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No-Arbitrage Criteria for Financial Markets with Transaction Costs and Incomplete Information

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Abstract This note deals with criteria of absence of arbitrage opportunities for an investor acting in a market with friction and having a limited access to the information flow. We develop a mathematical scheme covering major models of financial markets with transaction costs and prove several results including a criterion for the robust no-arbitrage property and a hedging theorem.

Key words: transaction costs, incomplete information, arbitrage, hedging.

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1 Introduction

In the classical arbitrage theory it is usually assumed that the investor makes his decisions using all market information and the majority of no-arbitrage criteria are developed in this framework. However, even though there is a vast amount of information available, an investor may base his decision only on a part of this information. On the other hand, mathematically, such an important feature as partial information used in the investor's decisions can be easily modelled, namely, by a subfiltration $\mathbf{G} = (G_t)$ of the main filtration $\mathbf{F} = (\mathcal{F}_t)$ describing the information flow. What are consequences of such modelling for the arbitrage theory?

Until recently, the only result in this more general framework was an extension of the Dalang–Morton–Willinger theorem for the model of the frictionless financial market in discrete-time given in the paper [7]. It happens that the no arbitrage property (shortly, NA-property) for the price process S holds if and only if there is a bounded strictly positive $\mathbf{F}$-martingale ρ such that the optional projection $(\rho S)^o$ is a $\mathbf{G}$-martingale, i.e. $\tilde{E}(S_{t+1}-S_t|\mathcal{G}_t)=0$ where $\tilde{P}:=\rho_T P$. On the other hand, the "global" (multi-step) NA-property is no longer equivalent to the NA-properties for all one-step sub-models. This explains why such a natural generalization was not obtained earlier: all proofs (of the"only if" part) except that given in [6] use a reduction to the one-step case.

The study of no-arbitrage properties for markets with friction was initiated by Jouini and Kallal [3] for a model with bid-ask spread and developed further in a number of papers: [4], [8], [5], [2] and others. There are several concepts of the no-arbitrage property. Equivalent conditions for them can be formulated in terms of the existence of martingales evolving in the duals to solvency cones (in the space used to represent the investor's positions in physical units) or in the interiors of these duals.

It is natural to consider as an arbitrage opportunity a self-financing portfolio strategy (with zero initial capital) yielding a positive outcome on a set of positive probability with no losses elsewhere. The absence of such "strict" arbitrage opportunities, i.e. the relation $\widehat{R}_T\cap L^0(\mathbf{R}^d_+)=\{0\}$ where $\widehat{R}_T$ is the set of the terminal values of portfolios, is called *weak no-arbitrage property* (shortly, NA^w-property). For the case of finite Ω the criterion for the NA^w-property was obtained in [4]: the latter holds if and only if there is a martingale evolving in the duals to solvency cones. For general Ω this equivalence holds only for the two-asset model, see [2]. An evaluation of the portfolio results without taking into account the transaction costs (as could be done by auditors) leads to a larger set of weak arbitrage opportunities. Their absence is referred to as *strict no-arbitrage property*, NA^s. In the case of arbitrary Ω and "efficient friction", i.e. non-emptiness of the interiors of dual cones, NA^s is equivalent to the existence of a martingale evolving in these interiors, see [5]. Without further assumptions, as was shown first in [8], the existence of a martingale evolving in relative interiors of duals to the solvency cones is equivalent to the so-called *robust no-arbitrage property*, NA^r. The latter means that there are no-arbitrage opportunities in strict sense even for smaller transaction costs.

The setting of market models with friction where the investor's information may be different from that given by the main filtration was investigated by Bruno Bouchard [1] who discovered some new phenomena. He showed that models with transaction costs and partial information not only necessitate important changes in the description of value processes but also appropriate modifications of the basic concepts. In particular, one cannot work on the level of portfolio positions, represented by a point in $\mathbf{R}^d$, but has to remain on the primary level, of the investor's decisions (orders), i.e. in a space of much higher dimension. In the model with partial information there is a difference between the investor's orders "exchange 1000 dollars for euros" and "exchange dollars to increase the holding in euros by 1000 euros": they are of a different nature. For the first type of orders the investor controls the decrease of the dollar holdings (hence, his debts), while for the second type, due to limited information, he may have no idea what is the resulting value of the eventually short position in dollars.

The model of [1] can be classified as that of a barter market but it covers also the case of the model with a numéraire by introducing auxiliary "fictive" assets. Bouchard suggested the coding of orders by real-valued $d \times d$-matrices (with zero diagonal) where the sign of each entry serves as in indicator of the order type ("to send" or "to get" an increment). His main result is the criterion for the NA^r-property of the market for a partially informed investor. It is necessary to recall that in models with full information there is no difference between "barter markets" and "financial markets". In the theory of markets with transaction costs it happens that it is much easier to analyze models where holdings are expressed in terms of physical units rather than in units of a numéraire. In the development of this idea in some recent papers, e.g., [8] and [2] the initial set-up is that of a "barter market", i.e. "conversion" matrices (π_t^{ij}) are specified. This is by no means a restriction: in the models with full information one can always construct prices S_t and matrices λ_t^{ij} of transaction costs coefficients (of course, not uniquely). However, the setting based on prices and transaction costs coefficients may lead to an information structure which seems not to be covered by models based on conversion matrices.

The aim of the present work is to simplify and extend the approach of [1] to include explicitly models with a numéraire. To this end we use an alternative coding of the investor's order and enjoy from the very beginning the linear structure of the problem which leads to a more transparent presentation. Our results include a criterion for the NA^w-property for the case of finite Ω (extending the criterion of [4]) and the criteria for the NA^r-property which is a generalization of those of [8] and [5]. We conclude with a version of the hedging theorem for the situation with partial information.

One should take into account that we are dealing here with a highly stylized mathematical model where orders should be executed, independently of the realized price movements. This means that we are working within the framework of linear control system. The practical situations might be much more complicated and depend on a market microstructure. Certain financial markets are organized as auctions where investors indicate reservation prices, when selling, and limit prices, when buying. A trading system equilibrates the supply and demand, generating asset prices. For example, during the following trading cycle, an order to buy may not be executed or executed partially if the price goes up above the limit price. Of course, an analysis of models incorporating such features as liquidity constraints, constrained orders etc. is of great interest and could be a subject of further studies.

Comment on notations: as usual $L^0(\mathbf{R}_+, \mathcal{F}_t)$ is the set of positive $\mathcal{F}_t$-measurable random variables (note that we prefer to say "positive" rather than "nonnegative") and, consistently, $\mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ stands for the set of martingales with strictly positive components; $\mathbf{1} := \sum e_i = (1, ..., 1)$.

2 Examples and mathematical framework

Example 1. Let us consider the barter market which is described by an $\mathbf{F}$-measurable conversion ("bid-ask") process $\Pi = (\pi^{ij}_t)$ taking values in the set of strictly positive $d \times d$ matrices such that $\pi^{ij}_t \pi^{ji}_t \geq 1$. The entry π^{ij}_t stands for a number of units of the ith asset needed to exchange, at time t, for one unit of the jth asset. The above inequality means that exchanging one unit of the ith asset for $1/\pi^{ij}_t$ units of the jth asset with simultaneous exchange back of the latter quantity results in decreasing of the ith position.

In the case of fully informed investor, the portfolio process is generated by an $\mathbf{F}$-adapted process (η^{ij}_t) with values in the set $\mathbf{M}^d_+$ of positive $d \times d$ matrices; the entry $\eta^{ij}_t \geq 0$ is the investor's order to increase the position j on η^{ij}_t units by converting a certain number of units of the ith asset. The investor has a precise idea about this "certain number": it is $\pi^{ij}_t \eta^{ij}_t$. The situation is radically different when the information available is given by a smaller filtration $\mathbf{G}$, i.e. η^{ij}_t is only $\mathcal{G}_t$-measurable. The decrease of the i-th asset implied by such an order, being $\mathcal{F}_t$-measurable, is unknown to the investor. However, one can easily imagine a situation where the latter is willing to control the lower level of investments in some assets in his portfolio. This can be done by using the $\mathbf{G}$-adapted order process $(\tilde{\eta}^{ij}_t)$ with the element $\tilde{\eta}^{ij}_t$ representing the number of units of the ith asset to be exchanged for the jth asset – the result of this transaction yields an increase of the jth position in $\tilde{\eta}^{ij}_t / \pi^{ij}_t$ units and, in general, now this quantity is unknown to the investor at time t. Of course, orders of both types, "to get", "to send", can be used simultaneously. In other words, the investor's orders form a $\mathbf{G}$-adapted process $[(\eta^{ij}_t), (\tilde{\eta}^{ij}_t)]$ taking values in the set of positive rectangular matrices $\mathbf{M}^{d\times 2d}_+ = \mathbf{M}^d_+ \times \mathbf{M}^d_+$. The dynamics of the portfolio processes is given by the formula

$$\Delta \widehat{V}_t = \widehat{\Delta B}^1_t + \widehat{\Delta B}^2_t, \tag{2.1}$$

where the coordinates of $\widehat{\Delta B}^1_t$ and $\widehat{\Delta B}^2_t$ are

$$\widehat{\Delta B}^{1,i}_t := \sum_{j=1}^d [\eta^{ji}_t - \pi^{ij}_t \eta^{ij}_t],$$

$$\widehat{\Delta B}^{2,i}_t := \sum_{j=1}^d [\tilde{\eta}^{ji}_t / \pi^{ji}_t - \tilde{\eta}^{ij}_t].$$

Let $(e^{ij}) \in \mathbf{M}^d_+$ be a matrix with all zero entries except the entry (i,j) which is equal to unity. The union of the elementary orders $[(e^{ij}), 0]$ and

$[0,(e^{ji})]$ forms a basis in $\mathbf{M}^{d\times 2d}$. The execution of the order $[(e^{ij}),(e^{ji})]$ (buying a unit of the jth asset in exchange for the ith asset and then exchanging it back) leads to a certain loss in the ith position while others remain unchanged, i.e. $\Delta\widehat{V}_t^i\le 0$, $\Delta\widehat{V}_t^j=0$, $j\ne i$. This observation will be used further, in the analysis of the NA^r property.

Example 2. Let us turn back to our basic model which is defined by a price process $S=(S_t)$ (describing the evolution of prices of units of assets in terms of some numéraire, e.g., the euro) and an $\mathbf{M}_+^d$-valued process $\Lambda=(\lambda_t^{ij})$ of transaction costs coefficients. This model admits a formulation in terms of portfolio positions in physical units: one can introduce the matrix Π by setting

$$\pi_t^{ij}=(1+\lambda_t^{ij})S_t^j/S_t^i,\qquad 1\le i,j\le d.$$

In the full information case the difference between two models is only in parametrizations: one can introduce in the barter market "money" by taking as the price process S an arbitrary one evolving in the duals to the solvency cones and non-vanishing and defining λ_t^{ij} from the above relations. On the other hand, from the perspective of partial information, the setting based on price quotes is more flexible and provides a wider range of possible generalizations.

Again, assume that the investor's information is described by a smaller filtration $\mathbf{G}$ while S and Λ are $\mathbf{F}$-adapted (note that these processes may be adapted with respect to different filtrations).

In contrast to the barter market, the investor now may communicate orders of four types: in addition to the orders (η_t^{ij}) and $(\tilde\eta_t^{ij})$ one can imagine also similar orders, "to get", "to send", but expressed in units of the numéraire and given by $\mathbf{G}$-adapted matrix-valued processes (α_t^{ij}) and $(\tilde\alpha_t^{ij})$ with positive components. The entry α_t^{ij} is the increment of value in the position j due to diminishing the position i, while the entry $\tilde\alpha_t^{ij}$ is a value of the ith asset ordered to be exchanged for the jth asset.

The dynamics of value processes in such a model, in physical units, is given by the formula

$$\Delta\widehat{V}_t=\widehat{\Delta B}_t^1+\widehat{\Delta B}_t^2+\widehat{\Delta B}_t^3+\widehat{\Delta B}_t^4, \tag{2.2}$$

where $\widehat{\Delta B}_t^{3,i}:=\Delta B_t^{3,i}/S_t^i$, $\widehat{\Delta B}_t^{4,i}:=\Delta B_t^{4,i}/S_t^i$ with

$$\Delta B_t^{3,i}:=\sum_{j=1}^d\alpha_t^{ji}-\sum_{j=1}^d(1+\lambda_t^{ij})\alpha_t^{ij},$$

$$\Delta B_t^{4,i}:=\sum_{j=1}^d\frac{\tilde\alpha_t^{ji}}{1+\lambda_t^{ji}}-\sum_{j=1}^d\tilde\alpha_t^{ij}.$$

Of course, in this case the dynamics can be expressed also in values, that is in units of the numéraire (using the relation $X^i=\widehat{X}^iS^i$).

Thus, in both cases the set of "results" (for portfolios with zero initial endowments) consists of the d-dimensional random variables

$$\xi = \sum_{t=0}^{T} \mathcal{L}_t \zeta_t, \qquad \zeta_t \in O_t := L^0(\mathbf{M}_+^{d\times m}, \mathcal{G}_t), \tag{2.3}$$

where m is either $2d$ or $4d$ and $\mathcal{L}_{\omega,t} : \mathbf{M}^{d\times m} \to \mathbf{R}^d$ are linear operators such that the mappings $\omega \mapsto \mathcal{L}_{\omega,t}$ are measurable with respect to the σ-algebra $\mathcal{F}_t$. We shall denote this set $\widehat{R}_T$ or, when needed, $\widehat{R}_T(\mathcal{L})$ to show the dependence on the defining operator-valued random process. As usual, we define the set of hedgeable claims $\widehat{A}_T(\mathcal{L}) := \widehat{R}_T(\mathcal{L}) - L^0(\mathbf{R}^d_+)$.

Let us associate with the random linear operator $\mathcal{L}_t$ (acting on elements of $\mathbf{M}^{d\times m}$) the linear operator $\mathbf{L}_t$ acting on $\mathbf{M}^{d\times m}$-valued random variables, $\mathbf{L}_t : L^0(\mathbf{M}^{d\times m}, \mathcal{G}_t) \to L^0(\mathbf{R}^d, \mathcal{F}_t)$, by setting $(\mathbf{L}_t\zeta)(\omega) = \mathcal{L}_{\omega,t}\zeta(\omega)$. With this notation,

$$\widehat{R}_T = \sum_{t=0}^{T} \mathbf{L}_t(O_t).$$

Sometimes, it is convenient to view $\mathbf{M}^{d\times m}$ as the set of linear operators defined by the corresponding matrices.

Unlike the case of a frictionless market the set $\widehat{R}_T$, in general, is not closed even for models with full information: see Example 1.3 in [2] (due to M. Ràsonyi) where the set $\widehat{R}_1 = \widehat{A}_1$ is not closed though the NA^w-condition is satisfied. However, as in the case of models with full information, we have the following result. We comment on its proof in the subsequent remark.

Proposition 2.1 *The sets $\mathbf{L}_t(O_t)$ are closed in probability.*

Proof. The arguments being standard, we only sketch them. In a slightly more general setting, consider a sequence of random vectors $\zeta^n = \sum_{i=1}^N c_i^n g_i$ in a finite-dimensional Euclidean space where g_i are $\mathcal{G}$-measurable random vectors and $c_i^n \in L^0_+(\mathcal{G})$. Let $\mathcal{L}$ be an $\mathcal{F}$-measurable random linear operator. Knowing that the sequence $\xi^n = \mathcal{L}\zeta^n$ converges to ξ, we want to show that $\xi = \mathcal{L}\zeta$ for some $\zeta = \sum_{i=1}^N c_i g_i$. Supposing that the result holds for $N-1$ (for $N=1$ it is obvious), we extend it to N. Indeed, it is easy to see, recalling the lemma on random subsequences[1], that we may assume without loss of generality that all sequences c_i^n converge to infinity and, moreover, the normalized sequences $\tilde{c}_i^n := c_i^n/|c^n|$, where $|c^n|$ is the sum of c_i^n, converge to some $\mathcal{G}$-measurable random variables $\tilde{c}_i$. For the random vector $\tilde{\zeta} := \sum_{i=1}^N \tilde{c}_i g_i$ we have that $\mathcal{L}\tilde{\zeta} = 0$. Put $\alpha^n := \min_i\{c_i^n/\tilde{c}_i : \tilde{c}_i > 0\}$. Note that $\bar{c}_i^n := c_i^n - \alpha^n \tilde{c}_i \geq 0$ and, for each ω, at least one of $\bar{c}_i^n(\omega)$ vanishes. For $\bar{\zeta}^n = \sum_{i=1}^N \bar{c}_i^n g_i$ we have that $\mathcal{L}\bar{\zeta}^n$ also tends to ξ. Considering the partition of Ω by disjoint $\mathcal{G}$-measurable subsets Γ_i constructed from the

[1] For any sequence of $\mathbf{R}^d$-valued random variables $\{\eta_n\}$ with $\liminf_n |\eta_n| < \infty$ one can find a sequence of random variables $\{\eta'_n\}$ such that $\{\eta'_n(\omega)\}$ is a convergent subsequence of $\{\eta_n(\omega)\}$ for almost all ω, see [6].

covering of Ω by sets $\{\liminf_n \bar{c}_i^n = 0\}$ and replacing on Γ_i the coefficients $\bar{c}_i^n$ by zero (without affecting the limit ξ), we obtain a reduction to the case with $N-1$ generators. $\Box$

Remark 2.1 We give the above assertion by methodological reasons, as a case study explaining the basic ideas and techniques. Though, formally, this result of independent interest will not be used in the sequel we recommend to the reader to make efforts to understand its proof. Its first idea is that we can consider a $\mathcal{G}$-measurable partition of Ω and prove the result separately for each elements of the partition. That is why we start with a two-element partition Ω_0, Ω_0^c such that on Ω_0 the result is obvious because, by virtue of the lemma on subsequences we can replace the initial sequence c^n by a convergent one defining the required representation for the limit. On Ω_0^c we can normalize the sequence and, using again the lemma on subsequences, obtain an identity which allows us to reduce the dimensionality of the problem (the number of generators in the considered case). The dimension reduction, resembling the Gauss algorithm of solving linear systems, can be done separately on elements of a subpartition. This type of reasoning, explained in details in [6], was used repeatedly in many proofs, and became standard. For multiperiod results the Gauss-type algorithm is imbedded in an induction in the number of periods and looks more involved but the principle remains the same. That is why we opt to present it in the case of a one-step assertion.

3 No Arbitrage Criteria: Finite Ω

The definition of the NA^w-property remains the same as in the model with full information: $\widehat{R}_T \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$ or $\widehat{A}_T \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$.

As always, criteria in the case of finite Ω are easy to establish using the finite-dimensional separation theorem.

Proposition 3.1 *Let Ω be finite. The following conditions are equivalent:*

(a) NA^w;

(b) there exists $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$.

Proof. $(a) \Rightarrow (b)$ Note that $\widehat{A}_T$ is a finite-dimensional polyhedral (thus, closed) cone containing $-L^0(\mathbf{R}^d_+)$. The NA^w-property implies that nonzero elements of $L^0(\mathbf{R}^d_+)$ can be separated from $\widehat{A}_T$ in a strict sense. Using a classical argument, we construct an $\mathbf{F}$-martingale $Z = (Z_t)$ with strictly positive components such that $EZ_T\xi \le 0$ for every $\xi \in \widehat{A}_T$. Namely, we can take Z_T equal to the sum of functionals negative on $\widehat{A}_T$ and strictly positive on e_iI_Γ with the summation index Γ running through the family of atoms of $\mathcal{F}_T$ and $i = 1,2,...,d$. It follows that $E(Z_t\mathcal{L}_t\zeta_t) \le 0$ for any $\zeta_t \in O_t$, implying the assertion.

$(b) \Rightarrow (a)$ This implication is obvious because for ζ admitting the representation (2.3) we have that

$$EZ_T\xi = \sum_{t=0}^{T} E[E(Z_t\mathcal{L}_t\zeta_t|\mathcal{G}_t)] \le 0$$

and, therefore, ξ cannot be an element of $L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ other than zero.

As we know, even in the case of full information, a straightforward generalization of the above criterion to an arbitrary Ω fails to be true, see [8], [2]. To get "satisfactory" theorems one needs either to impose extra assumptions, or to modify the concept of absence of arbitrage. We investigate here an analog of the NA^r-condition starting from the simple case when Ω is finite.

First, we establish a simple lemma which holds in a "very abstract" setting where the word "premodel" instead of "model" means that we do not suggest any particular properties of $(\mathcal{L}_t)$.

Fix a subset $\mathcal{I}_t$ of O_t. The elements of $\mathcal{I}_t$ will be interpreted later, in a more specific "financial" framework, as the reversible orders.

We say that the premodel has the *NA^r-property* if the NA^w-property holds for the premodel based on an $\mathbf{F}$-adapted process $\mathcal{L}' = (\mathcal{L}'_t)$ such that

(i) $\mathcal{L}'_t\zeta \ge \mathcal{L}_t\zeta$ componentwise for every $\zeta \in O_t$;

(ii) $\mathbf{1}\mathcal{L}'_t\zeta \ne \mathbf{1}\mathcal{L}_t\zeta$ if $\zeta \in O_t \setminus \mathcal{I}_t$ (i.e. the above inequality is not identity).

Lemma 3.2 *Let Ω be finite. If a premodel has the NA^r-property, then there is a process $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t \setminus \mathcal{I}_t$,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t. \tag{3.1}$$

Proof. According to Proposition 3.1 applied to the premodel based on the process $\mathcal{L}'$ from the definition of NA^r there exists $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}'_t\zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$. Hence, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ by virtue of (i). Again by (i) we have, for $\zeta \in O_t \setminus \mathcal{I}_t$, that

$$Z_t\mathcal{L}'_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \ge Z_t\mathcal{L}_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}.$$

If the order $\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}$ is not in $\mathcal{I}_t$, this inequality is strict on a non-null set. Thus, taking the expectation, we obtain

$$EZ_t\mathcal{L}'_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} > 0$$

which is contradiction. □

Now we give a precise meaning to the word "model" by imposing an **assumption** on the generating process (fulfilled in both our examples) and specifying the sets $\mathcal{I}_t$.

Namely, we suppose that in $\mathbf{M}^{d\times m}$ there is a basis formed by the union of two families of vectors $\{f_i\}$ and $\{\tilde{f}_i\}$, $1 \le i \le md/2$, belonging to $\mathbf{M}^{d\times m}_+$ and such that componentwise

$$\mathcal{L}_t f_i + \mathcal{L}_t \tilde{f}_i \le 0, \tag{3.2}$$

while $\mathcal{I}_t$ is the cone of (matrix-valued) random variables having the form $\sum_i(\eta_i f_i+\tilde\eta_i\tilde f_i)$ with $\eta_i,\tilde\eta_i\in L^0_+(\mathcal{G}_t)$ and such that $\mathcal{L}_t\sum_i(\eta_i+\tilde\eta_i)(f_i+\tilde f_i)=0$.

Note that the latter equality implies that $\mathbf{L}_t(\mathcal{I}_t)\subseteq\mathbf{L}_t(O_t)\cap(-\mathbf{L}_t(O_t))$. It is clear that the set $\mathcal{I}_t$ is stable under multiplication by elements of $L^0(\mathbf{R}_+,\mathcal{G}_t)$. This implies that the equality (3.1) for $\zeta\in\mathcal{I}_t$ always holds (cf. the formulations of Lemma 3.2 and the theorems below).

The inequality (3.2) means that the elementary transfers in opposite directions cannot lead to gains. The orders from $\mathcal{I}_t$, even symmetrized, do not incur losses.

For the models, in the definition of the NA^r the words "premodel" are replaced by "models", i.e. we require that the property (3.2) should hold also for the dominating process $\mathcal{L}'$.

Theorem 3.3 *Let Ω be finite. Then the following properties of the model are equivalent:*

(a) *NA^r;*

(b) *there is $Z\in\mathcal{M}(\operatorname{int}\mathbf{R}^d_+,\mathbf{F})$ such that $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)\le 0$ for every $\zeta\in O_t$ and, if $\zeta\in O_t$,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}\in\mathcal{I}_t.$$

Proof. To check the remaining implication $(b)\Rightarrow(a)$ we put $\mathcal{L}'_t\zeta:=\mathcal{L}_t\zeta-\bar{\mathcal{L}}_t\zeta$ defining the action of $\bar{\mathcal{L}}_t$ on the element $\zeta=\sum_i(\eta_i f_i+\tilde\eta_i\tilde f_i)$ by the formula $\bar{\mathcal{L}}_t\zeta:=\sum_i(\eta_i+\tilde\eta_i)\theta_i$ where $\theta_i=\theta_i(t)$ has the components

$$\theta_i^k:=\max\left\{\frac12[\mathcal{L}_t(f_i+\tilde f_i)]^k,\frac1d\frac{E(Z_t\mathcal{L}_tf_i|\mathcal{G}_t)}{E(Z_t^k|\mathcal{G}_t)},\frac1d\frac{E(Z_t\mathcal{L}_t\tilde f_i|\mathcal{G}_t)}{E(Z_t^k|\mathcal{G}_t)}\right\}.$$

The values $\theta_i^k(t)$ being negative, the condition (i) holds. The inequality (3.2) for $\mathcal{L}'_t$ is obviously fulfilled due to the first term in the definition of $\theta_i^k(t)$. Now let ζ be an element of $O_t\setminus\mathcal{I}_t$. This means that for some k and i the set

$$\Gamma:=\{(\eta_i+\tilde\eta_i)[\mathcal{L}_t(f_i+\tilde f_i)]^k<0\}=\{(\eta_i+\tilde\eta_i)Z_t^k[\mathcal{L}_t(f_i+\tilde f_i)]^k<0\}$$

is non-null. From elementary properties of conditional expectations it follows that $(\eta_i+\tilde\eta_i)E(Z_t^k[\mathcal{L}_t(f_i+\tilde f_i)]^k|\mathcal{G}_t)<0$ on Γ. The property (ii) holds because on Γ both $E(Z_t\mathcal{L}_tf_i|\mathcal{G}_t)$ and $E(Z_t\mathcal{L}_t\tilde f_i|\mathcal{G}_t)$ are strictly negative as follows from the coincidence of sets

$$\{E(Z_t\mathcal{L}_tf_i|\mathcal{G}_t)<0\}=\{E(Z_t\mathcal{L}_t\tilde f_i)|\mathcal{G}_t)<0\}=\{E(Z_t\mathcal{L}_t(f_i+\tilde f_i)|\mathcal{G}_t)<0\}$$

which can be established easily. Indeed, $f_iI_{\{E(Z_t\mathcal{L}_tf_i|\mathcal{G}_t)=0\}}\in\mathcal{I}_t$ and, by definition of $\mathcal{I}_t$,

$$I_{\{E(Z_t\mathcal{L}_tf_i|\mathcal{G}_t)=0\}}\mathcal{L}_t\tilde f_i=-I_{\{E(Z_t\mathcal{L}_tf_i|\mathcal{G}_t)=0\}}\mathcal{L}_tf_i.$$

Multiplying this identity by Z_t and taking the conditional expectation with respect to $\mathcal{G}_t$ we get that

$$I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)=0.$$

Similarly,

$$I_{\{E(Z_t\mathcal{L}_t \tilde{f}_i|\mathcal{G}_t)=0\}}E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0.$$

These two equalities imply the coincidence of sets where the conditional expectations (always negative) are zero, i.e. the required assertion.

Finally, we check the NA^w-property of $(\mathcal{L}'_t)$ using Proposition 3.1. For any $\zeta=\sum_i(\eta_i f_i+\tilde{\eta}_i\tilde{f}_i)$ from O_t we have:

$$\begin{aligned} E(Z_t\mathcal{L}'_t\zeta|\mathcal{G}_t) &= E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)-E\Big(\sum_i(\eta_i+\tilde{\eta}_i)\sum_{k=1}^d Z_t^k\theta_i^k\Big|\mathcal{G}_t\Big)\\ &\le E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)-\sum_i \eta_i E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)-\sum_i\tilde{\eta}_i E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)=0. \end{aligned}$$

It follows that $EZ_T\xi\le 0$ for every $\xi\in\widehat{R}_T(\mathcal{L}')\cap L^0(\mathbf{R}^d_+)$, excluding arbitrage opportunities for the model based on $\mathcal{L}'$.

The theorem is proven. □

Remark 3.1 One might find it convenient to view $\mathbf{M}^{d\times m}$ as the set of linear operators defined by corresponding matrices and consider the adjoint operators $\mathcal{L}^*_{\omega,t}:\mathbf{R}^d\to(\mathbf{M}^{d\times m})^*$. This gives a certain flexibility of notations, e.g., the property "$E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)\le 0$ for every $\zeta\in O_t$" can be formulated as "the operator $E(\mathcal{L}^*_t Z_t|\mathcal{G}_t)$ is negative" (in the sense of partial ordering induced by $\mathbf{M}^{d\times m}_+$), the inclusion $f_i\in\operatorname{Ker}E(\mathcal{L}^*_t Z_t|\mathcal{G}_t)$ can be written instead of the equality $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0$ and so on. However, the current notation has the advantage of being easier adjustable for more general situation where $\mathcal{L}_t$ is a concave positive homogeneous mapping from $\mathbf{M}^{d\times m}_+$ into $L^0(\mathbf{R}^d,\mathcal{F}_t)$.

Remark 3.2 The hypothesis on the structure of invertible claims may not be fulfilled for Examples 1 and 2. For the investor having access to full information, the set of all assets can be split into classes of equivalence within which one can do frictionless transfers though not necessary in one step. Our assumption means that all transfers within each class are frictionless, a hypothesis which, as was noted in [8], does not lead to a loss of generality as a fully informed "intelligent" investor will not lose money making charged transfers within an equivalence class. However, in the context of restricted information it seems that such an assumption means that the information on equivalence classes is available to the investor.

4 No Arbitrage Criteria: Arbitrary Ω

In the general case the assertion of Proposition 3.1 fails to be true though with a suitable modification its condition (b) remains sufficient for the NA^w-property. Namely, we have:

Proposition 4.1 *The NA^w-property holds if there exists $Z \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that all conditional expectations $E(|Z_t||\mathcal{L}_t f_i||\mathcal{G}_t)$ and $E(|Z_t||\mathcal{L}_t \tilde{f}_i||\mathcal{G}_t)$ are finite and $E(Z_t \mathcal{L}_t \zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$.*

This result is an obvious corollary of the following technical lemma dealing with integration issues.

Lemma 4.2 *Let $\Sigma_T = Z_T \sum_{t=0}^T \xi_t$ with $Z \in \mathcal{M}(\mathbf{R}^d_+, \mathbf{F})$ and $\xi_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $E(|Z_t||\xi_t||\mathcal{G}_t) < \infty$ and $E(Z_t \xi_t|\mathcal{G}_t) \le 0$. Put $\bar{\Sigma}_T := E(\Sigma_T|\mathcal{G}_T)$. If $\bar{\Sigma}_T^- \in L^1$, then $\bar{\Sigma}_T \in L^1$ and $E\bar{\Sigma}_T \le 0$.*

Proof. We proceed by induction. The claim is obvious for $T = 0$. Suppose that it holds for $T - 1$. Clearly,

$$Z_T \sum_{t=0}^{T-1} \xi_t = \Sigma_T - Z_T \xi_T.$$

By the martingale property $E(Z^i_T|\xi_t||\mathcal{G}_t) = E(Z^i_t|\xi_t||\mathcal{G}_t) < \infty$ implying that $E(|Z_T||\xi_t||\mathcal{G}_t) < \infty$ for any $t \le T$. Thus, $\bar{\Sigma}_T$ is well-defined and finite. Taking the conditional expectation with respect to $\mathcal{G}_T$ in the above identity we get, using the martingale property, that

$$E(\Sigma_{T-1}|\mathcal{G}_T) = E\left(Z_T \sum_{t=0}^{T-1} \xi_t \middle| \mathcal{G}_T \right) = \bar{\Sigma}_T - E(Z_T \xi_T|\mathcal{G}_T) \ge \bar{\Sigma}_T.$$

Therefore, the negative part of $E(\Sigma_{T-1}|\mathcal{G}_T)$ is dominated by the negative part of $\bar{\Sigma}_T$ which is integrable. Using Jensen's inequality we have:

$$\begin{aligned} \bar{\Sigma}^-_{T-1} &= [E(E(\Sigma_{T-1}|\mathcal{G}_T)|\mathcal{G}_{T-1})]^- \\ &\le E([E(\Sigma_{T-1}|\mathcal{G}_T)]^-|\mathcal{G}_{T-1}) \le E(\bar{\Sigma}^-_T|\mathcal{G}_{T-1}). \end{aligned}$$

Thus, $\bar{\Sigma}^-_{T-1} \in L^1$ and, by virtue of the induction hypothesis, $\bar{\Sigma}_{T-1} \in L^1$ and $E\bar{\Sigma}_{T-1} \le 0$. In the representation $\bar{\Sigma}_T = E(\bar{\Sigma}_{T-1}|\mathcal{G}_T) + E(\bar{\Sigma}^-_T|\mathcal{G}_{T-1})$ the first term is integrable and has negative expectation while the second is negative. Thus, $E\bar{\Sigma}_T \le 0$ and, automatically, $E\bar{\Sigma}^+_T < \infty$. □

The NA^r-criterion, suitably modified, remains true without any restriction on the probability space. Of course, in its formulation one needs to take care about the existence of the involved conditional expectations. This can be done as in the next result.

Theorem 4.3 *The following conditions are equivalent:*

(a) NA^r;

(b) there is $Z \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that all random variables $E(Z_t \mathcal{L}_t f_i|\mathcal{G}_t)$, $E(Z_t \mathcal{L}_t \tilde{f}_i|\mathcal{G}_t)$ are finite, $E(Z_t \mathcal{L}_t \zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t$,

$$\zeta I_{\{E(Z_t \mathcal{L}_t \zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t. \tag{4.1}$$

We have no trouble with the implication $(b) \Rightarrow (a)$: an inspection of the arguments given in the case of finite Ω shows that they work well until the concluding step which now can be done just by reference to Lemma 4.2.

The proof of the "difficult" implication $(a) \Rightarrow (b)$ follows the same line of ideas as in the case of full information.

Lemma 4.4 *Suppose that the equality*

$$\sum_{t=0}^{T} \mathcal{L}_t \tilde{\zeta}_t - \tilde{r} = 0 \tag{4.2}$$

with $\tilde{\zeta}_t \in O_t$ and $\tilde{r}_t \in L^0(\mathbf{R}^d_+)$ holds only if $\tilde{\zeta}_t \in \mathcal{I}_t$ and $\tilde{r} = 0$. Then $\widehat{A}_T$ is closed in probability.

Proof. For $T = 0$ the arguments are exactly the same as were used for Proposition 2.1 with obvious changes caused by the extra term describing the funds withdrawals. Namely, the difference is that for the limiting normalized order $\tilde{\zeta} := \sum_{i=1}^{N} \tilde{c}_i g_i$ we get the equality $\mathcal{L}\tilde{\zeta} - \tilde{r} = 0$ where $\tilde{r} \in L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ is the limit of normalized funds withdrawals. By hypothesis, $\tilde{r} = 0$ and we can complete the proof using the same Gauss-type reduction procedure.

Arguing by induction, we suppose that $\widehat{A}_{T-1}$ is closed and consider the sequence of order processes $(\zeta^n_t)_{t \le T}$ such that $\sum_{t=0}^{T} \mathcal{L}_t \zeta^n_t - r^n \to \eta$. There is an obvious reduction to the case where at least one of "elementary" orders at time zero tends to infinity. Normalizing and using the induction hypothesis we obtain that there exists an order process $(\tilde{\zeta}_t)_{t \le T}$ with nontrivial $\tilde{\zeta}_0$ such that $\sum_{t=0}^{T} \mathcal{L}_t \tilde{\zeta}_t - \tilde{r} = 0$ and we can use the assumption of the lemma. It ensures that $\tilde{r} = 0$ and there are $\zeta'_t \in O_t$ such that $\mathcal{L}_t \zeta'_t = -\mathcal{L}_t \tilde{\zeta}_t$. This allows us to reduce a number of non-zero coefficients (i.e. "elementary" orders) at the initial order by putting, $\bar{\zeta}^n_0 = \zeta^n_0 - \alpha^n \tilde{\zeta}_0$, as in the proof of Proposition 2.1, and $\bar{\zeta}^n_t = \zeta^n_t + \alpha^n \zeta'_t$ for $t \ge 1$. □

Lemma 4.5 *The NA^r-condition implies the hypothesis of the above lemma.*

Proof. Of course, $\tilde{r} = 0$ (otherwise, $(\tilde{\zeta}_t)$ is an arbitrage opportunity, i.e. even NA^w is violated). For the process $(\mathcal{L}'_t)$, from definition of NA^r we have that componentwise

$$\sum_{t=0}^{T} \mathcal{L}'_t \tilde{\zeta}_t \ge \sum_{t=0}^{T} \mathcal{L}_t \tilde{\zeta}_t = 0$$

and $\mathbf{1} \sum_{t=0}^{T} \mathcal{L}'_t \tilde{\zeta}_t > 0$ with strictly positive probability if at least one of $\tilde{\zeta}_t$ does not belong to $\mathcal{I}_t$. This means that $(\tilde{\zeta}_t)$ is an arbitrage opportunity for the model based on $(\mathcal{L}'_t)$. □

Lemma 4.6 *Assume that the hypothesis of Lemma 4.4 holds. Then for any "elementary" order f and every $t \le T$ one can find a bounded process $Z = Z^{(t,f)} \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that:*

1) $E(|Z_s||\mathcal{L}_s g|) < \infty$ *and* $E(Z_s \mathcal{L}_s g|\mathcal{G}_s) \le 0$ *for all* $s \le T$ *and all "elementary" orders* g,

2) $f I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}} \in \mathcal{I}_t$.

Proof. We may assume without loss of generality that all portfolio increments $\mathcal{L}_s g$ corresponding to the elementary orders g are integrable (otherwise we can pass to an equivalent measure P' with the bounded density ρ, find the process Z' with the needed properties under P' and take $Z = \rho \tilde{Z}'$).

Let $\mathcal{Z}$ be the set of all bounded processes $Z \in \mathcal{M}(\mathbf{R}^d_+, \mathbf{F})$ such that $EZ_T\xi \le 0$ whenever is $\xi \in \widehat{A}^1_T := \widehat{A}_T \cap L^1$. Let

$$c_t := \sup_{Z \in \mathcal{Z}} P(E(Z_t \mathcal{L}_t f|\mathcal{G}_t) < 0). \tag{4.3}$$

Let Z be an element for which the supremum is attained (one can take as Z a countable convex combination of any uniformly bounded sequence along which the supremum is attained).

If 2) fails, then the random vector $\mathcal{L}_t(f + \tilde{f}) I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}}$ (all components of which are negative) is not zero. This implies that the element $-\mathcal{L}_t \tilde{f} I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}}$ does not belong to $\widehat{A}^1_T$. Indeed, in the opposite case we would have the identity

$$\sum_{s=0}^{T} \mathcal{L}_s \zeta_s = -\mathcal{L}_t \tilde{f} I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}}.$$

The assumption of Lemma 4.4 ensures that the order $\tilde{f} I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}} + \zeta_t$ is in $\mathcal{I}_t$. Thus, for the symmetrized order we have that

$$\mathcal{L}_t(f + \tilde{f}) I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}} + \mathcal{L}_t(\zeta + \tilde{\zeta}) = 0.$$

Since the second term is also negative componentwise, both should be equal to zero and we get a contradiction.

By the Hahn–Banach theorem one can separate $\varphi := -\mathcal{L}_t \tilde{f} I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}}$ and $\widehat{A}^1_T$: that is we may find $\eta \in L^\infty(\mathbf{R}^d)$ such that

$$\sup_{\xi \in \widehat{A}^1_T} E\eta\xi < E\eta\varphi.$$

Since $\widehat{A}^1_T$ is a cone containing $-L^1(\mathbf{R}^d_+)$ the supremum above is equal to zero, $\eta \in L^1(\mathbf{R}^d_+)$ and $E\eta\varphi > 0$. The latter inequality implies that for $Z^\eta_t = E(\eta|\mathcal{G}_t)$ we have $EE(Z^\eta_t \mathcal{L}_t f|\mathcal{G}_t) I_{\{E(Z_t \mathcal{L}_t f|\mathcal{G}_t)=0\}} < 0$. Therefore, for the martingale $Z' := Z + Z^\eta$ we have that

$$P\big(E(Z'_t \mathcal{L}_t f|\mathcal{G}_t) < 0\big) > P\big(E(Z_t \mathcal{L}_t f|\mathcal{G}_t) < 0\big) = c_t.$$

This contradiction shows that 2) holds.

The process Z constructed in this way may be not in $\mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$. However, it can be easily "improved" to meet the latter property. To this end, fix $i \le d$ and consider, in the subset of $\mathcal{Z}$ on which the supremum c_t

in (4.3) is attained, a process Z with maximal probability $P(Z^i_T > 0)$ (such process does exist). Then $P(\bar{Z}^i_T > 0) = 1$. Indeed, in the opposite case, the element $e_i I_{\{Z^i_T = 0\}} \in L^1(\mathbf{R}^d_+)$ is not zero and, therefore, does not belong to $\widehat{A}^1_T$. So it can be separated from the latter set. The separating functional generates a martingale $Z' \in \mathcal{Z}$. Since $P(\bar{Z}_T + Z'_T > 0) > P(\bar{Z}_T > 0)$, we arrive to a contradiction with the definition of $\bar{Z}$. The set of $Z \in \mathcal{Z}$ satisfying 1) and 2) is convex and, hence, a convex combination of d processes obtained in this way for each coordinate has the required properties. □

The implication $(a) \Rightarrow (b)$ of the theorem follows from the lemmas above. Indeed, by virtue of Lemmas 4.5 – 4.6, NA^r ensures the existence of processes $Z^{(t,f)}$ satisfying 1) and 2) of Lemma 4.6. One can take as a required martingale Z the process $Z := \sum_{t,f} Z^{(t,f)}$ where $t = 0, 1, ..., T$ and f runs through the set of "elementary" orders. An arbitrary order $\zeta \in O_t$ is a linear combination of elementary orders with positive $\mathcal{G}_t$ measurable coefficients. The condition $E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \le 0$ follows from the property 1) of Lemma 4.6. To prove the inclusion (4.1) we note that $I_{\{\Sigma \xi_i = 0\}} = \prod I_{\{\xi_i = 0\}}$ when $\xi_i \le 0$. With this observation the required inclusion is an easy corollary of the property 2) of Lemma 4.6 and the stability of $\mathcal{I}_t$ under multiplication by positive $\mathcal{G}_t$-measurable random variables.

Remark 4.1 In the above proof we get from NA^r a condition which looks stronger than (b), with bounded Z and integrable random variables $|Z_t||\mathcal{L}_t f|$, but, in fact, it is equivalent to (b).

5 Hedging Theorem

Thanks to the previous development, hedging theorems in the model with partial information do not require new ideas. For the case of finite Ω the result can be formulated in our "very abstract" setting without additional assumptions on the structure of the sets $\mathcal{I}_t$.

We fix a d-dimensional random variable $\widehat{C}$, the contingent claim expressed in physical units. Define the set

$$\Gamma = \{v \in \mathbf{R}^d : \ \widehat{C} \in v + \widehat{A}_T\}.$$

Let $\mathcal{Z}$ be the set of martingales $Z \in \mathcal{M}_T(\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t \zeta_t | \mathcal{G}_t) \le 0$ for every $\zeta_t \in O_t$. Put

$$D := \left\{ v \in \mathbf{R}^d : \ \sup_{Z \in \mathcal{Z}} E(Z_T \widehat{C} - Z_0 v) \le 0 \right\}.$$

Proposition 5.1 *Let Ω be finite and $\mathcal{Z} \neq \emptyset$. Then $\Gamma = D$.*

In this theorem the inclusion $\Gamma \subseteq D$ is obvious while the reverse inclusion is an easy exercise on the finite-dimensional separation theorem. We leave it to the reader.

In the case of general Ω we should take care about integrability and closedness of the set $\widehat{A}_T$. To this end we shall work with the model in the "narrow" sense of the preceding sections assuming the NA^r-property. Now $\mathcal{Z}$ is the set of bounded martingales $Z \in \mathcal{M}_T(\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)$, $E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)$ are finite, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ and $E(Z_T\widehat{C})^- < \infty$. The definitions of the sets Γ and D remain the same.

Theorem 5.2 *Suppose that NA^r holds. Then $\Gamma = D$.*

Proof. The inclusion $\Gamma \subseteq D$ follows from the inequality

$$Z_T(\widehat{C} - v) \le Z_T \sum_{t=0}^{T} \mathcal{L}_t\zeta_t, \qquad \zeta_t \in O_t,$$

and Lemma 4.2.

To check the inclusion $D \subseteq \Gamma$ we take a point $v \notin \Gamma$ and show that $v \notin D$. It is sufficient to find $Z \in \mathcal{Z}$ such that $Z_0 v < EZ_T\widehat{C}$. Consider a measure $\tilde{P} \sim P$ with bounded density ρ such that $\widehat{C}$, and all $|\mathcal{L}_t||f_i|$ and $|\mathcal{L}_t||f_i|$ belong to $L^1(\tilde{P})$. Under NA^r the convex set $\tilde{A}^1 := A_0^T \cap L^1(\tilde{P})$ is closed and does not contain the point $\widehat{C} - v$. Thus, we can separate the latter by a functional η from L^∞. This means that

$$\sup_{\xi \in \tilde{A}^1} E\rho\eta\xi < E\eta\rho(\widehat{C} - v).$$

It is clear, that the bounded martingale $Z_t := E(\rho\eta|\mathcal{F}_t)$ satisfies the required properties. □

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