On a true submartingale property

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In a recent preprint [2] on utility maximization Walter Schachermayer revealed the importance of a class of integrands ("strategies" in financial terminology) H for which the stochastic integral $H \cdot S$ ("value process") is a supermartingale. His arguments used implicitly a criterion when a a nonnegative local submartingale is a (true) submartingale. We give here a short discussion of this result together with its applications. this test did not

We use the following standard notation: if σ is a stopping time and $A \in \mathcal{F}_\sigma$ then σ_A is the stopping time coinciding with σ on A and equal to ∞ on $\bar{A}$.

Lemma 1 *A local submartingale $Y \geq 0$ stopped at time $T < \infty$ is a submartingale iff for each sequence of stopping times (τ^n) with $P(\tau^n < \infty) \to 0$*

$$\liminf_n EY_{\tau^n} I_{\{\tau^n < \infty\}} = 0. \tag{1}$$

Proof. If Y is a submartingale, then $Y_\infty = Y_T$ is integrable and

$$EY_{\tau^n} I_{\{\tau^n < \infty\}} \leq EY_T I_{\{\tau^n < \infty\}} \to 0.$$

To prove the converse, put $\sigma^n := \inf\{t : Y_t \geq n\}$. It follows from (1) that Y^{σ^n} is a submartingale (being bounded by an integrable random variable). It remains to verify that for each $t \leq T$ the sequence of random variables $Y_{t \wedge \sigma^n} \geq 0$ contains a uniformly integrable subsequence, i.e., in virtue of the well-known criterion, that $EY_t < \infty$ and $Y_{t \wedge \sigma^n} \to EY_t$ along a subsequence. To this aim, let $A^n := \{\sigma^n \leq t\}$. Clearly,

$$EY_t \leq nP(\bar{A}^n) + EY_t I_{A^n} = nP(\bar{A}^n) + EY_{t_{A^n}} I_{\{t_{A^n} < \infty\}} < \infty$$

for sufficiently large n (due to (1) with $\tau^n = t_{A^n}$). Taking into account that

$$0 \leq EY_{\sigma^n} I_{A^n} \leq EY_{\sigma^n} I_{\{\sigma^n < \infty\}} \to 0$$

along a subsequence we infer that

$$EY_{t \wedge \sigma^n} = EY_{\sigma^n} I_{A^n} + EY_t I_{\bar{A}^n} \to EY_t$$

(along the same subsequence). $\square$

A semimartingale X is called σ-martingale (or a martingale of class (Σ_m)) if there exists an integrand G taking values in the interval $]0,1]$ such that the process $G \cdot X$ is a local martingale.

Proposition 2 *A σ-martingale X with $X_0 = 0$ and $X = X^T$ is a supermartingale iff for each sequence of stopping times (τ^n) with $P(\tau^n < \infty) \to 0$*

$$\liminf_n EX^-_{\tau^n} I_{\{\tau^n < \infty\}} = 0. \tag{2}$$

Proof. It follows from (2) by the Ansel–Stricker theorem [1] that X is a local martingale and hence $Y := X^-$ is a local submartingale satisfying (1). By the above lemma Y is a submartingale dominated by the martingale $M_t := E(Y_T|\mathcal{F}_t)$. Since $X \geq -M$, the local martingale X is a supermartingale. $\square$

In [2] the above criteria is used in the following form:

Proposition 3 *A σ-martingale X with $X_0 = 0$ and $X = X^T$ is not a supermartingale iff there is a sequence of stopping times (τ^n) with $P(\tau^n < \infty) \to 0$ such that $X_{\tau^n} I_{\{\tau^n < \infty\}} \leq 0$ and*

$$\lim_n EX_{\tau^n} I_{\{\tau^n < \infty\}} < 0.$$

This is just a reformulation of the previous assertion because

$$EX_{\tau_A} I_{\{\tau_A < \infty\}} = -EX^-_{\tau} I_{\{\tau < \infty\}}$$

where $A := \{X_\tau \leq 0\}$.

References

[1] Ansel J.-P., Stricker C. Couverture des actifs contingents. *Ann. Inst. Henri Poincaré*, **30** (1994), 2, 303-315.

[2] Schachermayer W. How potential investments may change the optimal portfolio for the exponential utility. Preprint, June 2001.