Noname manuscript No.
(will be inserted by the editor)

On the law of one price

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The date of receipt and acceptance will be inserted by the editor

Abstract We consider the standard discrete-time model of a frictionless financial market and show that the law of one price holds if and only if there exists a martingale density process with strictly positive initial value. In contrast to the classical no-arbitrage criteria, this density process may change its sign. We also give an application to the CAPM.

Key words Law of one price, Harrison–Pliska theorem, Dalang–Morton–Willinger theorem, market portfolio, CAPM

Mathematics Subject Classification (2000): 60G44

JEL Classification Numbers: G13, G11

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* This research was done during the stay of the author at Université de Franche-Comté.

1 Introduction

To explain our interest in the question discussed in this paper we recall several facts and concepts from the foundations of stochastic finance. The fundamental point of this theory is that there are basic securities with prices evolving as random processes; the randomness is inherited by the investors portfolios. In the simplest one-period model where we have $t \in \{0, T\}$ (the terminal date T may be denoted by 1), the numéraire is a traded asset, the price increments of d basic securities are described simply by a random d-dimensional vector ξ. The portfolio strategy is just a (deterministic) vector $h \in \mathbf{R}^d$ and the portfolio increment is the scalar product $h\xi$. One says that the market is arbitrage free (briefly: satisfies the NA property) if the inequality $h\xi \geq 0$ may hold only if $h\xi = 0$. This property plays a key role for the pricing of derivative securities. In the case of finite Ω it is an easy exercise on the use of the separation theorem to check that the NA property is equivalent to the existence of a (scalar) random variable Z which is strictly positive (a.s.), $EZ = 1$, and the vector $EZ\xi$ is zero. In the economic literature the probability space $(\Omega, \mathcal{F}, \tilde{P})$ with $\tilde{P} = ZP$ is called the "risk-neutral world". Rather remarkably, this simple exercise happened to be the germ of an important development in mathematical finance. The first steps of this development are now classical. The Harrison–Pliska theorem gives an extension to the multi-period model (with finite Ω) claiming that NA holds iff there exists a strictly positive (a.s.) martingale $Z = (Z_t)$ such that $EZ_t\xi_t = 0$ for all t. The Dalang–Morton–Willinger theorem, a result which is mathematically much more delicate, extends this assertion to an arbitrary Ω, adding also that one can always choose a bounded density process Z. At the moment, no-arbitrage criteria are obtained for numerous models (continuous-time models, models with transaction costs, models with constraints), see the handbook [2].

However, the NA property is not a single one isolated in the economic literature. A weaker property, called the law of one price (we shall use the abbreviation L1P), for the one-period model can be formulated as follows: the identity $x + h\xi = x' + h'\xi$ where $h, h' \in \mathbf{R}^d$ implies that $x = x'$, see, e.g. [7]. In other words, if a contingent claim has a price, namely, the replication price, this price is unique. Clearly, this property is always fulfilled if NA holds. Again, in this elementary case, it is an easy exercise on a finite-dimensional separation theorem to check that L1P holds iff there exists a random variable Z (not necessary positive) with $EZ = 1$ such that the vector $EZ\xi$ is zero. Similarly, an extension to the multi-period model with a finite number of states of nature (i.e. to the setting of the Harrison–Pliska theorem) does not pose new mathematical difficulties. Our aim here is to analyze the law of one price for the multi-period model with a general probability space in the same spirit as it was done in the recent note [5]. It is worthy to mention that similarly to NA criteria, there are several strategies of proof. We opt that of the mentioned note.

At last, we introduce the notion of normalized excess expected return of a strategy and show that such a week condition as LPT1 implies already the existence of a market portfolio.

2 The law of one price

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a finite discrete-time filtration $(\mathcal{F}_t)$, $t = 0, ..., T$, $\mathcal{F}_T = \mathcal{F}$, and let $S = (S_t)$ be an adapted d-dimensional process. Let $R_T := \{\xi : \ \xi = H \cdot S_T, \ H \in \mathcal{P}\}$ where $\mathcal{P}$ is the set of all predictable d-dimensional processes (i.e. H_t is $\mathcal{F}_{t-1}$-measurable) and let

$$H \cdot S_T := \sum_{t=1}^{T} H_t \Delta S_t, \qquad \Delta S_t := S_t - S_{t-1}.$$

We denote by $L^0(\mathcal{F}_t)$ the set of finite $\mathcal{F}_t$-measurable random variables.

The linear subspace R_T is closed in $L^0(\mathcal{F}_T)$. This fact, fundamental in the sequel, was established in [5].

We say that the model satisfies the law of one price (L1P) at the date $t = 0$ if the equality $\zeta + H \cdot S_T = \zeta' + H' \cdot S_T$ where $H, H' \in \mathcal{P}$ and $\zeta, \zeta' \in L^0(\mathcal{F}_0)$, implies that $\zeta = \zeta'$ (a.s).

It is easily seen that this condition, L1P, at $t = 0$, can be written as follows: $R_T \cap L^0(\mathcal{F}_0) = \{0\}$.

Theorem 1 *The following conditions are equivalent:*

(a) $R_T \cap L^0(\mathcal{F}_0) = \{0\}$*;*

(b) *there is a bounded martingale* $Z = (Z_t)_{t \le T}$ *with* $EZ_T = 1$ *and* $Z_0 > 0$ *such that the process* ZS *is a martingale.*

Proof. $(a) \Rightarrow (b)$ Take an arbitrary non-null $\mathcal{F}_0$-measurable set A. By the assumption $I_A \notin R_T$. Choose a probability measure $\tilde{P} \sim P$ with the bounded density $\rho := d\tilde{P}/dP$ and such that all S_t are integrable with respect to $\tilde{P}$. Since R_T is closed in probability, the set $R_T^1 := R_T \cap L^1(\tilde{P})$ is a closed linear space in $L^1(\tilde{P})$. Thus, there exists a bounded random variable $\tilde{Z}_T^A$ such that $\tilde{E}\tilde{Z}_T^A \eta = 0$ for all $\eta \in R_T^1$ but $\tilde{E}\tilde{Z}_T^A I_A > 0$. Putting $Z_T^A := \rho \tilde{Z}_T^A$ and normalizing, if necessarily, we can rephrase this as follows: there exists Z_T^A with $|Z_T^A| \le 1$ such that $EZ_T^A \eta = 0$ for all $\eta \in R_T^1$ but $EZ_T^A I_A > 0$. The set R_T^1 is stable under multiplication by the indicator functions of $\mathcal{F}_0$-measurable sets. It follows that the above properties remain valid if we replace Z_T^A by $I_A I_{\{E(Z_T^A|\mathcal{F}_0)>0\}}$. Hence, we may assume without loss of generality that $E(Z_T^A|\mathcal{F}_0) \ge 0$.

The usual exhaustion arguments ensure that there is a bounded random variable Z_T with $E(Z_T|\mathcal{F}_0) > 0$ such that $EZ_T \eta = 0$ for all $\eta \in R_T^1$. For the reader's convenience we recall them. Let $\mathcal{C}$ be the family formed by all sets of the form $\{E(Z_T^A|\mathcal{F}_0) > 0\}$, $A \in \mathcal{F}_0$. Let $a := \sup_{\Gamma \in \mathcal{C}} P(\Gamma)$. The supremum here is attained: it is sufficient to consider the set $A = \cup_n A_n$ and the bounded random variable $Z_T^A := \sum_n 2^{-n} Z_T^{A_n}$ where $A_n \in \mathcal{F}_0$ are such that $P(A_n) \to a$. It remains to notice that $P(A) = 1$ (otherwise we could increase the supremum with Z^{A^c}).

We conclude by putting $Z_t := E(Z_T|\mathcal{F}_t)$ and by observing that the martingale property of ZS holds because $\xi \Delta S_t \in R_T^1$ for every bounded $\mathcal{F}_{t-1}$-measurable random variable ξ.

$(b) \Rightarrow (a)$ This claim is obvious. Indeed, we should check that $\xi = 0$ whenever $\xi + H \cdot S_T = 0$. For the process $M := Z(\xi + H \cdot S)$ we have $M_{t-1} = E(M_t|\mathcal{F}_{t-1})$ for $t \geq 1$. As M is zero at the terminal date T, it is zero identically. In particular, its initial value $Z_0\xi = 0$. Since $Z_0 > 0$, this implies that $\xi = 0$. □

3 Ramifications and comments

1. It is quite natural to have a look at the situation where the portfolio strategies are subjects some trading constrains. For instance, shortselling may be prohibited. To be specific, suppose that we are given a closed convex cone $K \subseteq \mathbf{R}^d$ and at each date t the vector of holdings in risky assets H_t belongs to $L^0(K, \mathcal{F}_{t-1})$, i.e. H is a predictable d-dimensional process taking values in K. Let us denote the set of such processes by $\mathcal{P}^K$ and the corresponding set of "results" R^K_T.

The law of one price (at time zero), abbreviated in this model as L1P^K, means that the equality $\zeta + H \cdot S_T = \zeta' + H' \cdot S_T$ where $H, h' \in \mathcal{P}^K$ and $\zeta, \zeta' \in L^0(\mathcal{F}_0)$ may hold only if $\zeta = \zeta'$ (a.s).

This property can be expressed by the relation $(R^K_T - R^K_T) \cap L^0(\mathcal{F}_0) = \{0\}$.

Notice that

$$L^0(K, \mathcal{F}_t) - L^0(K, \mathcal{F}_t) = L^0(K - K, \mathcal{F}_t)$$

(to check the non-trivial implication $\supseteq$ it is sufficient to fix a basis $\{v_i\}$ in $K - K$ and consider an arbitrary representation $v_i = v'_i - v''_i$ with $v'_i, v''_i \in K$).

Recall that $K - K = \mathbf{R}^d$ if and only if $\operatorname{int} K \neq \emptyset$. Consequently, if $\operatorname{int} K \neq \emptyset$, then $R_T = R^K_T - R^K_T$ and we arrive at the following theorem which covers, in particular, the model where shortselling is not allowed.

Theorem 2 *Assume that* $\operatorname{int} K \neq \emptyset$*. Then the following conditions are equivalent:*

(a) L1P^K*;*

(b) *there is a bounded martingale* $Z = (Z_t)_{t \leq T}$ *with* $EZ_T = 1$ *and* $Z_0 > 0$ *such that the process* ZS *is a martingale.*

Without difficulties this result can be extended to the case where K_t are $\mathcal{F}_{t-1}$-measurable (as usual, this needs a bit of set-valued analysis).

2. It is economically reasonable that the law of one price holds globally, i.e. if a contingent claim is replicable when the trading starts at the date τ, then the replication price is unique. Formally, we can isolate the following GL1P property:

the equality $\zeta + H \cdot {}^{\tau}S_T = \zeta' + H' \cdot {}^{\tau}S_T$ where τ is a stopping time, $H, H' \in \mathcal{P}$, and $\zeta, \zeta' \in L^0(\mathcal{F}_\tau)$ may hold only if $\zeta = \zeta'$ (a.s).

Here we use the abbreviation ${}^{\tau}S_t := I_{]\tau,T]} \cdot S$.

The following assertion is a corollary of Theorem 1 (cf. with the formulation of Th. 4.5 in [6]).

Theorem 3 *The following conditions are equivalent:*

(a) GL1P^K*;*

(b) *for every stopping time* τ *there exists a bounded martingale* $Z^{(\tau)}$ *with* $EZ^{(\tau)}_T = 1$ *and* $Z^{(\tau)}_\tau > 0$ *such that the process* $Z^{(\tau)}\, {}^{\tau}S$ *is a martingale.*

Note that using in the formulation of GL1P property only deterministic times gives an equivalent definition.

3. As usual, one can find in the literature on linear inequalities related results corresponding to the one-step model and formulated as alternative theorems. As an example, we give a relevant one: either the equation $Ay = b$ has a solution or there is a vector $z \neq 0$ such that $zb = 1$ and $zA = 0$. In the case where $\Omega := \{\omega_1, ..., \omega_N\}$, the $N \times d$-matrix A is formed by the elements $a^{ij} := \Delta S^j(\omega_i)$ and $b = (1, ..., 1)^{\mathrm{T}}$.

4. Unfortunately, the above results have no natural counterparts for continuous-time models. "Natural" here means "for the standard concept of admissibility". The latter requires that the value process is bounded from below. To see this, consider the model where S is just a Wiener process, $T = \infty$. The process $V = e^{S - \frac{1}{2}\langle S \rangle}$ is a value process (corresponding to the admissible strategy $H = V$ and the initial value $x = 1$) with $V_\infty = 0$. Thus, we have that $1 + H \cdot S = 0 \cdot S$ violating the law of one price. Of course, an appropriate modification of this model provides an example where T is finite.

4 The law of one price and CAPM

The law of one price implies an interesting and important consequence: the existence of a market portfolio.

To show this we make an extra assumption that all $S_t \in L^2$ and define, in the Hilbert space L^2, the closed linear subspace $R_T^2 = R_T \cap L^2$ (recall that R_T is closed in L^0). It contains all terminal values of portfolios with bounded strategies and starting from zero. The random variable $x + H \cdot S_T$ is the terminal value (expressed in the units of the numéraire assumed to be a traded security) of a portfolio with the initial endowment x. Hence $\xi := H \cdot S_T$ is the surplus of this strategy with respect to holding x in the numéraire; it does not depend of the initial endowment, but for $x = 1$ it is usually referred in the economic literature as the excess return of the portfolio. We define as the (normalized) *excess expected return* of the strategy H the ratio $r_H := \frac{E\xi}{\sigma(\xi)}$ where σ is the standard deviation; r_H is called the Sharpe ratio. By convention, $\frac{0}{0} = 0$. Since $r_{\lambda H} = r_H$ for $\lambda > 0$, this normalization allows us to compare "quality" of the portfolio composition independently of its "size". We say that a strategy G with $\eta := G \cdot S_T \in R_T^2$ defines a *market portfolio* if for every strategy with the terminal value $\xi := H \cdot S_T$ in R_T^2 we have the equality

$$r_H = \beta_{GH} r_G \tag{1}$$

where β_{GH} is the correlation coefficient $\rho(\xi, \eta)$. Notice that (1) looks like the usual CAPM relation, see, e.g., [4].

For the case where P is a martingale measure the identity is trivial. We exclude this situation in the formulation below.

Proposition 4 *Suppose that P is not a martingale measure and L1P holds. Then the market portfolio does exist.*

Proof. The requirement is met by the strategy G corresponding to the projection η of the unit onto the subspace R^2_T. Indeed, by definition, $E(1-\eta)\xi = 0$ for each $\xi \in R^2_T$. In particular, $E(1-\eta)\eta = 0$ and, therefore, $E\eta = E\eta^2 \geq 0$. In fact, the inequality here is strict: otherwise η is zero and $E\xi = 0$ for each $\xi \in R^2_T$ implying that P is a martingale measure. The case $E\eta^2 = 1$ is also impossible: since $E\eta^2 + E(1-\eta)^2 = 1$ we would have that $\eta = 1$ violating L1P. Thus, $0 < E\eta < 1$. Now we write the relation $E(1-\eta)\xi = 0$ in a different way:

$$E\xi = E\xi\eta = \mathrm{Cov}\,(\xi,\eta) + E\xi E\eta$$

where $\mathrm{Cov}\,(\xi,\eta)$ is the covariance of ξ and η.

We deduce from here taking into account the equality $E\eta = E\eta^2$ that

$$E\xi = \frac{\mathrm{Cov}\,(\xi,\eta)}{1-E\eta} = \frac{\mathrm{Cov}\,(\xi,\eta)E\eta}{E\eta-(E\eta)^2} = \frac{\mathrm{Cov}\,(\xi,\eta)E\eta}{\sigma^2(\eta)}.$$

Hence,

$$\frac{E\xi}{\sigma(\xi)} = \frac{\mathrm{Cov}\,(\xi,\eta)}{\sigma(\xi)\sigma(\eta)}\frac{E\eta}{\sigma(\eta)}$$

and we obtain the required relation (1). □

Remark. In fact, the above proof is not needed: simple geometric considerations replace the above arguments. Indeed, r_H, r_G, β_{GH} are the cosines of angles between ξ and 1, η and 1, ξ and η. This observation shows that (1) means that the projection of the vector 1 onto ξ can be obtained in two steps: first we project 1 onto the plane of the vectors ξ and η and then project the result onto ξ. A picture makes this obvious.

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