In Discrete Time a Local Martingale is a Martingale under an Equivalent Probability Measure

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1 Result and Discussion

We consider a discrete-time infinite horizon model with an adapted d-dimensional process $S = (S_t)$ given on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t=0,1,\dots}, P)$. The notations used: $\mathcal{M}(P)$, $\mathcal{M}_{loc}(P)$ and $\mathcal{P}$ are the sets of d-dimensional martingales, local martingales and predictable (i.e. $(\mathcal{F}_{t-1})$-adapted) processes; $H \cdot S_t = \sum_{j \le t} H_j \Delta S_j$.

Theorem 1. *Let $S \in \mathcal{M}_{loc}(P)$. Then there is $\tilde{P} \sim P$ such that $S \in \mathcal{M}(\tilde{P})$.*

To our knowledge, this result was never formulated explicitly. On the other hand, it is well-known that if the stopped process $S^T = (S_{t \wedge T})$, $T \in \mathbf{N}$, belongs to $\mathcal{M}_{loc}(P)$ then there exists $\tilde{P}_T \sim P$ (and even with bounded density $d\tilde{P}_T/dP$) such that $S^T \in \mathcal{M}(\tilde{P}_T)$. This assertion is contained in the classical DMW criteria of absence of arbitrage, see the original paper [1] by Dalang–Morton–Willinger and more recent presentations in [3] and [4] with further references wherein. So, the news is: if $S \in \mathcal{M}_{loc}(P)$ then the intersection of the sets of true martingale measures for the processes S^T is non-empty.

Theorem 1 can be extracted from the old paper [6] by Schachermayer which merits a new reading. The proof given here uses the same approach of geometric functional analysis as in [6]. It is based on separation arguments in an ingeniously chosen countably-normed space where the separating functional happens to be the density of the needed martingale measure. The most involved part of the proof is to check that the conditions of the Krein–Šmulian theorem are verified. Since we assume that S is a local martingale, this can be done much faster than in the original paper (dealing with no arbitrage properties of S).

Our result sounds as a purely probabilistic one. It would be desirable to find a simpler proof which does not rely upon rather delicate theorems from functional analysis.

2 Prerequisites from Functional Analysis and Martingales

Let $(w_t)_{t\geq 0}$ be an increasing sequence of random variables with $w_0 = 1$.

Let L^1_w be the linear space of (classes of) random variables ξ with finite norms $||\xi||_t := Ew_t|\xi|$ defining the structure of locally convex metrizable topological vector space. A local base at zero of the topology is the family of sets $U_{t,\lambda} := \{\xi : \ ||\xi||_t < \lambda\}$, $\lambda > 0$, $t \geq 0$. The completion Φ_t of the subspace formed by the elements of L^1_w with respect to the norm $||.||_t$ is just the Lebesgue space $L^1(\mu_t)$ where $\mu_t := w_t P$. Usually, the dual Φ^*_t is identified with $L^\infty(\mu_t)$ but it is more convenient to identify the elements of Φ^*_t with the random variables η such that $\eta/w_t \in L^\infty(P)$. In such a case the result of the action of η on ξ is $E\xi\eta$. The dual $(L^1_w)^*$, denoted by L^∞_w, is the union of all Φ^*_t, i.e. the set of random variables η for which one can find an integer $t \geq 0$ and a constant c such that $|\eta| \leq cw_t$. The natural bilinear form $\langle \xi, \eta\rangle = E\xi\eta$ defines on L^∞_w the topology $\sigma(L^\infty_w, L^1_w)$ which separates the points of L^∞_w. We denote $(L^\infty_w)_+$ the set of positive elements of L^∞_w.

Note that $B_t := \{\eta : \ |\eta| \leq w_t\}$ is the polar of $U_{t,1}$, i.e.

$$B_t = \{\eta : |E\xi\eta| \leq 1 \ \forall\, \xi \in U_{t,1}\}.$$

The first fact we need is a version of the Kreps–Yan theorem which proof is literally the same as that given, e.g., in [4] for L^p-spaces.

Let C be a convex cone in L^∞_w closed in the topology $\sigma(L^\infty_w, L^1_w)$ and such that $C \supseteq -(L^\infty_w)_+$. If $C \cap (L^\infty_w)_+ = \{0\}$ then there exists a probability measure $\tilde P \sim P$ such that the density $d\tilde P/dP \in L^1_w$ and $\tilde E\eta \leq 0$ for all $\eta \in C$.

The second fact is the Komlós theorem, [5], A.7.1.

Let (η^n) be a bounded in L^1 sequence of random variables. Then there are a random variable $\eta \in L^1$ and a strictly increasing sequence of positive integers (n') such that for any further subsequence (n'') the sequence of random variables $(\eta^{n''})$ is Cesaro-convergent a.s. to η.

Recall that a sequence (a_m) is called Cesaro-convergent if the sequence of arithmetic means $\bar a_m = m^{-1}\sum_{k\leq m} a_k$ converges.

Using the diagonal procedure one can easily deduce from here a slightly stronger version of the Komlós theorem:

Let (η_u^n), u=1,2,..., be a countable family of sequences which elements belong to a bounded subset A of L^1. Then there are random variables $\eta_u \in L^1$ and a strictly increasing sequence of positive integers (n') such that for any further subsequence (n'') the sequences of random variables $(\eta_u^{n''})$ are Cesaro-convergent a.s. to η_u.

The third needed fact from the functional analysis is the Krein–Šmulian criterion of $\sigma(X^*, X)$-closedness of convex sets in the setting where X is a Frechet space and X^* is its dual, [2], Th. 3.10.2.

A convex set C in X^* is closed for the topology $\sigma(X^*, X)$ if and only if for every balanced convex $\sigma(X^*, X)$-closed equicontinuous subset B of X^* the intersection $C \cap B$ is closed for $\sigma(X^*, X)$.

For families of linear functionals the equicontinuity is equivalent to the equicontinuity at zero. It follows that a subset B in X^* is equicontinuous if and only if it is contained in a polar of a neighborhood of zero. So, in the case where $X = L^1_w$ it is sufficient to verify the $\sigma(X^*, X)$-closedness only for the intersections of C with the sets $\lambda^{-1}B_t$ and, if C is a cone, only with the sets B_t. The following lemma from [6] gives a "practical" condition.

Lemma 1. *A convex cone C in L^∞_w is $\sigma(L^\infty_w, L^1_w)$-closed if the sets $C \cap B_t$ are closed under convergence almost surely.*

Proof. Note that $C \cap B_t$ is $\sigma(L^\infty_w, L^1_w)$-closed if and only if $(w_t^{-1}C) \cap B_0$ is $\sigma(L^\infty(\mu_t), L^1(\mu_t))$-closed. But $\sigma(L^\infty(\mu_t), L^1(\mu_t))$ and $\sigma(L^\infty(\mu_t), L^2(\mu_t))$ coincides on B_0. This means that $(w_t^{-1}C) \cap B_0$ can be viewed as a subset of the Hilbert space in which for the convex subsets the weak closure coincides with the strong closure. So, $(w_t^{-1}C) \cap B_0$ is $\sigma(L^\infty(\mu_t), L^1(\mu_t))$-closed if and only if $(w_t^{-1}C) \cap B_0$ is strongly closed in $L^2(\mu_t)$. An $L^2(\mu_t)$-convergent sequence in $(w_t^{-1}C) \cap B_0$ is convergent in probability, so admits a subsequence convergent μ_t-almost surely, so P-a.s. and, by the assumption, its limit is an element of the considered set. □

Finally, we recall the very first theorem (due to P.-A. Meyer) from the chapter on martingales in Shiryaev's textbook [7], Th. VII.1.1, see also [3]:

Let $X = (X_t)_{t=0,1,\dots}$ be an adapted process with $X_0 = 0$. Then the following conditions are equivalent:

(a) X is a local martingale;

(b) X is a generalized martingale, i.e. $E(|X_{t+1}||\mathcal{F}_t) < \infty$, $E(X_{t+1}|\mathcal{F}_t) = X_t$ for all $t \ge 0$.

This characterization of discrete-time local martingales holds, clearly, also in the case when X_0 is integrable. It makes obvious the following assertion:

A local martingale $X = (X_t)_{t \le T}$ with $X_0 \in L^1$ and $X_T \ge 0$ is a martingale.

Indeed, by consecutive conditioning, $X_t \ge 0$ for all $t \le T$. By the Fatou lemma, a positive local martingale is a supermartingale. So, the integrability property of X_t, relaxed in the definition of generalized martingale, is fulfilled.

3 Proof

Put $w_t := 1 + \max_{r \le t} |S_r|$. Let $\mathcal{W}$ denote the class of processes $H \in \mathcal{P}$ for which there exist a date $t = t_H$ and a constant c_H such that $H \cdot S_u \ge c_H w_t$ for all $u \ge t$. For such a process there is a finite limit $H \cdot S_\infty$. Indeed, the process $M_u := H \cdot S_u - c_H w_t$, $u \ge t$, is a positive generalized martingale and so is the process $\tilde{M}_u := M_u/(1 + |M_t|)$. The latter, starting from a bounded random variable, is a martingale and, therefore, admits a finite limit at infinity. Suppose that $H \cdot S_\infty \ge c$ where c is a constant. Then $H \cdot S_u \ge E(H \cdot S_\infty | \mathcal{F}_u) \ge c$ for all $u \ge t$. It follows that $H \cdot S$ is a martingale dominating c. In particular, if $H \cdot S_\infty \ge 0$, then this martingale is identically equal to zero.

We introduce the set $R_{\mathcal{W}}$ formed by the random variables $H \cdot S_\infty$, $H \in \mathcal{W}$, and the set $A_{\mathcal{W}} := R_{\mathcal{W}} - L^0_+$. By the above observation, $A_{\mathcal{W}} \cap L^0_+ = \{0\}$ and, consequently, $A_{\mathcal{W}} \cap (L^\infty_{\mathcal{W}})_+ = \{0\}$. It remains to prove that $C_{\mathcal{W}} := A_{\mathcal{W}} \cap L^\infty_w$ is closed for the topology $\sigma(L^\infty_w, L^1_w)$ because the cited version of the Kreps–Yan theorem provides a separating measure $\tilde{P}$ which is a martingale one: whatever are u and $\Gamma \in \mathcal{F}_u$ the random variables $\pm I_\Gamma(S_{u+1} - S_u)$ belong to $C_{\mathcal{W}}$ and have zero $\tilde{P}$-expectations.

We check the closedness using Lemma 1 and the following simple remark. Let t_0 a positive integer and let $\zeta \in L^0(\mathcal{F}_{t_0})$. By the DMW theorem there exists a probability measure $P' \sim P$ with the density $dP'/dP \in L^\infty(\mathcal{F}_{t_0})$ such that the process $(S_u)_{u \le t_0}$ is a P'-martingale and $E'|\zeta| < \infty$. Note that S remains a local martingale with respect to P'. It follows that if $H \in \mathcal{W}$ and $H \cdot S_\infty \ge \zeta$ then $H \cdot S_u \ge E'(\zeta | \mathcal{F}_u)$ for all $u \ge t_H$, hence, for all $u \ge 0$, and the whole process $H \cdot S$ is a P'-martingale.

Consider a sequence $\xi^n \in (w_{t_0}^{-1} C_{\mathcal{W}}) \cap B_0$ convergent a.s. to some ξ. By definition, there are $H^n \in \mathcal{W}$ such that $H^n \cdot S_\infty \ge \xi^n w_{t_0} \ge -w_{t_0}$. Using the above remark with $\zeta = -w_{t_0}$ and passing to an equivalent probability measure we may assume without loss of generality that $w_{t_0} \in L^1$, the processes $H^n \cdot S$ are martingales and $H^n \cdot S \ge -M$ where $M_u = E(w_{t_0} | \mathcal{F}_u)$. It follows that $E|H^n \cdot S_u| \le 2Ew_{t_0}$ for every u. So, the extended version of the Komlós theorem is applicable. Replacing the initial H^n and ξ^n by the arithmetic means along the subsequence claimed in this theorem, we may assume, avoiding new notations, that for each u the sequence $H^n \cdot S_u$ converges a.s. to an integrable random variable. Using the closedness of the space of discrete-time stochastic integrals we infer that there exists a predictable process H such that $H^n \cdot S_u \to H \cdot S_u$ a.s. for all finite dates u. Obviously, $H \cdot S_u \ge E(w_{t_0} \xi | \mathcal{F}_u) \ge -M_u$ and, by the Lévy theorem, $H \cdot S_t \ge \xi w_{t_0}$. Note that $M_t = w_{t_0}$ for $t \ge t_0$. Thus, $\xi \in (w_{t_0}^{-1} C_{\mathcal{W}}) \cap B_0$, i.e. the set $(w_{t_0}^{-1} C_{\mathcal{W}}) \cap B_0$ is closed under convergence a.s. and we conclude. $\square$

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