in L^1(\operatorname{int} G_s^*, \mathcal{F}_s)$. We may assume without loss of generality that we have $E|\eta| \le 1/2$. We start with an arbitrary $\widetilde{Z}^1 \in \mathcal{M}_s^T(G^*\backslash\{0\}) \neq \emptyset$. Using the measurable selection, we find $\alpha_s^1 \in L^0(]0,1[, \mathcal{F}_s)$ such that the difference $\eta - \alpha_s^1\widetilde{Z}_s^1 \in \text{int } G_s^*$ a.s. The process $\alpha_s^1\widetilde{Z}^1 \in \mathcal{M}_s^T(G^*\backslash\{0\})$; we may assume that $E|\eta - \alpha_s^1\widetilde{Z}_s^1| \le 1$.

Now, we proceed by induction. Put $\bar{Z}^1 := \alpha_s^1\widetilde{Z}^1$. Since $\eta - \overline{Z}_s^1 \in \text{int } G_s^*$ a.s., we apply Lemma 3.2 and find $\widetilde{Z}^2 \in \mathcal{M}_s^T(G^*\backslash\{0\})$ such that

$$E|\eta - \overline{Z}_s^1 - \widetilde{Z}_s^2| \le 1/2.$$

Using measurable selection, we find $\alpha_s^2 \in L^0(]0,1[, \mathcal{F}_s)$ such that

$$\eta - \bar{Z}_s^1 - \alpha_s^2\widetilde{Z}_t^2 \in \text{int } G_s^*$$

where

$$\alpha_s^2\widetilde{Z}^2 \in \mathcal{M}_s^T(G^*\backslash\{0\}).$$

We put $\bar{Z}^2 := \bar{Z}^1 + \alpha_s^2\widetilde{Z}^2$. Let us suppose that we have already defined the processes $\bar{Z}^{n-1} = \sum_{i=1}^{n-1}\alpha_s^i\widetilde{Z}^i$, $\widetilde{Z}^{n-1}$ where $\widetilde{Z}^i \in \mathcal{M}_s^T(G^*\backslash\{0\})$ and $\alpha_s^i \in L^0(]0,1[, \mathcal{F}_s)$ such that

$$\eta - \overline{Z}_s^{n-1} \in \text{int } G_s^* \text{ a.s.}, \quad E|\eta - \bar{Z}_s^{n-2} - \widetilde{Z}_s^{n-1}| \le 2^{-(n-1)}.$$

By Lemma 3.2 there is $\widetilde{Z}^n \in \mathcal{M}_s^T(G^*\backslash\{0\})$ such that

$$E|\eta - \bar{Z}_s^{n-1} - \widetilde{Z}_s^n| \le 2^{-n}$$

and, by virtue of measurable selection arguments, there is $\alpha_s^n \in L^0(]0,1[, \mathcal{F}_s)$ such that

$$\eta - \bar{Z}_s^{n-1} - \alpha_s^n\widetilde{Z}_s^n \in \text{int } G_s^*, \quad \alpha_s^n\widetilde{Z}^n \in \mathcal{M}_s^T(G^*\backslash\{0\}).$$

We put $\bar{Z}^n := \bar{Z}^{n-1} + \alpha_s^n \widetilde{Z}^n$ and the induction step is done.

Due to our standing assumption $G_T^*\backslash\{0\}$ lays in the interior of R_+^d. It follows that $\bar{Z}_T^{n-1}$ is a componentwise increasing sequence bounded in L^1 and, therefore, this sequence converges a.s. and in L^1 to some random variable $\bar{Z}_T \in L^1(G_T^*\backslash\{0\}, \mathcal{F}_T)$. Automatically, $\bar{Z}_t^{n-1}$ converges (increasingly) to $E(\bar{Z}_T|\mathcal{F}_t)$ a.s. and in L^1 for each $t \geq s$. By construction, $\bar{Z}_s^{n-1} + \widetilde{Z}_s^n$ converges to η in L^1. The sequence of terminal values of martingales $Z^n := \bar{Z}^{n-1} + \widetilde{Z}^n$ evolving in $\mathcal{M}_s^T(G^*\backslash\{0\})$ is bounded in L^1 and the Komlós theorem can be applied. That is, passing to a sequence of Césaro, we may assume without loss of generality that $\widetilde{Z}_T^n \to \widetilde{Z}_T$ where $\widetilde{Z}_T \in L^1(G_T^*, \mathcal{F}_T)$. Hence, the properties claimed in the lemma holds for the sequence of Z^n. □

We need some further auxiliary results.

For $\eta \in L^1(\text{int } G_s^*, \mathcal{F}_s)$ we define the random half-space $\widetilde{G}_s$ by putting $\widetilde{G}_s^* = \mathrm{R}_+\eta$. Note that $(-\widetilde{G}_s) \cap G_s = \{0\}$.

Let $L_b^0(\mathrm{R}^d) := \{\xi \in \mathrm{R}^d : \exists \kappa_\xi \text{ such that } \xi + \kappa_\xi \mathbf{1} \in G_T\}$ and let

$$\widetilde{A}_{s,b}^T(T) := \Big(L^0(-\widetilde{G}_s, \mathcal{F}_s) + \mathcal{Y}_s^T(T)\Big) \cap L_b^0(\mathrm{R}^d).$$

Lemma 3.4 *Assume that* **B** *holds. If the set* $\mathcal{Y}_{s,b}^T(T)$ *is Fatou-closed, then* $\widetilde{A}_{s,b}^T(T)$ *is also Fatou-closed.*

Proof. We consider a sequence $Y_T^n := \xi_s^n + \gamma_T^n$ where

$$\xi_s^n \in L^0(-\widetilde{G}_s, \mathcal{F}_s), \quad \gamma_T^n \in \mathcal{Y}_s^T(T)$$

are such that $Y_T^n + k\mathbf{1} \in G_T$ a.s. for some constant k and $Y_T^n \to Y_T$ a.s. Define the set $\Gamma_s = \{\sup_n |\xi_s^n| = \infty\}$. According to the lemma on subsequences (see, e.g. [23]) there exists a strictly increasing sequence of integer-valued $\mathcal{F}_s$-measurable random variables θ_n such that $|\xi_s^{\theta_n}| \to \infty$ on Γ_s.

Put

$$\widetilde{\xi}_s^n := \frac{\xi_s^{\theta_n}}{|\xi_s^{\theta_n}| \vee 1} I_{\Gamma_s}, \qquad \widetilde{\gamma}_T^n := \frac{\gamma_T^{\theta_n}}{|\xi_s^{\theta_n}| \vee 1} I_{\Gamma_s}, \qquad \widetilde{Y}_T^n := \frac{Y_T^{\theta_n}}{|\xi_s^{\theta_n}| \vee 1} I_{\Gamma_s}.$$

The sequence $\widetilde{Y}_T^n$ is bounded from below (in the sense of partial ordering induced by G_T). Since $\widetilde{\xi}_s^n$ takes values in the unit ball, this implies that the sequence $\widetilde{\gamma}_T^n$ is bounded from below and its elements belong to $\mathcal{Y}_{s,b}^T(T)$ (note that the random variable $\gamma_T^{\theta_n} = \sum_k \gamma_T^k I_{\{\theta_n = k\}} \in \mathcal{Y}_s^T(T)$ due to our assumption). Applying again the lemma on subsequences (this time to $(\widetilde{\xi}_s^n)$) and taking into account that $\widetilde{Y}_T^n \to 0$ we may assume without loss of generality that

$$\widetilde{\xi}_s^n \to \widetilde{\xi}_s \in L^\infty(-\widetilde{G}_s, \mathcal{F}_s), \quad \widetilde{\gamma}_T^n \to \widetilde{\gamma}_T = -\widetilde{\xi}_s.$$

Due to the Fatou-closedness of $\mathcal{Y}_{s,b}^T(T)$, we have that $\widetilde{\gamma}_T \in \mathcal{Y}_{s,b}^T(T)$.

Let $Z \in \mathcal{M}_s^T(G^*)$ and let $\Gamma \in \mathcal{F}_s$. It follows from the hypothesis $\mathbf{S}_1$ that

$$0 = EZ_T I_\Gamma(\widetilde{\xi}_s + \widetilde{\gamma}_T) \leq EZ_T I_\Gamma \widetilde{\xi}_s = EZ_s I_\Gamma \widetilde{\xi}_s.$$

Therefore, $Z_s I_\Gamma \widetilde{\xi}_s \geq 0$ and, by virtue of the condition **B**, $\widetilde{\xi}_s \in L^\infty(G_s, \mathcal{F}_s)$. Hence, $\widetilde{\xi}_s \in L^\infty((-\widetilde{G}_s) \cap G_s, \mathcal{F}_s)$, i.e. $\widetilde{\xi}_s = 0$ a.s. But $|\widetilde{\xi}_s| = 1$ on Γ_s. Thus, $P(\Gamma_s) = 0$.

We may assume, passing to a subsequence, that $\xi_s^n \to \xi_s$ and $\gamma_T^n \to \gamma_T$ a.s. In the same spirit as above we define $\bar{Y}_T^n = \bar{\xi}_s^n + \bar{\gamma}_T^n$ where

$$\overline{\xi}_s^n := \frac{\xi_s^n}{|\xi_s^n|+1} \in L^\infty(-\widetilde{G}_s, \mathcal{F}_s), \qquad \overline{\gamma}_T^n := \frac{\gamma_T^n}{|\xi_s^n|+1} \in \mathcal{Y}_{s,b}^T(T).$$

By virtue of the Fatou-closedness of $\mathcal{Y}_b^s(T)$ we obtain that

$$\overline{\gamma}_T^n \to \overline{\gamma}_T = \frac{\gamma_T}{|\xi_s|+1} \in \mathcal{Y}_{s,b}^T(T).$$

Thus, $Y_T = \xi_s + (1+|\xi_s|)\,\overline{\gamma}_T$ is an element of $\widetilde{A}_{s,b}^T(T)$. □

Lemma 3.5 *Assume that* **B** *holds. If the set $\mathcal{Y}_{s,b}^T(T)$ is Fatou-closed, then the set $\widetilde{A}_{s,b}^T(T) \cap L^\infty$ is Fatou-dense in $\widetilde{A}_{s,b}^T(T)$.*

Proof. Let $Y_T = \xi_s + \gamma_T \in \widetilde{A}_{s,b}^T(T)$ and $Y_T + \kappa\mathbf{1} \geq_{G_T} 0$. Put

$$Y_T^n := Y_T I_{\{|Y_T| \leq n\}} - \kappa \mathbf{1} I_{\{|Y_T|>n\}}.$$

From the identity $Y_T - Y_T^n = (Y_T + \kappa\mathbf{1}) I_{\{|Y_T|>n\}} \in G_T$ we obtain that $Y_T^n \in \widetilde{A}_{s,b}^T(T)$. Clearly, Y_T^n form a sequence Fatou-convergent to Y_T. □

By virtue of the above lemmas we obtain the following dual characterization of the Fatou-closed set $\widetilde{A}_{s,b}^T(T)$ (see, e.g. Appendix in [24] and A.5 in [21]):

$$\widetilde{A}_{s,b}^T(T) = \left\{ \xi \in L_b^0(\mathrm{R}^d) : \; E\xi\eta \leq \sup_{X \in \widetilde{A}_{s,b}^T(T)} EX\eta, \quad \forall \eta \in L^1(G_T^*) \right\}. \tag{3.2}$$

In particular, it is closed in $\sigma\{L^\infty, L^1\}$.

Lemma 3.6 *Assume that* **B** *holds. If the set $\mathcal{Y}_{s,b}^T(T)$ is Fatou-closed, then*

$$\widetilde{A}_{s,b}^T(T) \cap L^\infty(\mathrm{R}_+^d) = \{0\}.$$

Proof. Let us consider

$$Y_T := \xi_s + \gamma_T \in \widetilde{A}_{s,b}^T(T) \cap L^\infty(\mathrm{R}_+^d).$$

Using the notation introduced above we rewrite Y_T in the form

$$Y_T := (1+|\xi_s|)\left(\overline{\xi}_s + \overline{\gamma}_T\right)$$

where $\overline{\gamma}_T \in \mathcal{Y}_{s,b}^T(T)$ and $\overline{\xi} \in L^\infty(-\widetilde{G}_s, \mathcal{F}_s)$. For the sequence Z^n from Lemma 3.3 we have by the Fatou lemma that

$$0 \leq E\left(\overline{\xi}_s + \overline{\gamma}_T\right) Z_T \leq \liminf_n (E\overline{\xi}_s Z_T^n + E\overline{\gamma}_T Z_T^n)$$

where

$$E\overline{\xi}_s Z_T^n = E\overline{\xi}_s Z_s^n \to E\overline{\xi}_s \eta \leq 0$$

and $E\overline{\gamma}_T Z_T^n \leq 0$ under the condition $\mathbf{S}_1$. This implies that $Y_T = 0$ a.s. □

With the above lemma we get the implication **B** $\Rightarrow$ **MCPS** by a standard argument. Indeed, the Kreps–Yan separation theorem ensures the existence of random variable $Z_T \in L^1(\mathrm{int}\,\mathrm{R}^d_+, \mathcal{F}_T)$ such that $EZ_T\xi \le 0$ for all $\xi \in \widetilde{A}^T_{s,b}(T)$. Define the martingale $Z_t := E(Z_T|\mathcal{F}_t)$, $s \ge t$, whose components are strictly positive. Since $\widetilde{A}^T_{s,b}(T)$ contains $L^\infty(-G_t, \mathcal{F}_t)$ for $t \ge s$ and $L^\infty(-\tilde{G}_s, \mathcal{F}_s)$, we infer that $Z_t \in L^1(G^*_t, \mathcal{F}_t)$ for $t \ge s$ and $Z_s \in L^1(\mathrm{R}_+\eta, \mathcal{F}_s)$. Since Z_T is defined up to a scalar strictly positive multiplier, we choose it to have the equality $Z_s = \eta$ and get a process claimed in the condition **MCPS**. □

Theorem 2.2 is proven. □

4 Discrete-Time Model

Let us consider a general discrete-time model with $\mathcal{Y}^T_s(T) = -\sum_{t=s}^T L^0(G_t, \mathcal{F}_t)$ where $G_t \cap (-G_t) = \{0\}$. All required hypotheses are fulfilled. The only non-trivial one, $\mathbf{S}_1$ follows from the following statement (Lemma 4 in [19]):

Lemma 4.1 *Let Z be an R^d-valued martingale and let $\Sigma_T := Z_T \sum_{s=0}^T \xi_s$ where $\xi_s \in L^0(\mathrm{R}^d, \mathcal{F}_s)$ are such that $Z_s\xi_s \le 0$. If $E\Sigma^-_T < \infty$, then all products $Z_s\xi_s$ are integrable, Σ_T is integrable and $E\Sigma_T \le 0$.*

For this model (which can be imbedded, as a very particular case, into our continuous-time framework) all five conditions of Theorem 2.2 are equivalent without extra hypotheses: the implication $\mathbf{B} \Rightarrow \mathbf{MCPS}$ holds without assuming the Fatou-closedness. The goal of this section is to show that our arguments can be appropriately modified using the specific feature of the discrete-time case to avoid this assumption. On the way we establish some other interesting properties of the model.

First, we recall the following property of R^d-valued martingales in discrete time.

Lemma 4.2 *Let $M = (M_t)_{t=0,\dots,T}$ be a martingale and let $\tilde{P} \sim P$. Then there exists an adapted strictly positive bounded process $\alpha = (\alpha_t)$ such that αM is a bounded $\tilde{P}$-martingale.*

Proof. Put $\zeta := 1 + \sup_{t \le T} |M_t|$. By the "easy" part of the Dalang–Morton–Willinger criteria (see, e.g., [23]), the R^{d+1}-valued martingale $(1, M)$ satisfies the **NA**-property (the first component serves as the *numéraire*). The latter, being invariant under equivalent change of measure, holds also with respect to the probability $P' := c\zeta^{-1}P$ where $c = 1/E\zeta^{-1}$. Using again the same theorem, we find a bounded density process $\rho_t > 0$ such that $\rho_t M_t$ is a P'-martingale or, equivalently, the process $M'_t = E(\zeta^{-1}|\mathcal{F}_t)\rho_t M_t$ is a bounded P-martingale. Applying the **NA**-criteria to $(1, M')$ with respect to $\tilde{P}$, we find a bounded $\tilde{P}$-martingale $\tilde{\rho} > 0$ such that $\tilde{\rho}_t M'_t$ is $\tilde{P}$-martingale and we get the result with $\alpha_t = \tilde{\rho}_t E(\zeta^{-1}|\mathcal{F}_t)\rho_t$. □

Now we introduce several conditions with interesting relations between them.

$\mathbf{B}^p$ If $\xi \in L^0(\mathrm{R}^d, \mathcal{F}_s)$ and $Z_s\xi \ge 0$ for any $Z \in \mathcal{M}^T_s(G^*)$ with $Z_T \in L^p$, then $\xi \in G_s$ (a.s.), $s = 0, \dots, T$.

Note that $\mathbf{B}^1$ is just $\mathbf{B}$ and, by the above lemma, $\mathbf{B} \Leftrightarrow \mathbf{B}^\infty$. So, all these conditions with $p \in [1, \infty]$ are equivalent, and moreover, they are invariant with respect to equivalent change of probability measure.

$\mathbf{NAA}^p$ $\overline{\mathcal{A}^T_{0,b}(T)}^{L^p} \cap L^p(\mathrm{R}^d_+, \mathcal{F}_T) = \{0\}$.

Accordingly to the existing terminology it is natural to call $\mathbf{NAA}^0$ by No Asymptotic Arbitrage (of the 1st Kind) — **NAA** and reserve for $\mathbf{NAA}^\infty$ the name No Free

Lunch with Vanishing Risk — **NFLVR**. Apparently, these two conditions are measure-invariant: they remain the same under equivalent change of probability measure.

Less trivial is the following fact.

Lemma 4.3 *The conditions* $\mathbf{NAA}^p$ *for* $p \in [1,\infty[$ *are measure-invariant and any of them is equivalent to* $\mathbf{NAA}^0$ *as well as to the condition* **NFL** *(which, in turn, is equivalent, to the existence of a bounded process* Z *in* $\mathcal{M}_s^T(G^* \setminus \{0\})$.

Proof. We apply the Kreps–Yan theorem in L^p, $p \in [1,\infty[$, and conclude, in the same way as in the proof of Proposition 2.1, that $\mathbf{NAA}^p$ holds if and only if there is a martingale $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$ with $|Z_T| \in L^q = (L^p)^*$. By virtue of Lemma 4.2 the latter condition implies that for any $\tilde{P} \sim P$ there exists a bounded martingale $Z \in \mathcal{M}_0^T(G^* \setminus \{0\}, \tilde{P})$.

Obviously, $\mathbf{NAA}^0$ implies the condition $\mathbf{NAA}^1(\tilde{P})$ whatever is $\tilde{P} \sim P$. To prove the converse implication, we suppose that $\mathbf{NAA}^1$ holds and consider the sequence $\xi_n \in \mathcal{A}_{0,b}^T(T)$ converging in probability to $\xi \in L^0(\mathrm{R}_+^d, \mathcal{F}_T)$. Taking a subsequence, we assume without loss of generality that $\xi_n \to \xi$ a.s. This sequence converges to ξ in $L^1(\tilde{P})$ where $\tilde{P} = c\exp\{-\sup_n |\xi_n|\}P$. The measure invariance of $\mathbf{NAA}^1$ ensures that $\xi = 0$ and, therefore, $\mathbf{NAA}^0$ holds.

At last, comparing the result of Proposition 2.1 with the equivalent characterization of $\mathbf{NAA}^1$ we obtain the remaining statement of the lemma. □

Clearly, for the considered discrete-time model the L^0-closures of $\mathcal{A}_{s,b}^T(T)$ and $\mathcal{Y}_s^T(T)$ coincide. Let us check that **NFL2** implies that $\mathcal{Y}_s^T(T)$ is closed in L^0. Indeed, for $s = T$ the claim is obvious. Suppose that it holds for $s \geq r+1$. Let

$$\zeta^n = \xi_r^n + ... + \xi_T^n, \qquad \xi_t^n \in L^0(-G_t, \mathcal{F}_t), \quad t \geq r, \tag{4.3}$$

and let $\zeta^n \to \zeta$ a.s. Put $\Gamma := \{\liminf_n |\xi_r^n| < \infty\}$. If $P(\Gamma) = 1$, then, applying the lemma on subsequences, one can find a strictly increasing sequence of integer-valued $\mathcal{F}_r$-measurable random variables τ_n such that $\xi_r^{\tau_n}$ tends a.s. to some $\xi_r \in L^0(-G_r, \mathcal{F}_r)$. Using the induction hypothesis we get from here that $\zeta \in \mathcal{Y}_r^T(T)$. So, it remains to check that the complement of Γ is a null set. Suppose that it is not the case. Without loss of generality we may assume that all ξ_t^n equal to zero on Γ. Dividing the equality (4.3) by $1 \vee |\xi_r^n|$ and applying again the lemma on subsequences, now to the random variables $\xi_r^n/(1 \vee |\xi_r^n|)$, we obtain, passing to a limit, that

$$\tilde{\xi}_r + ... + \tilde{\xi}_T = 0$$

for some $\tilde{\xi}_t \in L^0(-G_t, \mathcal{F}_t)$, $t \geq r$ with $|\xi_r| = 1$ on Γ^c a.s. By the **NFL2**-property this may happen only if $\tilde{\xi}_r \in L^0(G_r, \mathcal{F}_r)$. Since $-G_r \cap G_r = \{0\}$, we get that $P(\Gamma^c) = 0$.

The last arguments (with an obvious modification) show that **NFL2** implies that $\mathcal{Y}_0^T(T) \cap L^0(\mathrm{R}_+^d, \mathcal{F}_T) = \{0\}$, i.e. the $\mathbf{NA}^w$-property holds. It follows that for the discrete-time model we have the implication $\mathbf{B} \Longrightarrow \{\mathbf{B},\ \mathbf{NFL}\}$ which was not claimed for the continuous-time model.

Following the same logic, we define a family of $\mathbf{NAA2}^p$-conditions.

$\mathbf{NAA2}^p$ For each $s = 0, 1, ..., T-1$ and $\xi \in L^\infty(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + \overline{\mathcal{A}_{s,b}^T(T)}^{L^p}) \cap L^0(\mathrm{R}_+^d, \mathcal{F}_T) \neq \emptyset$$

only if $\xi \in L^\infty(G_s, \mathcal{F}_s)$.

Note that replacing L^∞ by L^0 leads to an equivalent condition.

Lemma 4.4 *The conditions* $\mathbf{NAA2}^p$ *for* $p \in [1,\infty[$ *are measure-invariant and any of them is equivalent to* $\mathbf{NAA2}^0$ *as well as to the condition* **NFL2** *(which, in turn, is equivalent to the condition* **B***).*

Proof. For $p \in [1,\infty[$ we have the equivalences

$$\{\mathbf{B}^q,\ \mathbf{NAA}^p\} \Leftrightarrow \mathbf{B}^q \Leftrightarrow \mathbf{NAA2}^p.$$

Their proofs use the same arguments as for corresponding equivalences in Theorem 2.2 but the duality (L^∞, L^1) should be replaced by the duality (L^p, L^q). But we already know that $\mathbf{B}^q \Leftrightarrow \mathbf{B} \Leftrightarrow \mathbf{NFL2}$ where the last condition is measure-invariant. The equivalence of $\mathbf{NAA2}^0$ and $\mathbf{NAA2}^1$ can be proven using the change of measure as in the previous lemma. □

With these preliminaries we can get the implication $\mathbf{B} \Rightarrow \mathbf{MCPS}$ using the same strategy of the proof as in the general case but without any additional hypothesis.

Lemma 4.5 *Assume that* **B** *holds. Then for any* $\eta \in L^2(\operatorname{int} G_s^*, \mathcal{F}_s)$ *there is a sequence* $Z^n \in \mathcal{M}_s^T(G^*\setminus\{0\})$ *with* $|Z_T^n| \in L^2$ *such that* $Z_s^n \to \eta$ *in* L^2*.*

The proof of this assertion is the same as of Lemma 3.2 but with the duality (L^2, L^2) replacing the duality (L^1, L^∞). Using Lemma 4.5 and repeating the arguments of Lemma 3.4 but now in L^2-norm instead of L^1-norm we get:

Lemma 4.6 *Assume that* **B** *holds. Then for any* $\eta \in L^\infty(\operatorname{int} G_s^*, \mathcal{F}_s)$ *there is a sequence of martingales* $Z^n \in \mathcal{M}_s^T(G^*\setminus\{0\})$ *with* $|Z_T^n| \in L^2$ *such that* $Z_s^n \to \eta$ *in* L^2 *and* $Z_T^n \to Z_T$ *a.s. where* $Z_T \in L^2(G_T^* \setminus \{0\})$*.*

For $\eta \in L^\infty(\text{int } G_s^*, \mathcal{F}_s)$ we define the random half-space $\widetilde{G}_s$ via its dual $\widetilde{G}_s^* = \mathrm{R}_+\eta$ and put

$$\widetilde{A}_s^T(T) := \Big(L^\infty(-\widetilde{G}_s, \mathcal{F}_s) + \mathcal{Y}_{s,b}^T(T)\Big) \cap L^2(\mathrm{R}^d).$$

Lemma 4.7 *Assume that* **B** *holds. Then*

$$\overline{\widetilde{A}_s^T(T)}^{L^2} \cap L^2(\mathbb{R}_+^d) = \{0\}.$$

Proof. Take an element Y_T from the set in the left-hand side of the above inequality. There exists a sequence

$$Y_T^n = \xi_s^n + \gamma_T^n \in \widetilde{A}_s^T(T)$$

with

$$\xi_s^n \in L^\infty(-\widetilde{G}_s, \mathcal{F}_s), \quad \gamma_T^n \in \mathcal{Y}_{s,b}^T(T) \cap L^2$$

such that converging $Y_T^n \to Y_T$ in L^2 and a.s. We claim that there is a subsequence for which $\sup_n |\xi_s^n| < \infty$ a.s. Indeed, suppose that it is not the case. Applying the lemma on subsequences, we may assume without loss of generality that there exists a non-null set A_s on which $\lim_n |\xi_s^n| \to \infty$. Noting that

$$\widetilde{\xi}_s^n := \frac{\xi_s^n}{|\xi_s^n| + 1} I_{A_s} \in L^\infty(-\widetilde{G}_s, \mathcal{F}_s), \qquad \widetilde{\gamma}_s^n := \frac{\gamma_T^n}{|\xi_s^n| + 1} I_{A_s} \in \mathcal{Y}_{s,b}^T(T) \cap L^2,$$

we consider the sequence $\widetilde{Y}_T^n := \widetilde{\xi}_s^n + \widetilde{\gamma}_s^n$ converging to zero a.s.

Applying again the lemma on subsequences may assume that

$$\widetilde{\xi}_s^n \to \widetilde{\xi}_s \in L^\infty(-\widetilde{G}_s, \mathcal{F}_s), \quad \widetilde{\gamma}_T^n \to \widetilde{\gamma}_T = -\widetilde{\xi}_s \in L^\infty(\mathcal{F}_s) \quad \text{a.s.}.$$

Thus, $\widetilde{\xi}_s + \widetilde{\gamma}^n_T \to 0$ a.s. By virtue of condition $\mathbf{NAA2}^0$ (equivalent to **B**) $\widetilde{\xi}_s \in G_s$ a.s. also. That is, $\widetilde{\xi}_s \in (-\widetilde{G}_s) \cap G_s = 0$. This is a contradiction, because $|\widetilde{\xi}_s| = 1$ on A_s.

Put $A^M_s := \{\sup_n |\xi^n_s| \le M\}$ and fix $\varepsilon > 0$. Let us consider the sequence of martingales Z^n given by Lemma 4.6. By the Fatou lemma, there exists $n_0 = n_0(\varepsilon)$ such that for all $n \ge n_0$ we have the inequality

$$EY_T Z_T I_{A^M_s} \le \varepsilon + EY_T Z^n_T I_{A^M_s}.$$

Since $Z^n_T \in L^2$, one can find m_n such that

$$EY_T Z^n_T I_{A^M_s} \le \varepsilon + EY^{m_n}_T Z^n_T I_{A^M_s}.$$

The hypothesis **S1** applied to $(Y^{m_n}_T - \xi^{m_n}_s) I_{A^M_s}$ implies that

$$EY^{m_n}_T Z^n_T I_{A^M_s} \le E\xi^{m_n}_s Z^n_s I_{A^M_s} \le E\xi^{m_n}_s (Z^n_s - \eta) I_{A^M_s}.$$

Since $Z^n_s \to \eta$ in L^2 and $\xi^{m_n}_s$ is bounded on A^M_s we obtain, due the arbitrariness of ε, that $EY_T Z_T I_{A^M_s} = 0$. This leads to the equality $Y_T Z_T = 0$. Because the components of Z_T are strictly positive, this is possible only if $Y_T = 0$. □

Using the same arguments as at the concluding step of the proof of Theorem 2.2 but now based on the Kreps–Yan theorem in L^2, we deduce from the last lemma that the condition **MCPS** is fulfilled.

Thus, for the discrete-time model with efficient friction we have that

$$\mathbf{MCPS} \Leftrightarrow \{\mathbf{B},\ \mathcal{M}^T_0(G^*\backslash\{0\}) \ne \emptyset\} \Leftrightarrow \{\mathbf{B},\ \mathbf{NFL}\} \Leftrightarrow \mathbf{B} \Leftrightarrow \mathbf{NFL2}$$

Formally, all properties above are different from those introduced in [28] where the main result is the equivalence **PCE** ⇔ **NGV**. The formulation of the latter property — No Sure Gain in Liquidation Value — is the following (with $A^T_s := \sum_{t=s}^T L^0(-G_t, \mathcal{F}_t)$):

NGV For each $s \in [0, T[$ and $\xi \in L^0(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + A^T_s) \cap L^0(\mathrm{R}^d_+, \mathcal{F}_T) \ne \emptyset$$

only if $\xi \in L^0(G_s, \mathcal{F}_s)$.

However, this equivalence follows from two simple observations.

First, **NFL2** ⇔ **NGV**. Indeed, due to Lemma 4.4 and the coincidence of L^0-closures of A^T_s and $\mathcal{A}^T_s(T)$, **NFL2** is equivalent to the property:

NGV′ For each $s \in [0, T[$ and $\xi \in L^0(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + \overline{A^T_s}^{L^0}) \cap L^0(\mathrm{R}^d_+, \mathcal{F}_T) \ne \emptyset$$

only if $\xi \in L^0(G_s, \mathcal{F}_s)$.

The latter is obviously stronger than **NGV**. On the other hand, successive application of **NGV** in combination with the efficient friction condition implies that the identity $\sum_{t=s}^T \xi_t = 0$ with $\xi_t \in L^0(-G_t, \mathcal{F}_t)$ may hold only if all $\xi_t = 0$. Therefore, A^T_s is closed in probability, [20] (Lemma 2), [21] (Lemma 3.2.8).

Second, **PCE** ⇔ **MCPS**. The implication ⇒ is trivial. The inverse implication can be proven by backward induction. Indeed, for $s = T$ there is nothing to prove. Suppose that for $s = t + 1 \le T$ the claim holds. In particular, there is $\tilde{Z} \in \mathcal{M}^T_{t+1}(\operatorname{int} G)$ with $|\tilde{Z}_{t+1}| = 1$. Put $\tilde{Z}_t := E(\tilde{Z}_{t+1}|\mathcal{F}_t)$. Let $\eta \in L^1(\mathcal{F}_t, G_t)$ with $|\eta| = 1$. Take

α be the $\mathcal{F}_t$-measurable random variable equal to the half of the distance of η_t to ∂G_t. Then $\eta - \alpha \tilde{Z}_t \in L^1(\operatorname{int} G_t, \mathcal{F}_t)$. By **MCPS** there exists $Z \in \mathcal{M}_t^T(G \setminus \{0\})$ with $Z_t \in \mathcal{M}_t^T(G \setminus \{0\})$ and $Z_t = \eta - \alpha \tilde{Z}_t$. Since $Z + \alpha \tilde{Z} \in \mathcal{M}_t^T(\operatorname{int} G)$ and $Z_t + \alpha \tilde{Z}_t = \eta$, we conclude.

Thus, our arguments in the discrete-time case lead to a new proof of the Rásonyi theorem (except the assertion that the "global" **NGV** is equivalent to one-step **NGV** conditions for each t).

Remark. As was indicated by the referee, the conclusion on the equivalence of conditions listed above can be obtained by combining the Rásonyi theorem with the chain of implications

$$\mathbf{MCPS} \Rightarrow \{\mathbf{B},\ \mathcal{M}_0^T(G^*\backslash\{0\}) \neq \emptyset\} \Rightarrow \{\mathbf{B},\ \mathbf{NFL}\} \Rightarrow \mathbf{B} \Rightarrow \mathbf{NFL2}$$

which was proven without using the Fatou-closedness.

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