

# Mean Square Error for the Leland–Lott Hedging Strategy

Moussa Gamys\*, Yuri Kabanov†

\*, † Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray,  
25030 Besançon, cedex, France  
E-mail: ykabanov@univ-fcomte.fr

The Leland strategy of approximate hedging of the call-option under proportional transaction costs prescribes to use, at equidistant instants of portfolio revisions, the classical Black–Scholes formula but with a suitably enlarged volatility. An appropriate mathematical framework is a scheme of series, i.e. a sequence of models  $M_n$  with the transaction costs coefficients  $k_n$  depending on  $n$ , the number of the revision intervals. The enlarged volatility  $\widehat{\sigma}_n$ , in general, also depends on  $n$ . Lott investigated in detail the particular case where the transaction costs coefficients decrease as  $n^{-1/2}$  and where the Leland formula yields  $\widehat{\sigma}_n$  not depending on  $n$ . He proved that the terminal value of the portfolio converges in probability to the pay-off. In the present note we show that it converges also in  $L^2$  and find the first order term of asymptotics for the mean square error. The considered setting covers the case of non-uniform revision intervals. We establish the asymptotic expansion when the revision dates are  $t_i^n = g(i/n)$  where the strictly increasing scale function  $g : [0, 1] \rightarrow [0, 1]$  and its inverse  $f$  are continuous with their first and second derivatives on the whole interval or  $g(t) = 1 - (1 - t)^\beta$ ,  $\beta \geq 1$ .

**Key words:** Black–Scholes formula, European option, transaction costs, Leland–Lott strategy, approximate hedging

## 1. Introduction

### 1.1 Formulation of the main result

To fix the notation we consider the classical Black–Scholes model, already under the martingale measure and with the maturity  $T = 1$ . So, let  $S = (S_t)$ ,  $t \in [0, 1]$ , be a geometric Brownian motion given by the formula

$$S_t = S_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

and satisfying the linear equation

$$dS_t = \sigma S_t dW_t$$

with a standard Wiener process  $W$  and constants  $S_0$ ,  $\sigma > 0$ . Let  $C(t, x)$  be the solution, in the domain  $[0, 1] \times ]0, \infty[$ , of the Cauchy problem

$$(1) \quad C_t(t, x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t, x) = 0, \quad C(1, x) = (x - K)^+,$$

where  $K > 0$ . The function  $C(t, x)$  admits an explicit expression and this is the famous Black–Scholes formula:

$$(2) \quad C(t, x) = C(t, x, \sigma) = x\Phi(d) - K\Phi(d - \sigma\sqrt{1-t}), \quad t < 1,$$

where  $\Phi$  is the Gaussian distribution function with the density  $\varphi$ ,

$$(3) \quad d = d(t, x) = d(t, x, \sigma) = \frac{1}{\sigma\sqrt{1-t}} \ln \frac{x}{K} + \frac{1}{2}\sigma\sqrt{1-t}.$$

Define the process

$$(4) \quad V_t = C(0, S_0) + \int_0^t C_x(u, S_u) dS_u.$$

In the Ito formula for  $C(t, S_t)$  the integral over  $dt$  vanishes. Hence,  $V_t = C(t, S_t)$  for all  $t \in [0, 1]$ . In particular,  $V_1 = (S_1 - K)^+$ : at maturity the value process  $V$  replicates the terminal pay-off of the call-option.

Modelling assumptions of the above formulation are, between others: frictionless market and continuous trading. The latter is a purely theoretical invention. Practically, an investor revises the portfolio at certain dates  $t_i$  and keeps  $C_x(t_i, S_{t_i})$  units of the stock until the next revision date  $t_{i+1}$ . The model becomes more realistic if the transactions are charged proportionally to their volume. The portfolio strategy suggested by Leland [6] generates the value process

$$(5) \quad V_t^n = \widehat{C}(0, S_0) + \int_0^t \sum_{i=1}^n H_{t_{i-1}}^n I_{]t_{i-1}, t_i]}(u) dS_u - \sum_{t_i < t} k_n S_{t_i} |H_{t_i}^n - H_{t_{i-1}}^n|,$$

where  $H_{t_i}^n = \widehat{C}_x(t_i, S_{t_i})$ ,  $t_i = i/n$ , the positive parameter  $k_n = k_0 n^{-1/2}$  is the transaction costs coefficient, and  $\widehat{C}(t, x)$  is the solution of (1) with  $\sigma$  replaced by  $\widehat{\sigma} > 0$  with

$$(6) \quad \widehat{\sigma}^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi}.$$

That is  $\widehat{C}(t, x) = C(t, x, \widehat{\sigma})$  and for such a strategy there is no need in a new software: traders can use their old one, changing only one input parameter, the volatility.

In his paper Leland claimed, without providing arguments, that  $V_1^n$  converges to  $V_1 = (S_1 - K)^+$  in probability as  $n \rightarrow \infty$ . This assertion was proven by Lott in his thesis [7] and we believe that the result could be referred to as the Leland–Lott theorem. In fact,  $V_1^n$  converges also in  $L^2$  and the following statement gives the rate of convergence:

**Theorem 1.1.** *The mean square approximation error of the Leland–Lott strategy has the following asymptotics:*

$$(7) \quad E(V_1^n - V_1)^2 = A_1 n^{-1} + o(n^{-1}), \quad n \rightarrow \infty,$$

where the coefficient

$$(8) \quad A_1 = \int_0^1 \left[ \frac{\sigma^4}{2} + \sigma^3 k_0 \sqrt{\frac{2}{\pi}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \right] \Lambda_t dt$$

with  $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t, S_t)$ . Explicitly,

$$(9) \quad \Lambda_t = \frac{K^2}{2\pi\widehat{\sigma} \sqrt{1-t} \sqrt{2\sigma^2 t + \widehat{\sigma}^2(1-t)}} \exp \left\{ -\frac{\left( \ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 t - \frac{1}{2}\widehat{\sigma}^2(1-t) \right)^2}{2\sigma^2 t + \widehat{\sigma}^2(1-t)} \right\}.$$

The main result of this note is slightly more general and covers also a model with non-uniform grids given as follows.

Let  $f$  be a strictly increasing smooth function on  $[0, 1]$  with  $f(0) = 0$ ,  $f(1) = 1$  and let  $g := f^{-1}$  denote its inverse. For each fixed  $n$  we define the revision dates  $t_i = t_i^n = g(i/n)$ ,  $1, \dots, n$ . The enlarged volatility now depends on  $t$  and is given by the formula

$$(10) \quad \widehat{\sigma}_t^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi} \sqrt{f'(t)}$$

while the function  $\widehat{C}(t, x)$  given by the formula

$$\widehat{C}(t, x) = x\Phi(\rho_t^{-1} \ln(x/K) + \rho_t/2) - K\Phi(\rho_t^{-1} \ln(x/K) - \rho_t/2), \quad t < 1,$$

with  $\rho_t^2 = \int_t^1 \widehat{\sigma}_s^2 ds$  solves the Cauchy problem

$$(11) \quad \widehat{C}_t(t, x) + \frac{1}{2}\widehat{\sigma}_t^2 x^2 \widehat{C}_{xx}(t, x) = 0, \quad \widehat{C}(1, x) = (x - K)^+,$$

The following bounds are obvious:

$$\sigma^2(1-t) \leq \rho_t^2 \leq \sigma^2(1-t) + \sigma k_0 \sqrt{8/\pi} (1-t)^{1/2} (1-f(t))^{1/2}.$$

**Assumption 1:**  $g, f \in C^2([0, 1])$ .

**Assumption 2:**  $g(t) = 1 - (1 - t)^\beta, \beta \geq 1$ .

Note that in the second case where  $f(t) = 1 - (1 - t)^{1/\beta}$  the derivative  $f'$  for  $\beta > 1$  explodes at the maturity date and so does the enlarged volatility.

**Theorem 1.2.** *Under any of the above assumptions*

$$(12) \quad E(V_1^n - V_1)^2 = A_1(f)n^{-1} + o(n^{-1}), \quad n \rightarrow \infty,$$

where the coefficient

$$(13) \quad A_1(f) = \int_0^1 \left[ \frac{\sigma^4}{2} \frac{1}{f'(t)} + k_0 \sigma^3 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{f'(t)}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \right] \Lambda_t dt,$$

with  $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t, S_t)$ . Explicitly,

$$(14) \quad \Lambda_t = \frac{1}{2\pi\rho_t} \frac{K^2}{\sqrt{2\sigma^2 t + \rho_t^2}} \exp \left\{ -\frac{\left(\ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 t - \frac{1}{2}\rho_t^2\right)^2}{2\sigma^2 t + \rho_t^2} \right\}.$$

The case  $f(t) = t$  corresponds to the uniform grid and  $A_1(f) = A_1$ .

The above result makes plausible the conjecture that the normalized difference  $n^{1/2}(V_1^n - V_1)$  converges in law. Indeed, this is the case, see [4].

## 1.2 Comments on the Grannan–Swindle paper

The Leland method based on the Black–Scholes formula is amongst a few practical recipes how to price options under transaction costs. It has an advantage to rely upon well-known and well-understood formulae from the theory of frictionless markets. The method gave rise to a variety of other schemes. Of course, the precision of the resulting approximate hedging is an important issue, see [5], [2], [8], [9] and a survey [10] for related development.

The idea to parameterize the non-uniform grids by increasing functions and consider the family of strategies with the enlarged volatilities given by (10) is due to Grannan and Swindle, [3]. The mentioned paper claims that the asymptotics (12) holds for more general option with the pay-off of the form  $G(S_1)$ . In such a case the function  $\widehat{C}(t, x)$  is the solution of the Cauchy problem

$$\widehat{C}_t(t, x) + \frac{1}{2}\widehat{\sigma}_t^2 x^2 \widehat{C}_{xx}(t, x) = 0, \quad \widehat{C}(1, x) = G(x).$$

To our opinion, the formulations and arguments given in [3] are not satisfactory. In particular, the hypothesis that for any nonnegative integers  $m, n, p$

$$\|\widehat{C}\|_{m,n,p} = \sup_{x>0, t \in [0,1]} \left[ x^m \frac{\partial^{n+p} \widehat{C}(t, x)}{\partial x^n \partial t^p} \right] < \infty$$

is not fulfilled for the call-option with  $G(x) = (x - K)^+$  (even for the uniform grid): explicit formulae show that derivatives of  $\widehat{C}(t, x)$  have singularities at the point  $(1, K)$ . So, the mathematical results of the original paper [3] do not cover practically interesting cases. Nevertheless, the formula for  $A_1(f)$  is used in numerical analysis of the approximate hedging error of call-options. Note also that the authors of [3] do not care about the eventual divergence of the integral (13) due to singularities of  $1/f'$  which are not excluded by their assumptions. Neglecting the singularities may lead to an erroneous answer (recall the unfortunate error in Leland's paper corrected in [5]). That is why we are looking here for a rigorous proof to build a platform for further studies. The asymptotic analysis happens to be more involved comparatively with the arguments in [3] and we restrict ourselves to the case of the classical call-option.

The paper [3] contains another interesting idea: to minimize the functional  $A_1(f)$  with respect to the scale  $f$  in a hope to improve the performance of the strategy by an appropriate choice of the revision dates<sup>1</sup>. We alert the reader that the reduction to a classical variational problem is not correct as well as the derived Euler–Lagrange equation. That is why the whole paper [3] can be considered only as one giving useful heuristics but leaving open mathematical problems of practical importance.

## 2. Proof of Theorem 1.2

### 2.1 Preparatory Manipulations

First of all, we represent the deviation of the approximating portfolio from the pay-off in an integral form which is instructive how to proceed further.

**Lemma 2.1.** *We have the representation  $V_1^n - V_1 = F_1^n + F_2^n$  where*

$$F_1^n = \sigma \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)) S_t dW_t,$$

$$F_2^n = k_0 \sqrt{\frac{2}{\pi}} \sigma \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)} dt - \frac{k_0}{\sqrt{n}} \sum_{i=1}^{n-1} |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})| S_{t_i}.$$

<sup>1</sup>Even in the frictionless case the choice of an optimal scale to minimize the hedging error is an important and nontrivial problem, especially, for irregular pay-off functions, see, e.g., [1] and references therein.

*Proof.* Applying the Ito formula and taking into account that the functions  $C(t, x)$  and  $\widehat{C}(t, x)$  satisfy the Cauchy problems with the same terminal condition, we get:

$$\begin{aligned}\widehat{C}(0, S_0) - C(0, S_0) &= C(1, S_1) - C(0, S_0) - [\widehat{C}(1, S_1) - \widehat{C}(0, S_0)] \\ &= \int_0^1 C_x(t, S_t) dS_t - \int_0^1 \widehat{C}_x(t, S_t) dS_t \\ &\quad + \int_0^1 \frac{1}{2} (\widehat{\sigma}_t^2 - \sigma^2) S_t^2 \widehat{C}_{xx}(t, S_t) dt.\end{aligned}$$

Taking the difference of (4) and (5) with  $t = 1$  and using the above formula, we obtain the required representation.  $\square$

Note that the sum in the expression for  $F_2^n$  does not include the term with  $i = n$ . Having in mind singularities of derivatives at the maturity, it is convenient to isolate the last summands in other sums and treat them separately.

A short inspection of the above formulae using the well-known helpful heuristics  $\Delta S_t \approx \sigma S_t \Delta W_t \approx \sigma S_t \sqrt{dt}$  reveals that the main contributions in the first order Taylor approximations of increments originate from the derivatives in  $x$ . This consideration allows us to specify the principal terms of a particularly transparent structure. Namely, we put

$$\begin{aligned}P_1^n &:= \sum_{i=1}^{n-1} \sigma \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (1 - S_t/S_{t_{i-1}}) S_t/S_{t_{i-1}} dW_t, \\ P_2^n &:= k_0 \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[ \sigma \sqrt{2/\pi} \sqrt{f'(t_{i-1})} \Delta t_i - |S_{t_i}/S_{t_{i-1}} - 1|/\sqrt{n} \right].\end{aligned}$$

To establish Theorem 1.2 we check that  $nE(P_1^n + P_2^n)^2 \rightarrow A_1(f)$  as  $n \rightarrow \infty$  and the residual terms  $R_i^n := F_i^n - P_i^n$  are negligible, i.e.  $nE(R_i^n)^2 = o(1)$ .

The first residual term is of the following form:

$$(15) \quad R_1^n = (R_{1n}^n - R_{1t}^n - (1/2)\tilde{R}_1^n)\sigma,$$

where

$$\begin{aligned}R_{1n}^n &= \int_{t_{n-1}}^1 (\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) - \widehat{C}_x(t, S_t)) S_t dW_t, \\ R_{1t}^n &= \sum_{i=1}^{n-1} \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (t - t_{i-1}) S_t dW_t, \\ \tilde{R}_1^n &= \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \tilde{U}_t^i dW_t,\end{aligned}$$

where

$$\begin{aligned}\tilde{U}_t^i &= \widehat{C}_{xxx}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(S_t - S_{t_{i-1}})^2 S_t + \widehat{C}_{xtt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(t - t_{i-1})^2 S_t \\ &\quad + 2\widehat{C}_{xxt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(t - t_{i-1})(S_t - S_{t_{i-1}})S_t,\end{aligned}$$

$\tilde{t}_{i-1}$  and  $\tilde{S}_{t_{i-1}}$  are random variables with values in the intervals  $[t_{i-1}, t_i]$  and  $[S_{t_{i-1}}, S_t]$ , respectively. The structure of the above representation of  $R_1^n$  is clear: the term  $R_{1n}^n$  corresponds to the  $n$ -th revision interval (it will be treated separately because of singularities at the left extremity of the time interval), the term  $R_{1t}^n$  involving the first derivatives of  $\widehat{C}_x$  in  $t$  at points  $(t_{i-1}, S_{t_{i-1}})$  comes from the Taylor formula and the “tilde” term is due to the remainder of latter.

It is important to note that the integrals involving in the definition of  $P_1^n$  depend only on the increments of the Wiener process on the intervals  $[t_{i-1}, t_i]$  and, therefore, are independent on the  $\sigma$ -algebras  $\mathcal{F}_{t_{i-1}}$ . This helps to calculate the expectation of the squared sum: according to Lemma 2.2 below it is the sum of expectations of the squared terms. We define  $P_2^n$  in a way to enjoy the same property. The second residual term includes the term  $R_{2n}^n$  corresponding to the last revision interval; the term  $R_{21}^n$  represents the approximation error arising from replacement of the integral by the Riemann sum; the remaining part of the residual we split in a natural way into summands  $R_{22}^n$  and  $R_{23}^n$ . After these explanations we write the second residual term as follows:

$$(16) \quad R_2^n = (R_{2n}^n + R_{21}^n + R_{22}^n + R_{23}^n + R_{24}^n)k_0,$$

with

$$\begin{aligned}R_{2n}^n &= \sqrt{\frac{2}{\pi}}\sigma \int_{t_{n-1}}^1 S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)} dt, \\ R_{21}^n &= \sqrt{\frac{2}{\pi}}\sigma \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)} - S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \sqrt{f'(t_{i-1})}) dt, \\ R_{22}^n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) |S_{t_{i-1}} - S_{t_i}| (S_{t_{i-1}} - S_{t_i}), \\ R_{23}^n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} [\dots]_i (S_{t_i} - S_{t_{i-1}}), \\ R_{24}^n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} [\dots]_i S_{t_{i-1}},\end{aligned}$$

where

$$(17) \quad [\dots]_i = \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) |S_{t_i} - S_{t_{i-1}}| - |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})|.$$

## 2.2 Tools

In our computations we shall use frequently the following two assertions. The first one is a standard fact on square integrable martingales in discrete time.

**Lemma 2.2.** *Let  $M = (M_i)$  be a square-integrable martingale with respect to a filtration  $(\mathcal{G}_i)$ ,  $i = 0, \dots, k$ , and let  $X = (X_i)$  be a predictable process with  $EX^2 \cdot \langle M \rangle_k < \infty$ . Then*

$$E(X \cdot M_k)^2 = EX^2 \cdot \langle M \rangle_k = \sum_{i=1}^k EX_i^2 (\Delta M_i)^2,$$

where, as usual,  $\Delta \langle M \rangle_i := E((\Delta M_i)^2 | \mathcal{G}_{i-1})$ ,

$$X \cdot M_k := \sum_{i=1}^k X_i \Delta M_i, \quad X^2 \cdot \langle M \rangle_k := \sum_{i=1}^k X_i^2 \langle M \rangle_i.$$

**Lemma 2.3.** *Suppose that  $g', f' \in C([0, 1])$ . Let  $p > 0$  and  $a \geq 0$ . Then*

$$\sum_{i=1}^{n-1} \frac{(\Delta t_i)^{p+a}}{(1-t_i)^p} = \begin{cases} O(n^{1-p-a}), & p < 1, \\ O(n^{-a} \ln n), & p = 1, \\ O(n^{-a}), & p > 1. \end{cases}$$

If  $g(t) = 1 - (1-t)^\beta$ ,  $\beta \geq 1$ , then

$$\sum_{i=1}^{n-1} \frac{(\Delta t_i)^{p+a}}{(1-t_i)^p} = \begin{cases} O(n^{1-p-a}), & p < 1 + a(\beta - 1), \\ O(n^{-a\beta} \ln n), & p = 1 + a(\beta - 1), \\ O(n^{-a}), & p > 1 + a(\beta - 1). \end{cases}$$

*Proof.* We consider first the case where  $g', f' \in C([0, 1])$ , i.e.  $g'$  is not only bounded but also bounded away from zero. By the finite increments formula  $\Delta t_i = g'(x_i)n^{-1}$  where  $x_i \in [(i-1)/n, i/n]$  and, hence,  $\Delta t_i \leq \text{const } n^{-1}$ . Applying again the finite increments formula and taking into account that  $\min g'(t) > 0$ , it is easy to check that there is a constant  $c$  such that

$$\frac{1-t_{i-1}}{1-t_i} \leq c, \quad 1 \leq i \leq n-1.$$

Thus,

$$\sum_{i=1}^{n-1} \frac{\Delta t_i}{(1-t_i)^p} \leq c \sum_{i=1}^{n-1} \frac{\Delta t_i}{(1-t_{i-1})^p} \leq c \int_0^{t_{n-1}} \frac{dt}{(1-t)^p}.$$

Since

$$n^{-1} \min g'(t) \leq 1 - g(1 - 1/n) \leq n^{-1} \max g'(t),$$

the asymptotic of the last integral is  $O(1)$ , if  $p < 1$  (the integral converges),  $O(\ln n)$ , if  $p = 1$ , and  $O(n^{p-1})$ , if  $p > 1$ . This implies the claimed property.



In the second case where  $g(t) = 1 - (1 - t)^\beta$ ,  $\beta \geq 1$ , we have

$$\sum_{i=1}^{n-1} \frac{(\Delta t_i)^{p+a}}{(1 - t_i)^p} = \frac{\beta^{p+a}}{n^{p-1+a}} \sum_{i=1}^{n-1} \frac{(1 - x_i)^{(\beta-1)(p+a)}}{(1 - i/n)^{\beta p}} \frac{1}{n}.$$

The sum in the right-hand side is dominated, up to a multiplicative constant, by

$$\sum_{i=1}^{n-1} \frac{1}{(1 - (i-1)/n)^{p+a-\beta a}} \frac{1}{n} \leq \int_0^{1-1/n} \frac{dt}{(1-t)^{p+a-\beta a}}.$$

Using the explicit formulae for the integral we infer that the required property holds whatever are the parameters  $p > 0$ ,  $a \geq 0$ , and  $\beta \geq 1$ .  $\square$

### 2.3 Explicit Formulae and Useful Bounds

We consider the function  $\widehat{C}(t, x)$  corresponding to the “artificial”, in general, time-varying volatility. This function, solving the Cauchy problem (11), is given by the formula

$$(18) \quad \widehat{C}(t, x) = x\Phi(\widehat{d}(t, x)) - K\Phi(\widehat{d}(t, x) - \rho_t), \quad t < 1,$$

where

$$(19) \quad \widehat{d}(t, x) := \frac{1}{\rho_t} \ln \frac{x}{K} + \frac{1}{2}\rho_t$$

and  $\rho_t > 0$ ,

$$(20) \quad \rho_t^2 := \int_t^1 \widehat{\sigma}_s^2 ds = \int_t^1 (\sigma^2 + \sigma k_0 \sqrt{8/\pi} \sqrt{f'(s)}) ds.$$

It is easy to verify that

$$\begin{aligned} \widehat{C}_x(t, x) &= \Phi(\widehat{d}(t, x)), \\ \widehat{C}_{xx}(t, x) &= \frac{1}{x\rho_t} \varphi(\widehat{d}(t, x)). \end{aligned}$$

The first and the second derivatives of  $\widehat{d}(t, x)$  in  $t$  are, respectively,

$$\begin{aligned} \widehat{d}_t(t, x) &= \frac{\widehat{\sigma}_t^2}{2\rho_t^3} \ln \frac{x}{K} - \frac{\widehat{\sigma}_t^2}{4\rho_t} = \frac{\widehat{\sigma}_t^2}{2\rho_t^2} (\widehat{d}(t, x) - \rho_t), \\ \widehat{d}_{tt}(t, x) &= \frac{\sigma k_0 \sqrt{2/\pi} f''(t)}{2\rho_t^2 \sqrt{f'(t)}} (\widehat{d}(t, x) - \rho_t) + \frac{3\widehat{\sigma}_t^4}{4\rho_t^4} (\widehat{d}(t, x) - \rho_t) + \frac{\widehat{\sigma}_t^4}{4\rho_t^3}. \end{aligned}$$

For the analysis of residual terms we need also the following derivatives:

$$\begin{aligned}\widehat{C}_{xt}(t, x) &= \frac{\widehat{\sigma}_t^2}{2\rho_t^2}(\widehat{d}(t, x) - \rho_t)\varphi(\widehat{d}(t, x)), \\ \widehat{C}_{xxx}(t, x) &= -\frac{1}{x^2\rho_t^2}(\widehat{d}(t, x) + \rho_t)\varphi(\widehat{d}(t, x)), \\ \widehat{C}_{xtt}(t, x) &= \left[ \widehat{d}_{tt}(t, x) - \frac{\widehat{\sigma}_t^4}{4\rho_t^4}(\widehat{d}(t, x) - \rho_t)^2\widehat{d}(t, x) \right] \varphi(\widehat{d}(t, x)), \\ \widehat{C}_{xxt}(t, x) &= \frac{1}{x} \left[ \frac{\widehat{\sigma}_t^2}{2\rho_t^3} + \frac{\widehat{\sigma}_t^2}{2\rho_t^3}(\widehat{d}(t, x) - \rho_t)\widehat{d}(t, x) \right] \varphi(\widehat{d}(t, x)), \\ \widehat{C}_{xxxx}(t, x) &= \frac{1}{x^3\rho_t^3} \left[ 2\rho_t(\widehat{d}(t, x) + \rho_t) - 1 + (\widehat{d}(t, x) + \rho_t)\widehat{d}(t, x) \right] \varphi(\widehat{d}(t, x)).\end{aligned}$$

Note that under any of our assumptions on the scale transformation the ratio  $\widehat{\sigma}_t^2/\rho_t^2$  has a singularity  $c(1-t)^{-1}$  as  $t \rightarrow 1$ . Since for every  $p \geq 0$  the function  $|y|^p\varphi(y)$  is bounded, we obtain from the above formulae the following estimates:

$$(21) \quad |\widehat{C}_{xt}(t, x)| \leq \kappa \frac{1}{1-t},$$

$$(22) \quad |\widehat{C}_{xxx}(t, x)| \leq \kappa \frac{1}{x^2(1-t)},$$

$$(23) \quad |\widehat{C}_{xtt}(t, x)| \leq \kappa \frac{1}{(1-t)^2},$$

$$(24) \quad |\widehat{C}_{xxt}(t, x)| \leq \kappa \frac{1}{x} \frac{1}{(1-t)^{3/2}},$$

$$(25) \quad |\widehat{C}_{xxxx}(t, x)| \leq \kappa \frac{1}{x^3(1-t)^{3/2}}.$$

Now we obtain a formula which gives, in particular, an expression for  $\Lambda_t$ .

Let  $\xi \in \mathcal{N}(0, 1)$  and let  $a \neq 0$ ,  $b$ ,  $c$  be arbitrary constants. Then

$$(26) \quad E e^{c\xi} e^{-(a\xi+b)^2} = \frac{1}{\sqrt{2a^2+1}} \exp \left\{ -\frac{\tilde{b}^2}{2a^2+1} + \tilde{b}^2 - b^2 \right\}.$$

where  $\tilde{b} := b - c/(2a)$ .

The distribution of the random variable  $2\pi S_t^p \widehat{C}_{xx}^2(t, S_t)$  is the same as of

$$S_0^{p-2} e^{-\frac{1}{2}(p-2)\sigma^2 t} \rho_t^{-2} e^{c_t \xi} e^{-(a_t \xi + b_t)^2}$$

where  $c_t = (p-2)\sigma t^{1/2}$ ,  $a_t = \frac{1}{\rho_t}\sigma t^{1/2}$ ,

$$b_t = \frac{1}{\rho_t} \left( \ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 t \right) + \frac{1}{2}\rho_t, \quad \tilde{b}_t = b_t - \frac{1}{2}(p-2)\rho_t.$$

Since

$$\tilde{b}_t^2 - b_t^2 = -(p-2) \left[ \left( \ln \frac{S_0}{K} - \frac{1}{2} \sigma^2 t \right) + \rho_t^2 - \frac{1}{4} p \rho_t^2 \right],$$

we obtain from above that

$$(27) \quad ES_t^p \widehat{C}_{xx}^2(t, S_t) = \frac{1}{2\pi\rho_t} \frac{K^{p-2}}{\sqrt{2\sigma^2 t + \rho_t^2}} e^{-B_t},$$

where

$$(28) \quad B_t := \frac{\left( \ln \frac{S_0}{K} - \frac{1}{2} \sigma^2 t - \frac{1}{2} (p-3) \rho_t^2 \right)^2}{2\sigma^2 t + \rho_t^2} - \frac{(p-2)(p-4)}{4} \rho_t^2.$$

In particular, with  $p = 4$ , we have the following formula:

$$(29) \quad \Lambda_t = \frac{1}{2\pi\rho_t} \frac{K^2}{\sqrt{2\sigma^2 t + \rho_t^2}} \exp \left\{ - \frac{\left( \ln \frac{S_0}{K} - \frac{1}{2} \sigma^2 t - \frac{1}{2} \rho_t^2 \right)^2}{2\sigma^2 t + \rho_t^2} \right\}.$$

It is easily seen that  $\Lambda_t$  is a continuous function on  $[0, 1[$  tending to infinity as  $t \rightarrow 1$ . The singularity at infinity is integrable since

$$(30) \quad \frac{1}{\kappa} \frac{1}{(1-t)^{1/4}} \leq \Lambda_t \leq \kappa \frac{1}{(1-t)^{1/2}}$$

for some constant  $\kappa > 0$  (of course, under Assumption 1 we have the lower bound of the same order as the upper one).

It is worth to notice that the upper bound above is better than one could get using the straightforward estimate  $\widehat{C}_{xx}^2(t, x) \leq \kappa/(x^2 \rho_t^2)$ .

Sharper bounds for the expectations will be of frequent use in our analysis. To get them we observe that for  $p \in \mathbf{R}$ ,  $m \geq 0$ ,  $r > 0$ , and  $r' \in ]0, r[$  we have the bound

$$ES_t^p \widehat{d}^m(t, S_t) \varphi(r\widehat{d}(t, S_t)) \leq \kappa ES_t^p \varphi(r'\widehat{d}(t, S_t)).$$

Exploiting again the identity (26) to estimate the right-hand side we get the inequality

$$ES_t^p \widehat{d}^m(t, S_t) \varphi(r\widehat{d}(t, S_t)) \leq \kappa \rho_t.$$

Using the expressions for the derivatives of  $\widehat{C}$  we obtain the following bounds:

$$(31) \quad ES_t^p \widehat{C}_{xt}^{2m}(t, S_t) \leq \kappa \frac{1}{(1-t)^{2m-1/2}},$$

$$(32) \quad ES_t^p \widehat{C}_{xxx}^{2m}(t, S_t) \leq \kappa \frac{1}{(1-t)^{2m-1/2}},$$

$$(33) \quad ES_t^p \widehat{C}_{xxt}^{2m}(t, S_t) \leq \kappa \frac{1}{(1-t)^{3m-1/2}},$$

$$(34) \quad ES_t^p \widehat{C}_{xxxx}^{2m}(t, S_t) \leq \kappa \frac{1}{(1-t)^{3m-1/2}},$$

where the constant  $\kappa$  depends on  $p$  and  $m$ .

#### 2.4 Analysis of the Principal Terms

Let us check that  $nE(P_1^n + P_2^n) \rightarrow A_1(f)$  as  $n \rightarrow \infty$ . To this aim we put

$$\tilde{P}_1^n := \frac{1}{2} \sigma^2 \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 [\Delta t_i - (\Delta W_{t_i})^2],$$

$$\tilde{P}_2^n := k_0 \sigma n^{-1/2} \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 [\sqrt{2/\pi} \sqrt{\Delta t_i} - |\Delta W_{t_i}|],$$

where, as usual,  $\Delta t_i := t_i - t_{i-1}$  and  $\Delta W_{t_i} := W_{t_i} - W_{t_{i-1}}$ . It is sufficient to verify that  $nE(\tilde{P}_1^n + \tilde{P}_2^n)^2 \rightarrow A_1(f)$  while  $nE(P_j^n - \tilde{P}_j^n)^2 \rightarrow 0$ ,  $j = 1, 2$ .

Recall that  $E(\xi^2 - 1)^2 = 2$  and  $E|\xi|^3 = 2E|\xi| = 2\sqrt{2/\pi}$  for  $\xi \in \mathcal{N}(0, 1)$ . Using Lemma 2.2 we obtain the representation

$$\begin{aligned} nE(\tilde{P}_1^n + \tilde{P}_2^n)^2 &= \frac{\sigma^4}{2} n \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} (\Delta t_i)^2 + k_0 \sigma^3 \sqrt{\frac{2}{\pi}} n^{1/2} \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} (\Delta t_i)^{3/2} \\ &\quad + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} \Delta t_i. \end{aligned}$$

By the finite increments formula  $\Delta t_i = g(i/n) - g((i-1)/n) = g'(x_i)/n$  where  $x_i \in [(i-1)/n, i/n]$ . We substitute this expression into the sums above. Let us introduce the function  $F_n$  (depending on  $p$ ) by the formula

$$F_n(t) := \sum_{i=1}^{n-1} \Lambda_{g((i-1)/n)} [g'(x_i)]^p I_{[(i-1)/n, i/n]}(t).$$

For  $p \geq 1$  we have:

$$\sum_{i=1}^{n-1} \Lambda_{g((i-1)/n)} [g'(x_i)]^p \frac{1}{n} = \int_0^1 F_n(t) dt \rightarrow \int_0^1 \Lambda_{g(t)} [g'(t)]^p dt.$$

The needed uniform integrability of the sequence  $\{F_n\}$  with respect to the Lebesgue measure follows from the de la Vallée-Poussin criterion because the estimate  $\Lambda_t \leq \kappa(1-t)^{-1/2}$  and the boundedness of  $g'$  imply that

$$\int_0^1 F_n^{3/2}(t) dt \leq \text{const} \int_0^1 \frac{dg(t)}{(1-g(t))^{3/4}} = \text{const} \int_0^1 \frac{ds}{(1-s)^{3/4}} < \infty.$$

By the change of variable, taking into account that  $g'(t) = 1/f'(g(t))$ , we transform the limiting integral into the form used in the formulations of the theorem:

$$\int_0^1 \Lambda_{g(t)} [g'(t)]^p dt = \int_0^1 \Lambda_{g(t)} [g'(t)]^{p-1} dg(t) = \int_0^1 \Lambda_t [f'(t)]^{1-p} dt.$$

The first claimed property on the convergence to  $A_1(f)$  is verified.

Using again Lemma 2.2 we get that

$$E(P_1^n - \tilde{P}_1^n)^2 = \sigma^2 \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} \int_{t_{i-1}}^{t_i} E \left[ \left( \frac{S_t}{S_{t_{i-1}}} - 1 \right) \frac{S_t}{S_{t_{i-1}}} - \sigma(W_t - W_{t_{i-1}}) \right]^2 dt.$$

It is a simple exercises to check that

$$E((e^{\sigma t^{1/2} \xi - \frac{1}{2} \sigma^2 t} - 1) e^{\sigma t^{1/2} \xi - \frac{1}{2} \sigma^2 t} - \sigma t^{1/2} \xi)^2 = O(t^2), \quad t \rightarrow 0.$$

Therefore,

$$nE(P_1^n - \tilde{P}_1^n)^2 \leq \text{const} n \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} (\Delta t_i)^3 \rightarrow 0, \quad n \rightarrow \infty.$$

The sum  $P_2^n$  is not centered and, therefore, Lemma 2.2 cannot be directly applied. Let us check that under our assumptions the bias is negligible. Indeed, put

$$P_2^{n'} = k_0 n^{-1/2} \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[ E|S_{t_i}/S_{t_{i-1}} - 1| - |S_{t_i}/S_{t_{i-1}} - 1| \right].$$

We have:

$$n^{1/2} \|P_2^n - P_2^{n'}\|_{L^2} \leq k_0 \sum_{i=1}^{n-1} \Lambda_{t_{i-1}}^{1/2} B_i,$$

where

$$B_i := \left| \sigma \sqrt{2/\pi} \sqrt{nf'(t_{i-1})} \Delta t_i - E|S_{t_i}/S_{t_{i-1}} - 1| \right|.$$

Using the Taylor formula it is easy to verify that for  $u > 0$

$$E[e^{u\xi - \frac{1}{2}u^2} - 1] = 2[\Phi(u/2) - \Phi(-u/2)] = \sqrt{2/\pi} u + O(u^3), \quad u \rightarrow 0,$$

It follows that

$$B_i = \sigma \sqrt{2/\pi} (\Delta t_i)^{1/2} \left| \sqrt{n f'(t_{i-1}) \Delta t_i} - 1 \right| + O((\Delta t_i)^{3/2}).$$

By the Taylor formula

$$\Delta t_i = g(i/n) - g((i-1)/n) = g'((i-1)/n) \frac{1}{n} + \frac{1}{2} g''(y_i) \frac{1}{n^2},$$

where the point  $y_i \in [(i-1)/n, i/n]$ . Since the function  $f$  is the inverse of  $g$  we have  $f'(t_{i-1}) = 1/g'((i-1)/n)$ . Using these identities and the elementary inequality  $|\sqrt{1+a} - 1| \leq |a|$  for  $a \geq -1$  we obtain that

$$B_i \leq \text{const} \frac{|g''(y_i)|}{g'((i-1)/n)} (\Delta t_i)^{1/2} \frac{1}{n} + O((\Delta t_i)^{3/2}).$$

Fix  $\varepsilon \in ]0, 1/4[$ . Substituting the finite increments formula  $\Delta t_i = g'(x_i)/n$  with an intermediate point  $x_i$  in  $[(i-1)/n, i/n]$ , we infer that

$$B_i \leq \text{const} a_n \frac{g'(x_i)}{[1 - g((i-1)/n)]^{3/4-\varepsilon}} \frac{1}{n} + O((\Delta t_i)^{3/2}).$$

where

$$a_n = \frac{1}{n^{1/2}} \sup_{i \leq n-1} \sup_{x_i, y_i} \frac{|g''(y_i)| [1 - g((i-1)/n)]^{3/4-\varepsilon}}{g'((i-1)/n) (g'(x_i))^{1/2}}.$$

Recall that

$$\sum_{i=1}^{n-1} \frac{g'(x_i)}{[1 - g((i-1)/n)]^{1-\varepsilon}} \frac{1}{n} \rightarrow \int_0^1 \frac{dg(t)}{[1 - g(t)]^{1-\varepsilon}} = \int_0^1 \frac{dt}{(1-t)^{1-\varepsilon}} < \infty$$

and  $a_n \rightarrow 0$  under each of our assumptions. These observations lead to the conclusion that

$$\sum_{i=1}^{n-1} \Lambda_{t_{i-1}}^{1/2} B_i \rightarrow 0.$$

Thus, we have the convergence  $nE(P_2^n - P_2^{n'})^2 \rightarrow 0$ .

Noticing that

$$E(|e^{u\xi - \frac{1}{2}u^2} - 1| - u|\xi|)^2 = O(u^4), \quad u \rightarrow 0,$$

we infer that

$$E\left[\left(E|S_{t_i}/S_{t_{i-1}} - 1| - |S_{t_i}/S_{t_{i-1}} - 1| - \sigma\left(E|\Delta W_{t_i}| - |\Delta W_{t_i}|\right)\right)^2\right] = O((\Delta t_i)^2).$$

Using again Lemma 2.2 we get that

$$nE(P_2^{n'} - \tilde{P}_2^n)^2 \leq \text{const} \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} (\Delta t_i)^2 \rightarrow 0, \quad n \rightarrow \infty.$$

It follows that  $nE(P_2^n - \tilde{P}_2^n)^2 \rightarrow 0$  and the proof of the second claimed property is completed.

## 2.5 Analysis of the Residual $R_1^n$

1. Calculating the expectation of the squared stochastic integral we obtain that

$$E(R_{1n}^n)^2 = \int_{t_{n-1}}^1 E(\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) - \widehat{C}_x(t, S_t))^2 S_t^2 dt \leq \kappa_n(1 - t_{n-1}),$$

where  $\kappa_n$  is the supremum of the integrand over  $[t_{n-1}, 1]$ . Note that the process  $\widehat{d}(t, S_t)$  diverges a.s. when  $t \rightarrow 1$ . On the set where it diverges to  $-\infty$ , we have  $\widehat{C}_x(t, S_t) = \Phi(\widehat{d}(t, S_t)) \rightarrow 0$  while  $\widehat{C}_x(t, S_t) \rightarrow 1$  a.s. on the complement of this set. It follows that  $\kappa_n \rightarrow 0$ . Since  $1 - t_{n-1} \leq \kappa n^{-1}$  (due to the boundedness of  $g'$ ), we conclude that  $nE(R_{1n}^n)^2 \rightarrow 0$ .

2. Let us consider the term

$$R_{1t}^n = \sum_{i=1}^{n-1} \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (t - t_{i-1}) S_t dW_t.$$

According to (31)

$$E\widehat{C}_{xt}^2(t, S_t) S_t^2 \leq \kappa \frac{1}{(1-t)^{3/2}}.$$

Therefore,

$$\begin{aligned} E(R_{1t}^n)^2 &= \sum_{i=1}^{n-1} E\widehat{C}_{xt}^2(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (t - t_{i-1})^2 E(S_t / S_{t_{i-1}})^2 dt \\ &\leq \text{const} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{(1 - t_{i-1})^{3/2}} = O(n^{-3/2}), \quad n \rightarrow \infty, \end{aligned}$$

in virtue of Lemma 2.3. Hence,  $nE(R_{1t}^n)^2 \rightarrow 0$ .

3. Now we estimate the expectation  $E(\tilde{R}_1^n)^2$  corresponding to the terms arising from the residual in the Taylor formula for  $\widehat{C}_x$ . We have:

$$E(\tilde{R}_1^n)^2 = \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E(\tilde{U}_t^i)^2 dt.$$

Since  $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ , it is sufficient to check that each of the following sums converge to zero as  $o(n^{-1})$ :

$$\begin{aligned} \Sigma_1^n &= \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E\widehat{C}_{xxx}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) (S_t - S_{t_{i-1}})^4 S_t^2 dt, \\ \Sigma_2^n &:= \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E\widehat{C}_{xxt}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) (t - t_{i-1})^4 S_t^2 dt, \\ \Sigma_3^n &:= \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E\widehat{C}_{xxt}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) (t - t_{i-1})^2 (S_t - S_{t_{i-1}})^2 S_t^2 dt. \end{aligned}$$

Taking into account that  $y\varphi(y)$  is bounded,  $\widehat{d}(t, x)\varphi(\widehat{d}(t, x)) \rightarrow 0$  as  $t \rightarrow 1$  whatever is  $x \neq K$ , and  $S_1 \neq K$  (a.s), we deduce from the explicit formula for  $\widehat{C}_{xxx}$  that for any  $\varepsilon > 0$ ,  $m \geq 1$ , there exists  $a \in ]0, 1[$  such that

$$(35) \quad E|\widehat{C}_{xxx}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})|^{2m} \leq \varepsilon \frac{1}{(1-t_i)^{2m}}$$

for every  $t_{i-1} \geq a$ . For  $t_{i-1} < a$  the above expectation is bounded by a constant which does not on  $n$ .

Let  $\xi \sim N(0, 1)$  and let  $b \in [0, 1]$ . Using the elementary bound

$$|e^{bx} - 1| \leq b(e^{|x|} - 1)$$

which follows from the Taylor expansion, we obtain, for  $m \geq 1$ , the estimate

$$E(e^{u\sigma\xi - (1/2)\sigma^2 u^2} - 1)^{2m} \leq \kappa u^{2m}$$

where the constant  $\kappa$  depends on  $m$  and  $\sigma$ . Applying the Cauchy–Schwarz inequality and this estimate we get that

$$E(S_t - S_{t_{i-1}})^{2m} S_t^p \leq \kappa(t - t_{i-1})^m.$$

Manipulating again with the Cauchy–Schwarz inequality we obtain with the help of the above bounds that

$$\Sigma_1^n \leq \kappa \sum_{t_{i-1} < a} (\Delta t_i)^3 + \kappa \varepsilon \sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{(1-t_i)^2}.$$

The first sum in the right-hand side is of order  $O(n^{-2})$ . According to Lemma 2.3 the second one is of order  $O(n^{-1})$ . Since  $\varepsilon > 0$  is arbitrary, it follows that  $\lim_n n \Sigma_1^n = 0$ .

Similarly to the bound (35), we can establish that for any  $\varepsilon > 0$  there is a threshold  $a \in ]0, 1[$  such that for any  $t_{i-1} \geq a$  the following inequalities hold:

$$(36) \quad E|\widehat{C}_{xxt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})|^{2m} \leq \varepsilon \frac{1}{(1-t_i)^{3m}}$$

and

$$(37) \quad E|\widehat{C}_{xtt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})|^{2m} \leq \varepsilon \frac{1}{(1-t_i)^{4m}}.$$

With these bounds we prove, making obvious changes in arguments, that  $\lim_n n \Sigma_2^n = 0$  and  $\lim_n n \Sigma_3^n = 0$ .



## 2.6 Analysis of the Residual $R_2^n$

1. Noting that  $\|S_t^2 \widehat{C}_{xx}(t, S_t)\|_{L^2} = \Lambda_t^{1/2}$ , we have:

$$\|R_{2n}^n\|_{L^2} \leq c \int_{t_{n-1}}^1 \Lambda_t^{1/2} \sqrt{f'(t)} dt \leq c \left( \int_{t_{n-1}}^1 \Lambda_t dt \right)^{1/2} (1 - f(t_{n-1}))^{1/2}$$

with  $c = \sqrt{2/\pi}\sigma$ . Since  $f(t_{n-1}) = f(g((n-1)/n)) = 1 - 1/n$  and the function  $\Lambda$  is integrable, it follows that  $nE(R_{2n}^n)^2 \rightarrow 0$ .

2. The term  $R_{21}^n$  describes the error in approximation of an integral by Riemann sums. To analyze the approximation rate we need the following auxiliary result.

**Lemma 2.4.** *Let  $X = (X_t)_{t \in [0,1]}$  be a process with*

$$dX_t = \mu_t dt + \vartheta_t dW_t, \quad X_0 = 0,$$

where  $\mu = (\mu_t)_{t \in [0,1]}$  and  $\vartheta = (\vartheta_t)_{t \in [0,1]}$  are predictable processes such that

$$\int_0^1 (|\mu_t| + \vartheta_t^2) dt < \infty.$$

Let  $X_t^n := \sum_{i=1}^n X_{t_{i-1}} I_{[t_{i-1}, t_i]}(t)$ . Then

$$\begin{aligned} E \left( \int_0^1 (X_t - X_t^n) dt \right)^2 &\leq 2 \int_0^1 \sum_{i=1}^n (t_i - u)^2 I_{[t_{i-1}, t_i]}(u) E \vartheta_u^2 du \\ &\quad + 2 \left( \int_0^1 \sum_{i=1}^n (t_i - u) I_{[t_{i-1}, t_i]}(u) (E \mu_u^2)^{1/2} du \right)^2. \end{aligned}$$

*Proof.* It is sufficient to work assuming that the right-hand side of the inequality is finite and consider separately the cases where one of the coefficients is zero. Let us start with the case where  $\mu = 0$ . Using the stochastic Fubini theorem, we have:

$$\int_{t_{i-1}}^{t_i} (X_t - X_{t_{i-1}}) dt = \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \vartheta_u I_{[t_{i-1}, t_i]}(u) dW_u dt = \int_{t_{i-1}}^{t_i} (t_i - u) \vartheta_u dW_u.$$

It follows that

$$E \left( \int_0^1 (X_t - X_t^n) dt \right)^2 = \int_0^1 \sum_{i=1}^n (t_i - u)^2 I_{[t_{i-1}, t_i]}(u) E \vartheta_u^2 du.$$

In the case where  $\vartheta = 0$  we have, this time by the ordinary Fubini theorem, that

$$\int_{t_{i-1}}^{t_i} (X_t - X_{t_{i-1}}) dt = \int_{t_{i-1}}^{t_i} (t_i - u) \mu_u du$$

and this representation allows us to transform the left-hand side of the required inequality to the following form:

$$\int_0^1 \int_0^1 \sum_{i,j=1}^n (t_i - u)(t_j - v) I_{[t_{i-1}, t_i]}(u) I_{[t_{j-1}, t_j]}(v) E\mu_u \mu_v dudv.$$

Using the Cauchy–Schwarz inequality  $E\mu_u \mu_v \leq (E\mu_u^2)^{1/2} (E\mu_v^2)^{1/2}$  and once again the Fubini theorem we obtain the needed bound.  $\square$

Note that

$$E(R_{21}^n)^2 = E\left(\int_0^{t_{n-1}} (X_t - X_t^n) dt\right)^2$$

where  $X_t := S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)}$  has the coefficients

$$\begin{aligned} \vartheta_t &= [2S_t \widehat{C}_{xx}(t, S_t) + S_t^2 \widehat{C}_{xxx}(t, S_t)] \sqrt{f'(t)} \sigma S_t, \\ \mu_t &= \frac{1}{2} [2\widehat{C}_{xx}(t, S_t) + 4S_t \widehat{C}_{xxx}(t, S_t) + S_t^2 \widehat{C}_{xxxx}(t, S_t)] \sqrt{f'(t)} \sigma^2 S_t^2 \\ &\quad + \frac{1}{2} S_t^2 \widehat{C}_{xx}(t, S_t) \frac{f''(t)}{\sqrt{f'(t)}} + S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)}. \end{aligned}$$

In the case where  $g'$  is bounded away from zero (hence,  $f'$  is bounded), the estimates (27) and (32) imply that  $E\vartheta_t^2 \leq \kappa/(1-t)^{3/2}$ . If also  $f''$  is bounded, then the estimates (27) and (32)–(34) ensure that  $E\mu_t^2 \leq \kappa/(1-t)^{5/2}$ .

Applying the previous lemma we have:

$$E(R_{21}^n)^2 \leq \kappa \sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{(1-t_i)^{3/2}} + \kappa \left( \sum_{i=1}^{n-1} \frac{(\Delta t_i)^2}{(1-t_i)^{5/4}} \right)^2.$$

According to Lemma 2.3 the right-hand side is  $O(n^{-3/2})$  as  $n \rightarrow \infty$ .

In the case where  $g(t) = 1 - (1-t)^\beta$ ,  $\beta > 1$ , we obtain in the same way that  $E\vartheta_t^2 \leq \kappa/(1-t)^{5/2-1/\beta}$ ,  $E\mu_t^2 \leq \kappa/(1-t)^{7/2-1/\beta}$ , and

$$E(R_{21}^n)^2 \leq \kappa \sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{(1-t_i)^{5/2-1/\beta}} + \kappa \left( \sum_{i=1}^{n-1} \frac{(\Delta t_i)^2}{(1-t_i)^{7/4-1/(2\beta)}} \right)^2.$$

By Lemma 2.3 the first sum in the right-hand side can be of order  $O(n^{-2})$ ,  $O(n^{-2} \ln n)$ , or  $O(n^{-(\beta/2+1)})$ , that is  $o(n^{-1})$  as  $n \rightarrow \infty$ . The second sum can be  $O(n^{-1})$ ,  $O(n^{-1} \ln n)$ , or  $O(n^{-(\beta/4+1/2)})$ , i.e.  $o(n^{-1/2})$ . In all cases  $nE(R_{21}^n)^2 \rightarrow 0$ .

**3.** The analysis of the term  $R_{22}^n$  is based on the first claim of Lemma 3.1 given in the section on asymptotics of Gaussian integrals.

We need to check that

$$E \left( \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 (S_{t_i}/S_{t_{i-1}} - 1)^2 \text{sign}(S_{t_i}/S_{t_{i-1}} - 1) \right)^2$$

tends to zero.

For the expectation of the sum of squared terms we have:

$$\sum_{i=1}^{n-1} \Lambda_{t_{i-1}} E(S_{t_i}/S_{t_{i-1}} - 1)^4 = O(n^{-1}),$$

since

$$E(e^{u\xi - \frac{1}{2}u^2} - 1)^4 = O(u^4), \quad u \rightarrow 0,$$

$\Lambda_t$  is an integrable function on  $[0, 1]$ , and  $g'(t)$  is bounded.

Let us consider a “generic” cross term with indices  $i < j$ . It can be split in the product of two independent random variables. The expectation of the first one,  $(S_{t_j}/S_{t_{j-1}} - 1)^2 \text{sign}(S_{t_j}/S_{t_{j-1}} - 1)$ , by virtue of Lemma 3.1 is dominated by  $\kappa \Delta t_j^{3/2}$  where  $\kappa$  is a constant. The second one is the product of  $\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 (S_{t_i}/S_{t_{i-1}} - 1)^2 \text{sign}(S_{t_i}/S_{t_{i-1}} - 1)$  and  $\widehat{C}_{xx}(t_{j-1}, S_{t_{j-1}}) S_{t_{j-1}}^2$  and we dominate the absolute value of its expectation using the Cauchy–Schwarz inequality which gives the following bound:

$$\Lambda_{t_{i-1}}^{1/2} (E(S_{t_i}/S_{t_{i-1}} - 1)^4)^{1/2} \Lambda_{t_{j-1}}^{1/2} \leq \kappa \Lambda_{t_{i-1}}^{1/2} \Lambda_{t_{j-1}}^{1/2} \Delta t_i.$$

Since the function  $\Lambda_t^{1/2}$  is integrable and  $g'$  is bounded, this implies that the sum of absolute values of the expectations of the cross terms decreases to zero as  $n^{-1/2}$  and, hence,  $n(R_{22}^n)^2 = O(n^{-1/2})$ .

**3.** We verify that  $nE(R_{23}^n)^2 \rightarrow 0$ . Recall that

$$E(S_{t_i} - S_{t_{i-1}})^{2m} \leq c_m (\Delta t_i)^m.$$

Using (31) we obtain the bound

$$E \widehat{C}_{xt}^2(t_{i-1}, S_{t_{i-1}}) (\Delta t_i)^2 (S_{t_i} - S_{t_{i-1}})^2 \leq c \frac{(\Delta t_i)^3}{(1 - t_{i-1})^{3/2}}.$$

To estimate the terms coming from the residual term of the Taylor expansion we use the Cauchy–Schwarz inequality and the bounds (22)–(24). This yields in the following:

$$\begin{aligned} E \widehat{C}_{xxx}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) (S_{t_i} - S_{t_{i-1}})^6 &\leq c \frac{(\Delta t_i)^3}{(1 - t_i)^2}, \\ E \widehat{C}_{xxt}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) (S_{t_i} - S_{t_{i-1}})^4 (\Delta t_i)^2 &\leq c \frac{(\Delta t_i)^4}{(1 - t_i)^3}, \\ E \widehat{C}_{xtt}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) (\Delta t_i)^4 (S_{t_i} - S_{t_{i-1}})^2 &\leq c \frac{(\Delta t_i)^5}{(1 - t_i)^4}. \end{aligned}$$

We dominate the  $L^2$ -norm of  $n^{1/2}R_{23}^n$  by the sum of the  $L^2$ -norm of the random variables  $[\dots]_i(S_{t_i} - S_{t_{i-1}})$ , where  $[\dots]_i$  is defined in (17). Taking into account that  $\widehat{C}_{xx}(t, x) > 0$  and using the inequality  $\| |a| - |b| \| \leq |a - b|$  we can write that

$$\|[\dots]_i(S_{t_i} - S_{t_{i-1}})\|_{L^2} \leq c \left( \|\widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}})(t_i - t_{i-1})(S_{t_i} - S_{t_{i-1}})\|_{L^2} + \dots \right)$$

where we denote by dots the  $L^2$ -norms of the residual term in the first order Taylor expansion of the difference  $\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})$ . Summing up and using the above estimates we conclude, applying Lemma 2.3, that the right-hand side of the above inequality tends to zero as  $n \rightarrow \infty$  and we conclude.

4. It remains to check that  $nE(R_{24}^n)^2 \rightarrow 0$  and this happens to be the most delicate part of the proof. The expression for  $nE(R_{24}^n)^2$  involves the sum of expectations of squared terms and the sum of expectations of cross terms which we analyze separately.

For the squared terms the arguments are relatively straightforward. We apply the Ito formula to the function  $\widehat{C}_x(t, x)$ . Using the positivity of  $\widehat{C}_{xx}(t, x)$  and the inequality  $\| |a| - |b| \| \leq |a - b|$  we dominate the absolute value of the square bracket  $[\dots]_i$  in the definition of  $R_{24}^n$ , given by the formula (17), by the absolute value of

$$\int_{t_{i-1}}^{t_i} (\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_{xx}(t, S_t)) dS_t - \int_{t_{i-1}}^{t_i} (\widehat{C}_{xt}(t, S_t) + \frac{\sigma^2}{2} S_t^2 \widehat{C}_{xxx}(t, S_t)) dt.$$

We check that

$$(38) \quad \sum_{i=1}^{n-1} ES_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_{xx}(t, S_t))^2 S_t^2 dt = O(n^{-1/4}),$$

$$(39) \quad \sum_{i=1}^{n-1} \Delta t_i ES_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (\widehat{C}_{xt}^2(t, S_t) + S_t^4 \widehat{C}_{xxx}^2(t, S_t)) dt = O(n^{-1/2}).$$

A generic term of the first sum is dominated by

$$\Delta t_i E \sup_{t \leq 1} S_t^4 \sup_{t_{i-1} \leq t \leq t_i} (\widehat{C}_{xx}(t, S_t) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}))^2.$$

The Cauchy–Schwarz inequality allows us to separate the terms under the sign of expectation and reduce the problem to the estimation of the fourth power of the difference  $\widehat{C}_{xx}(t, S_t) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})$ . The Ito formula transforms this difference into the sum of a stochastic integral and an ordinary integral. Using consecutively

the Burkholder and Cauchy–Schwarz inequalities and the bound (32) we have:

$$\begin{aligned} E \sup_{t \in [t_{i-1}, t_i]} \left[ \int_{t_{i-1}}^t \widehat{C}_{xxx}(u, S_u) S_u dS_u \right]^4 &\leq c_4 E \left[ \int_{t_{i-1}}^{t_i} \widehat{C}_{xxx}^2(u, S_u) S_u^4 du \right]^2 \\ &\leq c_4 \Delta t_i E \int_{t_{i-1}}^{t_i} \widehat{C}_{xxx}^4(u, S_u) S_u^8 du \\ &\leq c \frac{(\Delta t_i)^2}{(1 - t_i)^{7/2}}. \end{aligned}$$

To estimate the ordinary integral we use the Jensen inequality for  $f(x) = x^4$  and the bounds (33) and (34) and get that

$$E \sup_{t \in [t_{i-1}, t_i]} \left[ \int_{t_{i-1}}^t \left( \widehat{C}_{xx}(u, S_u) + \frac{1}{2} \sigma^2 S_u^2 \widehat{C}_{xxx}(u, S_u) \right) du \right]^4 \leq c \frac{(\Delta t_i)^4}{(1 - t_i)^{11/2}}.$$

Using these estimates we obtain that the sum in (38) is dominated, up to a multiplicative constant, by

$$\sum_{i=1}^{n-1} \left[ \frac{(\Delta t_i)^2}{(1 - t_i)^{7/4}} + \frac{(\Delta t_i)^3}{(1 - t_i)^{11/4}} \right]$$

and the claimed asymptotics follows from Lemma 2.3.

Finally, similar arguments using the inequalities (31) and (32) give us the second asymptotic formula.

From the same estimates we obtain that

$$\sum_{i=1}^{n-1} (ES_{t_{i-1}}^2 [\dots]_i^2)^{1/2} \leq c \sum_{i=1}^{n-1} \frac{\Delta t_i}{(1 - t_i)^{7/8}} + c \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/2}}{(1 - t_i)^{11/8}}.$$

The second sum in the right-hand side converges to zero while for the first one we can say only that it is dominated by a convergent integral. Using this observation we conclude that the sum of expectations of cross terms over indices  $i, j$  with  $i < j$  and  $t_j > a$  also can be done arbitrary small by choosing  $a$  sufficiently close to one.

Unexpectedly, the most difficult part of the proof is in establishing the convergence to zero of the sum of cross terms corresponding to the dates of revisions before  $a < 1$ , i.e. bounded away from the singularity.

Using the Taylor expansion we can reduce the problem to the case where the difference  $\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})$  is replaced by terms involving the derivatives  $C_{xx}(t_{i-1}, S_{t_{i-1}})$ ,  $C_{xt}(t_{i-1}, S_{t_{i-1}})$  and  $C_{xxx}(t_{i-1}, S_{t_{i-1}})$ .

To formulate the claim we introduce “reasonable” notations. Put

$$\begin{aligned} \alpha_i &:= \widehat{C}_{xx}(S_{t_{i-1}}, t_{i-1}) S_{t_{i-1}}^2 \left( \frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right), \\ \beta_i &:= S_{t_{i-1}} \widehat{C}_{xt}(S_{t_{i-1}}, t_{i-1}) \Delta t_i + \frac{1}{2} S_{t_{i-1}}^3 \widehat{C}_{xxx}(S_{t_{i-1}}, t_{i-1}) \left( \frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right)^2, \end{aligned}$$

$\gamma_i := |\alpha_i + \beta_i| - |\alpha_i|$ . Let us define also the random variable  $\chi_i := \text{sign}(\alpha_i \beta_i)$  and the set  $A_i := \{|\beta_i| < |\alpha_i|\}$ .

The assertion needed to conclude is the lemma below. It is based on asymptotic analysis of expectations of some Gaussian integrals which are given in the next section and the following identities:

$$\begin{aligned} |\alpha + \beta| - |\alpha| &= |\beta| \chi I_A + |\beta| I_{\{\chi > 0\}} I_{A^c} + (|\beta| - 2|\alpha|) I_{\{\chi \leq 0\}} I_{A^c} \\ &= |\beta| \chi + 2(|\beta| - |\alpha|) I_{\{\chi \leq 0\}} I_{A^c} - |\beta| I_{\{\chi = 0\}} I_{A^c} \end{aligned}$$

where  $\alpha, \beta$  are arbitrary random variables,  $\chi := \text{sign}(\alpha\beta)$ ,  $A := \{|\beta| < |\alpha|\}$ .

**Lemma 2.5.** For every fixed  $a \in ]0, 1[$

$$\left| \sum_{i < j, t_j \leq a} E \gamma_i \gamma_j \right| = o(1), \quad n \rightarrow \infty.$$

*Proof.* The routine estimation  $|E \gamma_i \gamma_j| \leq E |\gamma_i| |\gamma_j|$  does not work in our case. But for  $i < j$

$$|E \gamma_i \gamma_j| = |E(\gamma_i E(\gamma_j | \mathcal{F}_{t_{j-1}}))| \leq E(|\gamma_i| |E(\gamma_j | \mathcal{F}_{t_{j-1}})|) \leq E(|\beta_i| |E(\gamma_j | \mathcal{F}_{t_{j-1}})|).$$

According to the above identity,

$$|E(\gamma_j | \mathcal{F}_{t_{j-1}})| \leq |E(|\beta_j| \chi_j | \mathcal{F}_{t_{j-1}})| + 2E(|\beta_j| I_{A_j^c} | \mathcal{F}_{t_{j-1}}).$$

Using Lemma 3.2 of the next section with  $\eta_u = S_{t_j}/S_{t_{j-1}} - 1$ ,  $u = (\Delta t_j)^{1/2}$ , we dominate the first term in the right-hand side by

$$\kappa(S_{t_{j-1}} |\widehat{C}_{xt}(S_{t_{j-1}}, t_{j-1})| + S_{t_{j-1}}^3 |\widehat{C}_{xxx}(S_{t_{j-1}}, t_{j-1})|) (\Delta t_j)^{3/2}$$

It is easily seen from the explicit formulae that the coefficients above when  $t_j \leq a$  can be dominated uniformly by  $c_a(1 + \sup_{t \leq 1} S_t)$ , i.e. by a random variable having all moments. In the same range of indices we have the bound  $E(\beta_i^2 | \mathcal{F}_{t_{i-1}}) \leq \zeta_a (\Delta t_i)^2$  where  $\zeta_a$  a random variable having all moments. It follows from here that

$$\sum_{i < j, t_j \leq a} E(|\beta_i| |E(|\beta_j| \chi_j | \mathcal{F}_{t_{j-1}})|) = O(n^{-1/2}).$$

We estimate  $P(A_j^c | \mathcal{F}_{t_{j-1}})$  applying Lemma 3.3 of the next section with

$$c_1(t_{j-1}) := \frac{S_{t_{j-1}}^3 \widehat{C}_{xxx}(S_{t_{j-1}}, t_{j-1})}{S_{t_{j-1}}^2 \widehat{C}_{xx}(S_{t_{j-1}}, t_{j-1})}, \quad c_2(t_{j-1}) := \frac{S_{t_{j-1}} \widehat{C}_{xt}(S_{t_{j-1}}, t_{j-1})}{S_{t_{j-1}}^2 \widehat{C}_{xx}(S_{t_{j-1}}, t_{j-1})},$$

and  $c(t_{j-1}) := 2(|c_1(t_{j-1})| + |c_2(t_{j-1})| + 1)$ . On the interval  $[0, a]$  the continuous process  $c(t)$  can be dominated by a random variable  $\xi_a$ . Fix  $\varepsilon > 0$  and choose  $N$  such that  $P(\xi_a > N) < \varepsilon$ . Lemma 3.3 implies that

$$P(A_j^c | \mathcal{F}_{t_{j-1}}) \leq L_N (\Delta t_j)^{1/2} I_{\{c(t_{j-1}) \leq N\}} + I_{\{c(t_{j-1}) > N\}}$$

and, therefore,  $P(A_j^c) \leq L_N (\Delta t_j)^{1/2} + \varepsilon \leq 2\varepsilon$  when  $n$  is large enough. Using the Cauchy–Schwarz and Jensen inequalities we get that

$$\begin{aligned} \sum_{i < j, t_j \leq a} E(|\beta_i| |E(|\beta_j| I_{A_j^c} | \mathcal{F}_{t_{j-1}})|) &\leq \sum_{t_i \leq a} (E\beta_i^2)^{1/2} \sum_{t_j \leq a} (E\beta_j^4)^{1/4} (P(A_j^c))^{1/4} \\ &\leq (2\varepsilon)^{1/4} \sum_{t_i \leq a} (E\beta_i^2)^{1/2} \sum_{t_j \leq a} (E\beta_j^4)^{1/4}. \end{aligned}$$

Note that both sums in the right-hand side are bounded due to the inequalities  $E\beta_j^2 \leq \kappa(\Delta t_j)^2$  and  $E\beta_j^4 \leq \kappa(\Delta t_j)^4$ . By the choice of  $\varepsilon$  the right-hand side can be made arbitrarily small. Thus,  $nE(R_{24}^n)^2 \rightarrow 0$ .  $\square$

### 3. Asymptotics of Gaussian Integrals

Let  $\xi \in \mathcal{N}(0, 1)$  and let  $\eta_u := e^{u\xi - \frac{1}{2}u^2} - 1$ ,  $u \in [0, 1]$ .

**Lemma 3.1.** *The following asymptotical formulae holds as  $u \rightarrow 0$ :*

$$\begin{aligned} E[\eta_u^2 - \eta_{-u}^2] I_{\{\eta_u > 0\}} &= \frac{2}{\sqrt{2\pi}} u^3 + O(u^4), \\ E\eta_u^2 \text{sign } \eta_u &= \frac{2}{\sqrt{2\pi}} u^3 + O(u^4), \\ E \text{sign } \eta_u &= -\frac{1}{\sqrt{2\pi}} u + O(u^3). \end{aligned}$$

*Proof.* Put

$$Z(u) := (e^{u\xi - \frac{1}{2}u^2} - 1)^2 - (e^{-u\xi - \frac{1}{2}u^2} - 1)^2.$$

Then  $Z(0) = Z'(0) = Z''(0) = 0$ ,  $Z'''(0) = 12(\xi^3 - \xi)$ , and the function  $Z^{(4)}(u)$  is bounded by a random variable having moments of any order. Using the Taylor formula we obtain that

$$EZ(u) I_{\{\xi \geq \frac{1}{2}u\}} = 2u^3 E(\xi^3 - \xi) I_{\{\xi \geq \frac{1}{2}u\}} + O(u^4), \quad u \rightarrow 0,$$

and we obtain the first formula. The second formula is a corollary of the first one since

$$E\eta_u^2 \text{sign } \eta_u = EZ(u) I_{\{\xi \geq \frac{1}{2}u\}} - E\eta_u^2 I_{\{|\xi| \leq \frac{1}{2}u\}}$$

and the last term is  $O(u^4)$  as  $u \rightarrow 0$ . Finally,

$$\begin{aligned} E \operatorname{sign} \eta_u &= P(\xi > u/2) - P(\xi < u/2) = 2(\Phi(0) - \Phi(u/2)) \\ &= -\frac{1}{\sqrt{2\pi}}u + \frac{1}{4}\varphi(\tilde{u})\tilde{u}u^2, \end{aligned}$$

where  $\tilde{u} \in [0, u/2]$ .  $\square$

**Lemma 3.2.** *There exists a constant  $\kappa$  such that for any real  $A$*

$$(40) \quad |E[\eta_u^2 - Au^2] \operatorname{sign}(\eta_u^2 - Au^2)\eta_u| \leq \kappa(1 + |A|)u^3.$$

*Proof.* Note that  $|x| \operatorname{sign} xy = x \operatorname{sign} y$ . Therefore the left-hand side of (40) is dominated by

$$|E\eta_u^2 \operatorname{sign} \eta_u| + |A|u^2 |E \operatorname{sign} \eta_u|$$

and the result holds by virtue of the previous lemma.  $\square$

**Lemma 3.3.** *For every  $N > 0$  there is a constant  $L_N$  such that for all  $u \in [0, 1]$*

$$P(|c_1\eta_u^2 + c_2u^2| > |\eta_u|) \leq L_N I_{\{c \leq N\}}u + I_{\{c > N\}}.$$

for any constants  $c_1, c_2$  and  $c := 2(|c_1| + |c_2| + 1)$ .

*Proof.* Suppose that  $N \geq c > 2$ , the only case where the work is needed. It is easy to see that

$$\begin{aligned} P(|c_1\eta_u^2 + c_2u^2| > |\eta_u|) &\leq P((c/2)\eta_u^2 + (c/2)u^2 > |\eta_u|) \\ &\leq P(c|\eta_u| > 1) + P(|\eta_u| < cu^2). \end{aligned}$$

The probabilities in the right-hand side as functions of  $c$  are increasing and it remains to dominate their values at the point  $c = N$ . The required bound holds for the first probability in the right-hand side (and even with a constant which does not depend on  $N$ ). Indeed, using the Chebyshev inequality, finite increments formula, and the bound  $\varphi(x) \leq 1/\sqrt{2\pi}$  we have:

$$P(N|\eta_u| > 1) \leq \frac{1}{N} E|\eta_u| \leq \frac{1}{2} E|\eta_u| = \Phi(u/2) - \Phi(-u/2) \leq \frac{1}{\sqrt{2\pi}}u.$$

For  $u \geq 1/\sqrt{2N}$  the second probability is dominated by linear functions with  $L_N \geq \sqrt{2N}$ . For  $u < 1/\sqrt{2N}$  we write it as

$$P(u/2 \leq \xi < (1/u) \ln(1 + Nu^2) + u/2) + P((1/u) \ln(1 - Nu^2) + u/2 < \xi < u/2).$$

Using again the finite increments formula we obtain that

$$P(u/2 \leq \xi < (1/u) \ln(1 + Nu^2) + u/2) \leq \frac{1}{\sqrt{2\pi}}Nu.$$



On the interval  $]0, 1/\sqrt{2N}[$  we have the bound  $(1/u)\ln(1 - Nu^2) \geq -\kappa Nu$  where  $\kappa > 0$  is the maximum of the function  $-\ln(1 - x)/x$  on the interval  $]0, 1/2]$ . It follows that

$$P((1/u)\ln(1 - Nu^2) + u/2 < \xi < u/2) \leq \frac{1}{\sqrt{2\pi}}\kappa Nu.$$

Thus, the second probability also admits a linear majorant on the whole interval  $[0, 1]$ .  $\square$

### Acknowledgement

The authors express their thanks to Emmanuel Denis and anonymous referee for constructive criticism and helpful suggestions.

### References

1. Geiss S. Quantitative approximation of certain stochastic integrals. *Stochastics and Stochastic Reports*, **73** (2002), 3-4, 241–270.
2. Granditz P., Schachinger W. Leland's approach to option pricing: The evolution of discontinuity. *Mathematical Finance*, **11** (2001), 3, 347–355.
3. Grannan E.R., Swindle G.H. Minimizing transaction costs of hedging strategies. *Mathematical Finance*, **6** (1996), 4, 341–364.
4. Denis E., Kabanov Yu.M. Functional limit theorem for Leland–Lott hedging strategy. Working paper.
5. Kabanov Yu.M., Safarian M. On Leland's strategy of option pricing with transaction costs. *Finance and Stochastics*, **1**, 3, 1997, 239–250.
6. Leland H. Option pricing and replication with transactions costs, *Journal of Finance*, **XL** (1985), 5, 1283–1301.
7. Lott K. Ein Verfahren zur Replikation von Optionen unter Transaktionskosten in stetiger Zeit, Dissertation. Universität der Bundeswehr München. Institut für Mathematik und Datenverarbeitung, 1993.
8. Pergamenschikov S. Limit theorem for Leland's strategy. *The Annals of Applied Probability*, **13** (2003), 1099–1118.
9. Sekine J., Yano J. Hedging errors of Leland's strategies with time-inhomogeneous rebalancing. Preprint.
10. Zariphopoulou, T. Stochastic control methods in asset pricing. In: *Handbook of Stochastic Analysis and Applications*, Eds. Kannan D. and Lakshmikantham V. Marcel Dekker, New York–Basel, 2002.