A teachers' note on no-arbitrage criteria

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Abstract

We give a new proof of the classical Dalang–Morton–Willinger theorem.

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1. Introduction. The Dalang–Morton–Willinger theorem asserts, for a discrete-time model of security market, that there is no arbitrage if and only if the price process is a martingale with respect to an equivalent probability measure. This remarkable result sometimes is referred to as the First Fundamental Theorem of mathematical finance, [9]. A simple statement suggests a simple proof and many attempts were made to find a such one, cf. [1], [10], [8], [6], [7], [4], [2]. Various aspects were investigated in details and the theorem was augmented by additional equivalent conditions revealing its profound difference from the Harrison–Pliska theorem [3] which is the same criterion but for the model with finite Ω. Unfortunately, all existing proofs are too cumbersome for lecture courses. This note is a new attempt to provide a concise proof which uses only results from the standard syllabus.

2. No-arbitrage criteria. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a finite discrete-time filtration $(\mathcal{F}_t)$, $t = 0, ..., T$, $\mathcal{F}_T = \mathcal{F}$, and let $S = (S_t)$ be an adapted d-dimensional process. Let $R_T := \{\xi : \ \xi = H \cdot S_T, \ H \in \mathcal{P}\}$ where $\mathcal{P}$ is the set of all predictable d-dimensional processes (i.e. H_t is $\mathcal{F}_{t-1}$-measurable) and

$$H \cdot S_T := \sum_{t=1}^{T} H_t \Delta S_t, \qquad \Delta S_t := S_t - S_{t-1}.$$

Put $A_T := R_T - L^0_+$; $\bar{A}_T$ is the closure of A_T in probability, L^0_+ is the set of non-negative random variables.

Theorem 1 *The following conditions are equivalent:*

(a) $A_T \cap L^0_+ = \{0\}$;

(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$;

(c) $\bar{A}_T \cap L^0_+ = \{0\}$;

(d) *there is a probability* $\tilde{P} \sim P$ *with* $d\tilde{P}/dP \in L^\infty$ *such that* S *is a* $\tilde{P}$*-martingale.*

In the context of mathematical finance this model corresponds to the case where the "numéraire" is a traded security, S describes the evolution of prices of risky assets, and $H \cdot S_T$ is the terminal value of a self-financing portfolio. Condition (a) is interpreted as the absence of arbitrage; it can be written in the obviously equivalent form $R_T \cap L^0_+ = \{0\}$ (or $H \cdot S_T \geq 0 \Rightarrow H \cdot S_T = 0$). We include in the formulation only the basic equivalences: various other ones known in the literature can easily be deduced from the listed above.

If Ω is finite then A_T is closed being a polyhedral cone in a finite-dimensional space. For infinite Ω the set A_1 may be not closed, see an example in [8], while R_T is always closed (this can be checked in a similar way as the implication $(a) \Rightarrow (b)$ in the proof below).

3. Auxiliary results. The following observation is elementary.

Lemma 2 *Let $\eta^n \in L^0(\mathbf{R}^d)$ be such that $\underline{\eta} := \liminf |\eta^n| < \infty$. Then there are $\tilde{\eta}^k \in L^0(\mathbf{R}^d)$ such that for all ω the sequence of $\tilde{\eta}^k(\omega)$ is a convergent subsequence of the sequence of $\eta^n(\omega)$.*

Proof. Let $\tau_0 := 0$. Define the random variables $\tau_k := \inf\{n > \tau_{k-1} : \, ||\eta^n| - \underline{\eta}| \leq k^{-1}\}$. Then $\tilde{\eta}^k_0 := \eta^{\tau_k}$ is in $L^0(\mathbf{R}^d)$ and $\sup_k |\tilde{\eta}^k_0| < \infty$. Working further with the sequence of $\tilde{\eta}^n_0$ we construct, applying the above procedure to the first component, a sequence of $\tilde{\eta}^k_1$ with convergent first component and such that for all ω the sequence of $\tilde{\eta}^k_1(\omega)$ is a subsequence of the sequence of $\tilde{\eta}^n_0(\omega)$. Passing on each step to the newly created sequence of random variables and to the next component we arrive to a sequence with the desired properties. □

Remark. The above claim can be formulated as follows: there exists an increasing sequence of integer-valued random variables σ_k such that η^{σ_k} converges a.s.

For the sake of completeness, we recall the proof of the well-known result due to Kreps and Yan, [5], [11].

Lemma 3 *Let $K \supseteq -L^1_+$ be a closed convex cone in L^1 such that $K \cap L^1_+ = \{0\}$. Then there is a probability $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that $\tilde{E}\xi \leq 0$ for all $\xi \in K$.*

Proof. By the Hahn–Banach theorem for any $x \in L^1_+$, $x \neq 0$, there is $z_x \in L^\infty$ such that $Ez_x\xi < Ez_x x$ for all $\xi \in K$. Necessarily, $z_x \geq 0$ and $Ez_x x > 0$. Normalizing, we assume that $z_x \leq 1$. The Halmos–Savage theorem asserts that the family of measures $\{z_x P\}$ contains a countable equivalent subfamily $\{z_{x_i} P, \ i \in \mathbf{N}\}$ (i.e., both vanish on the same sets). Put $\rho := \sum 2^{-n} z_{x_n}$. The measure $\tilde{P} := c\rho P$ where $c = 1/E\rho$ meets the requirements. □

Remark. The Halmos–Savage theorem is simple and the reference can be replaced by its proof which is as follows. Consider the larger family $\{yP\}$ where y are convex combinations of z_x. Then $\operatorname{ess\,sup} I_{\{y>0\}}$ can be attained on an increasing sequence of $I_{\{y_k>0\}}$. Clearly, $\{y_k P\}$ is a countable equivalent subfamily of $\{yP\}$ and it is a convex envelope of a countable family $\{z_{x_i} P\}$ we are looking for.

4. Proof of Theorem 1. $(a) \Rightarrow (b)$ To show that A_T is closed we proceed by induction. Let $T = 1$. Suppose that $H^n_1 \Delta S_1 - r^n \to \zeta$ a.s. where H^n_1 is $\mathcal{F}_0$-measurable and $r^n \in L^0_+$. It is sufficient to find $\mathcal{F}_0$-measurable random variables $\tilde{H}^k_1$ which are convergent a.s. and $\tilde{r}^k \in L^0_+$ such that $\tilde{H}^k_1 \Delta S_1 - \tilde{r}^k \to \zeta$ a.s. convergent.

Let $\Omega_i \in \mathcal{F}_0$ form a finite partition of Ω. Obviously, we may argue on each Ω_i separately as on an autonomous measure space (considering the restrictions of random variables and traces of σ-algebras).

Let $\underline{H}_1 := \liminf |H_1^n|$. On the set $\Omega_1 := \{\underline{H}_1 < \infty\}$ we can take, using Lemma 2, $\mathcal{F}_0$-measurable $\tilde{H}_1^k$ such that $\tilde{H}_1^k(\omega)$ is a convergent subsequence of $H_1^n(\omega)$ for every ω; $\tilde{r}^k$ are defined correspondingly. Thus, if Ω_1 is of full measure, the goal is achieved.

On $\Omega_2 := \{\underline{H}_1 = \infty\}$ we put $G_1^n := H_1^n/|H_1^n|$ and $h_1^n := r_1^n/|H_1^n|$ and observe that $G_1^n \Delta S_1 - h_1^n \to 0$ a.s. By Lemma 2 we find $\mathcal{F}_0$-measurable $\tilde{G}_1^k$ such that $\tilde{G}_1^k(\omega)$ is a convergent subsequence of $G_1^n(\omega)$ for every ω. Denoting the limit by $\tilde{G}_1$, we obtain that $\tilde{G}_1 \Delta S_1 = \tilde{h}_1$ where $\tilde{h}_1$ is non-negative, hence, in virtue of (a), $\tilde{G}_1 \Delta S_1 = 0$.

As $\tilde{G}_1(\omega) \neq 0$, there exists a partition of Ω_2 into d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $\tilde{G}_1^i \neq 0$ on Ω_2^i. Define $\bar{H}_1^n := H_1^n - \beta^n \tilde{G}_1$ where $\beta^n := H_1^n/\tilde{G}_1^i$ on Ω_2^i. Then $\bar{H}_1^n \Delta S_1 = H_1^n \Delta S_1$ on Ω_2. We repeat the procedure on each Ω_2^i with the sequence $\bar{H}_1^n$ knowing that $\bar{H}_1^{ni} = 0$ for all n. Apparently, after a finite number of steps we construct the desired sequence.

Let the claim be true for $T-1$ and let $\sum_{t=1}^{T} H_t^n \Delta S_t - r^n \to \zeta$ a.s. where H_t^n are $\mathcal{F}_t$-measurable and $r^n \in L_+^0$. By the same arguments based on the elimination of non-zero components of the sequence H_1^n and using the induction hypothesis we replace H_t^n and r^n by $\tilde{H}_t^k$ and $\tilde{r}^k$ such that $\tilde{H}_1^k$ converges a.s. This means that the problem is reduced to the one with $T-1$ steps.

$(b) \Rightarrow (c)$ Trivial.

$(c) \Rightarrow (d)$ Notice that for any random variable η there is an equivalent probability P' with bounded density such that $\eta \in L^1(P')$ (e.g., one can take $P' = Ce^{-|\eta|}P$). Property (c) (as well as (a) and (b)) is invariant under equivalent change of probability. This consideration allows us to assume that all S_t are integrable. The convex set $A_T^1 := \bar{A}_T \cap L^1$ is closed in L^1. Since $A_T^1 \cap L_+^1 = \{0\}$, Lemma 3 ensures the existence of $\tilde{P} \sim P$ with bounded density and such that $\tilde{E}\xi \leq 0$ for all $\xi \in A_T^1$, in particular, for $\xi = \pm H_t \Delta S_t$ where H_t is bounded and $\mathcal{F}_{t-1}$-measurable. Thus, $\tilde{E}(\Delta S_t | \mathcal{F}_{t-1}) = 0$.

$(d) \Rightarrow (a)$ Let $\xi \in A_T \cap L_+^0$, i.e. $0 \leq \xi \leq H \cdot S_T$. As $\tilde{E}(H_t \Delta S_t | \mathcal{F}_{t-1}) = 0$, we obtain by conditioning that $\tilde{E} H \cdot S_T = 0$. Thus, $\xi = 0$. □

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