The Harrison–Pliska arbitrage pricing theorem under transaction costs

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Abstract

We consider a simple multi-asset discrete-time model of a currency market with transaction costs assuming the finite number of states of the nature. Defining two kinds of arbitrage opportunities we study necessary and sufficient conditions for the absence of arbitrage. Our main result is a natural extension of the Harrison–Pliska theorem on asset pricing. We prove also a hedging theorem without supplementary hypotheses.

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1 Introduction

The famous result of Harrison–Pliska [?], known also as the Fundamental Theorem on Asset (or Arbitrage) Pricing (FTAP) asserts that a frictionless financial market is free of arbitrage if and only if the price process is a martingale under a probability measure equivalent to the objective one. The original formulation involved the assumption that the underlying probability space $(\Omega, \mathcal{F}, P)$ (in other words, the number of states of the nature) is finite; it has been removed in the subsequent study of Dalang–Morton–Willinger [?]. It is worth to note that the passage from finite to infinite Ω is by no means trivial: instead of purely geometric considerations (which make the Harrison–Pliska theorem so attractive for elementary courses in financial economics) much more delicate topological or measure-theoretical arguments must be used. These mathematical aspects were investigated in details by a number of authors (see, e.g., [?], [?], [?], [?], [?]). The aim of this note is to present an extension of the arbitrage pricing theorem for a multi-asset multi-period model with finite Ω and proportional transaction costs. We use the geometric formalism developed in [?], [?], and [?]. Our result makes clear that the concept of the equivalent martingale measure, though useful in the context of frictionless market models, has no importance (and even misleading) in a more realistic situation of transaction costs. We track how the dual variables in our more general model "degenerate" into densities of martingale measures. Our paper contains also a hedging theorem which is free from any auxiliary assumptions (cf. with results in [?], [?], and [?]). In spite that the results are mathematically simple, they are not deductible, to the best of our knowledge, from the existing in the literature. Their extensions for the arbitrary probability space and, further, to the continuous-time setting, require more sophisticated tools and will be published elsewhere.

The reader is invited to compare the suggested approach to that of the important article by Jouini and Kallal [?], conceptually different not only at the level of modeling (continuous-time setting with bid and ask prices) but also in the formulation of the no-arbitrage criteria, see the end of our note. An attempt to find the arbitrage pricing theorem (for the binomial model) can be found in the preprint [?].

Do not assuming that the reader has any background in random processes above the definitions and elementary properties of martingales and supermartingales, we explain standard (and very convenient) notations of stochastic calculus used throughout the paper. Namely, for $X = (X_t)$ and $Y = (Y_t)$ we define $X_- := (X_{t-1})$, $\Delta X_t := X_t - X_{t-1}$, and, at last,

$$X \cdot Y_t := \sum_{k=0}^{t} X_k \Delta Y_k,$$

for the discrete-time integral (here X and Y can be scalar or vector-valued). For finite Ω, if X is a predictable process (i.e., X_- is adapted) and Y is a martingale, then $X \cdot Y$ is a martingale. The product formula $\Delta(XY) = X\Delta Y + Y_-\Delta X$ is obvious. The books [?] and [?] may serve as references in convex analysis.

2 Market with transaction costs

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space equipped with a discrete-time filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T}$ and an adapted d-dimensional process $S = (S_t)$, $S_0 = s$, having the strictly positive components and describing the dynamics of prices of d assets, e.g., currencies quoted in some reference asset (say, in "euro"). We assume that $\mathcal{F}_0$ is trivial.

We consider a model with proportional transaction costs given by a $d \times d$ matrix $\Lambda = (\lambda^{ij})$ with $\lambda^{ij} \ge 0$ and $\lambda^{ii} = 0$.

The agent's positions at time t can be described either by the vector of their values in "euros"

$$V_t = (V_t^1, \dots, V_t^d)$$

or by the vector of "physical" quantities

$$\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d);$$

the obvious relation $\widehat{V}_t^i := V_t^i / S_t^i$ is just a definition of the "hat" operator.

Formalizing the notion of self-financing portfolio, we define its dynamics as follows:

$$V^i = v^i + \widehat{V}_-^i \cdot S^i + B^i, \quad i \le d, \tag{1}$$

where

$$B^i := \sum_{j=1}^{d} L^{ji} - \sum_{j=1}^{d} (1 + \lambda^{ij}) \cdot L^{ij}, \tag{2}$$

the adapted increasing process L^{ij} represents the net cumulative transfers from the i-th to the j-th asset under transaction costs. By convention, $S_{0-} = s$, $L_{0-} = 0$, and, hence, $V_{0-} = 0$.

The above relations have an obvious sense: the increment ΔV_t^i of the value invested into the i-th asset (at the end of the trading date t) consists of two parts: $\widehat{V}_{t-1}^i \Delta S_t^i$, due to the price movements, and ΔB_t, due to the agent's actions at the date t.

In our setting, the agent can choose at the time t, using only the information available to this date, a matrix formed by positive $\mathcal{F}_t$-measurable random variables ΔL^{ij} which are interpreted as the net values "arriving" to the position i from the position j. To increase the value in the i-th position for one unit by moving funds from the position j, the agent should decrease the value of the latter for $1 + \lambda^{ji}$ units, paying for this transfer λ^{ji} units as the transaction costs.

Of course, the dynamics of the portfolio expressed in units of assets depends exclusively on agent's actions. Thus, instead of (**??**), the model can be specified by the simpler relations

$$\widehat{V}^i = \widehat{v}^i + (1/S^i) \cdot B^i, \quad i \le d, \tag{3}$$

where the constant $\widehat{v}^i := v^i / S_0^i$ is the initial endowment in units. The obvious equivalence of these two descriptions can be easily justified by formal calculations.

Indeed, it follows from the definition of the "hat" operator, the product formula, and the above relation that

$$V^i = \widehat{V}^i S^i = \widehat{v}^i S_0^i + \widehat{V}_-^i \cdot S^i + S^i \cdot \widehat{V}^i = v^i + \widehat{V}_-^i \cdot S^i + B^i.$$

Since the dynamics of a portfolio depends, in fact, on the changes in positions (i.e. on ΔB) rather than on chosen transfers ΔL (which can be done, in general, with much larger flexibility), it is natural to define "controls" or "strategies" in this model directly in terms of the process B. To this aim we introduce the set $M \subseteq \mathbf{R}^d$ consisting of all x for which there exists a "transfer" matrix (a^{ij}) with nonnegative entries such that

$$x^i = -\sum_{j=1}^d a^{ji} + \sum_{j=1}^d (1+\lambda^{ij})a^{ij}, \quad i \le d. \tag{4}$$

The set M, being the image of the polyhedral cone $\mathbf{R}_+^{d^2}$ under a linear mapping, is a polyhedral cone. Its dual positive cone

$$M^* := \{w \in \mathbf{R}^d : \inf_{x \in K} wx \ge 0\}$$

is also a polyhedral cone which can be easily described by homogeneous linear inequalities. Indeed, for x given by (**??**)

$$wx = \sum_{i=1}^d w^i \left(-\sum_{j=1}^d a^{ji} + \sum_{j=1}^d (1+\lambda^{ij})a^{ij} \right) = \sum_{i,j=1}^d [-w^j + (1+\lambda^{ij})w^i]a^{ij}.$$

Thus,

$$M^* = \{w \in \mathbf{R}^d : \ w^j - (1+\lambda^{ij})w^i \le 0, \ 1 \le i,j \le d\}.$$

We define as a strategy any adapted process B such that for all $t \ge 0$ the random variables ΔB_t take values in $-M$. To make explicit the dependence on the strategy and the initial point we shall use the notation $V^{v,B}$ (or V^B if $v = 0$).

The sets of strategies and corresponding value processes starting from v will be denoted, respectively, by $\mathcal{B}$ and $\mathcal{V}^v$. In the case of finite Ω, the $\mathcal{F}_t$-measurable functions are those which are constants on the sets of the partition generating the σ-algebra $\mathcal{F}_t$. It follows that any $B \in \mathcal{B}$ admits the representation (**??**) with a certain adapted matrix-valued process L with the increasing components; in general, L is not uniquely defined by B.

The *solvency region* is the polyhedral cone $K = M + \mathbf{R}_+^d$. By definition, $x \in K$ if there exists a "transfer" matrix (a^{ij}) with nonnegative entries and a vector $l \in \mathbf{R}_+^d$ such that

$$x^i = -\sum_{j=1}^d a^{ji} + \sum_{j=1}^d (1+\lambda^{ij})a^{ij} + l^i, \quad i \le d. \tag{5}$$

The closed convex cone K induces the partial ordering: $x_1 \succeq x_2$ if $x_1 - x_2 \in K$. The dual positive cone of K is

$$K^* = M^* \cap \mathbf{R}_+^d = \{w \in \mathbf{R}_+^d : \ w^j - (1+\lambda^{ij})w^i \le 0, \ i,j \le d\}. \tag{6}$$

Let e_i be the i-th unit vector of the canonical base in $\mathbf{R}^d$ and $\mathbf{1} := \sum_{i=1}^d e_i$. In the case of the frictionless market, where all λ^{ij} are equal to zero, M is the hyperplane $\{x: \ x\mathbf{1} = 0\}$ orthogonal to the line $M^* = \{x: \ x = \alpha\mathbf{1}, \ \alpha \in \mathbf{R}\}$, and K is a halfspace $\{x: \ x\mathbf{1} \geq 0\}$ orthogonal to the ray $K^* = \{x: \ x = \alpha\mathbf{1}, \ \alpha \in \mathbf{R}_+\}$.

In the general case we classify the assets by inducing in the set $J := \{1, \dots, d\}$ a structure of a directed graph with the corresponding equivalence relation using the Boolean matrix (ε^{ij}) with $\varepsilon^{ij} := I_{\{\lambda^{ij}=0\}}$.

We say that the asset i is (freely) convertible into the asset j (notation: $i \to j$) if there are $i_0, \dots, i_q$ such that $i_0 = i$, $i_q = j$, and $\lambda^{i_0 i_1} = \lambda^{i_1 i_2} = \dots = \lambda^{i_{q-1} i_q} = 0$, i.e.

$$\varepsilon^{i_0 i_1} \varepsilon^{i_1 i_2} \dots \varepsilon^{i_{q-1} i_q} = 1. \tag{7}$$

By definition, $i \sim j$ if $i \to j$ and $j \to i$. This equivalence relation splits J into classes of equivalence $J_1, \dots, J_r$, $r \leq d$. Each J_k consists of all assets which are mutually convertible free of charge (though, maybe, not by a single transfer).

This simple and well-known construction (cf. with the classification of states of a Markov chain with a transition matrix (p^{ij}) where $\varepsilon^{ij} = I_{\{p^{ij}>0\}}$) gives more flexibility in describing properties of the model.

Let us consider the set $F := K \cap (-K)$ which is a linear subspace of $\mathbf{R}^d$. It is not difficult to prove (see Proposition 5.2 in [**?**]) that

$$F = \Big\{x \in \mathbf{R}^d: \ \sum_{i \in J_k} x^i = 0, \ k \leq r\Big\}.$$

Since $F^* = K^* - K^*$, we conclude that the linear space spanned by the cone K^* is generated by the orthogonal vectors $\mathbf{1}_k := \sum_{i \in J_k} e_i$, $k \leq r$.

The components w^i and w^j of the vector $w \in K^*$ coincide when i and j belong to the same class of equivalence. This follows from (**??**) because there is a loop containing i and j with consecutive frictionless transfers. Denoting by $\tilde{i}$ the class containing i we can write that

$$K^* = \{w \in \mathbf{R}^d_+: \ w^j - (1+\lambda^{ij})w^i \leq 0, \ \text{if } \tilde{i} \neq \tilde{j}, \ w^j = w^i, \ \text{if } \tilde{i} = \tilde{j}, \ i,j \leq d\}.$$

It is easy to see that the relative interior of K^* is given by

$$\operatorname{ri} K^* = \{w \in \mathbf{R}^d_+: \ w^j - (1+\lambda^{ij})w^i < 0, \ \text{if } \tilde{i} \neq \tilde{j}, \ w^j = w^i > 0, \ \text{if } \tilde{i} = \tilde{j}, \ i,j \leq d\}.$$

If $r = 1$ then $F = \{x: \ x\mathbf{1} = 0\}$ but M coincides with F only when all $\lambda^{ij} = 0$; if it is not the case, $M = K$.

We introduce the linear mapping $\Theta: \mathbf{R}^d \to \mathbf{R}^r$ putting $(\Theta x)^m := \sum_{i \in J_m} x^i$, $m \leq r$. For $\Theta^*: \mathbf{R}^r \to \mathbf{R}^d$ we have, by definition of the dual operator, $\Theta^* y = w$ with $w^i = y^k$ when $i \in J_k$.

Define the polyhedral cone $\tilde{K} := \Theta K$ generating the partial ordering $\tilde{\succeq}$. Then $\tilde{K}^* = (\Theta^*)^{-1} K^*$ and

$$\tilde{K}^* = \{y \in \mathbf{R}^r_+: \ y^m - (1+\tilde{\lambda}^{lm})y^l \leq 0, \ l,m \leq r\}$$

where

$$\tilde{\lambda}^{lm} := \min\{\lambda^{ij}: \ i \in J_l, \ j \in J_m\}.$$

The cone $\tilde{K}^*$ has the non-empty interior

$$\operatorname{int} \tilde{K}^* = \{y \in \mathbf{R}^r_+ : \ y^m - (1+\tilde{\lambda}^{lm})y^l < 0, \ l,m \le r, l \ne m\}.$$

Clearly, $x \succeq 0$ if and only if $\Theta x \tilde{\succeq} 0$ expressing the fact that the agent may account his capital using "aggregated" positions within each class. In the case with only one class of equivalence (e.g., when there is no friction), $\tilde{K}^* = \mathbf{R}_+$ and all positions can be aggregated.

If $y \in \operatorname{int} \tilde{K}^*$ then there exists a constant κ_y such that for all $x \in \tilde{K}$ we have the inequality

$$\kappa_y |x| \le yx. \tag{8}$$

For the reason of simplicity, the above notions were developed with $\Lambda = (\lambda^{ij})$ being a constant matrix. There are no changes if Λ is an adapted matrix-valued process (with positive entries). In such a case the introduced objects will depend on ω and t but in conventional notations of stochastic processes they are often omitted if there is no ambiguity.

We denote by $L^0(K_t, \mathcal{F}_t)$ the set of all $\mathcal{F}_t$-measurable random variables ξ such that $\xi(\omega) \in K_t(\omega)$ and alleviate the notation by omitting $\mathcal{F}_T$.

Let $\Omega = \{\omega_1, ..., \omega_N\}$. One can identify, by a linear isomorphism, the set of all mappings from Ω into $\mathbf{R}^d$ with the Euclidean space $(\mathbf{R}^d)^{\otimes N}$. It is an easy exercise to verify that $L^0(K_T)$ is a polyhedral cone in this space; the set

$$\Theta_T L^0(K_T) = L^0(\Theta_T K_T) = L^0(\tilde{K}_T)$$

is a polyhedral cone in $(\mathbf{R}^r)^{\otimes N}$.

For $v \in \mathbf{R}^d$ we define the convex sets

$$R^v_T := \{V^{v,B}_T : \ B \in \mathcal{B}\}$$

of the attainable wealth and

$$A^v_T := \{U \in L^0(\mathbf{R}^d) : \ V^{v,B}_T \succeq U \text{ for some } B \in \mathcal{B}\}$$

of all hedgeable claims. It is easy to see that R^0_T and $A^0_T = R^0_T - L^0(K_T)$ are also polyhedral cones.

3 Arbitrage pricing theorem.

From now on we assume that Λ is an adapted random process.

We consider as a *weak arbitrage opportunity at T* any strategy B such that $V^B_T \in L^0(K_T)$ but $V^B_T \notin L^0(F_T)$, or, equivalently, $\Theta_T V^B_T \tilde{\succeq}_T 0$ but $\Theta_T V^B_T \ne 0$.

For such a strategy the wealth V_T^B in spite that after liquidating the negative positions, possibly, is equal to zero, may have an interest for certain agents, e.g., for those who are not constrained by transaction costs.

We say that a strategy $B \in \mathcal{B}_T$ is a *strict arbitrage opportunity at* T if $V_T^B \in L^0(K_T)$ and $P(V_T^B \in \operatorname{int} K_T) > 0$.

We may formulate the "strict no-arbitrage" condition at the date T (excluding weak arbitrage opportunities) in two forms: either as

$\mathbf{NA}_T^s$. $R_T^0 \cap L^0(K_T) \subseteq L^0(F_T)$

or as

$\mathbf{NA}_T^s$. $A_T^0 \cap L^0(K_T) \subseteq L^0(F_T)$.

At first glance, the second version is stronger since $R_T^0 \subseteq A_T^0$. In fact, they are equivalent: if $V_T^B - k \in L^0(K_T)$ where $k \in L^0(K_T)$ then $V_T^B \in L^0(K_T)$ as well and the first property implies the second. In spite that the use of R_T^0 is more natural from the point of view of finance, the formulation involving A_T^0 is not only convenient mathematically but may serve as a starting point for deep generalizations of FTAP in the theory of continuous trading.

Respectively, the "weak no-arbitrage" condition at the date T (excluding strict arbitrage opportunities) is:

$\mathbf{NA}_T^w$. $A_T^0 \cap L^0(K_T) \subseteq L^0(\partial K_T)$

or, equivalently,

$\mathbf{NA}_T^w$. $R_T^0 \cap L^0(K_T) \subseteq L^0(\partial K_T)$.

The equivalence of these two versions is also easy to verify using the following simple geometric observation: if $k_1 \in \partial K$, $k_2 \in K$, $k_1 - k_2 \in K$, then $k_1 - k_2 \in \partial K$.

Notice that $\mathbf{NA}_T^w \iff \mathbf{NA}_t^w\ \forall t \le T$: the absence of strict arbitrage opportunities at T implies that they do not exist on the whole interval $[0,T]$. Indeed, let $B = (B_s)_{s \le t}$, $t < T$, be a strict arbitrage opportunity at t. Liquidating in a suitable way all positions by transferring values to the first one, we may assume that the first coordinate of V_t^B (as well as $\widehat{V}_t^B$) is positive and different from zero while all other coordinates vanish. Put $\Delta B_s := 0$ for $s > t$. Then $B = (B_s)_{s \le T}$ is a strict arbitrage opportunity at T.

Let $\mathcal{D}_T$ be the set of $\mathbf{R}_+^d$-valued martingales $Z = (Z_t)_{t \le T}$ with $\widehat{Z}_t \in L^0(K_t^*, \mathcal{F}_t)$ for all $t \in [0,T]$. We introduce the conditions

$\mathbf{H}_T^s$. There exists $Z^{(T)} \in \mathcal{D}_T$ such that $\widehat{Z}_T^{(T)} \in L^0(\operatorname{ri} K_T^*)$.

$\mathbf{H}_T^w$. There exists $Z^o \in \mathcal{D}_T$ such that $|\widehat{Z}_T^o| > 0$.

Lemma 3.1 *For a process $Z \in \mathcal{D}_T$ the following properties are equivalent:*

(a) $|\widehat{Z}_T| > 0$*;*

(b) $\widehat{Z}_T^i > 0$ *for every* $i \le d$*;*

(c) $\widehat{Z}_T \xi > 0$ *for every* $\xi \in L^0(\operatorname{int} K_T)$*;*

(d) $Z_T^i > 0$ *for every* $i \le d$*;*

(e) $Z_t^i > 0$ *for every* $i \le d$ *and* $t \le T$*.*

Proof. The equivalence of (b) and (d) is obvious. The properties (d) and (e) hold simultaneously, because Z^i, being positive martingales, after hitting zero remain zero forever. Of course, $(b) \Rightarrow (a)$. Since $e_i \in \operatorname{int} K_T$ we have $(c) \Rightarrow (b)$. At last, to get the implication $(a) \Rightarrow (c)$, notice that for $\xi \in L^0(\operatorname{int} K_T)$

$$\widehat{Z}_T \xi \geq \kappa_\xi |\widehat{Z}_T|$$

for some $\kappa_\xi > 0$. $\square$

Thus, in particular, $\mathbf{H}^w_T \iff \mathbf{H}^w_t \ \forall\, t \leq T$.

Theorem 3.2 *(i)* $\mathbf{NA}^s_T \iff \mathbf{H}^s_T$*;*
(ii) $\mathbf{NA}^w_T \iff \mathbf{H}^w_T$.

Before the proof we explain why the above assertion is a direct generalization of the Harrison–Pliska theorem. Indeed, in the case of the frictionless market $F = \partial K$ (hence, $\mathbf{NA}^s_T = \mathbf{NA}^w_T$) and the set $L^0(F)$ consists of random vectors ξ such that $\xi\mathbf{1} = 0$ (i.e. the sum of all components is zero). Thus, if the terminal value of a portfolio is a random vector with values in the solvency cone K, i.e. the sum of its components is non-negative, then this sum is equal to zero. This is exactly the classic definition of the no-arbitrage property. On the other side, all the components of $Z^{(T)}$ are identical. Define on $[0,T]$ the process $\rho^{(T)} := \widehat{Z}^{(T)} S^1$ which is a strictly positive martingale (being strictly positive at T). Normalizing, we may assume that $E\rho^{(T)}_T = 1$. Hence, $\tilde{P} := \rho^{(T)}_T P$ is the equivalent martingale measure in the sense that all the processes S^i/S^1 are $\tilde{P}$ martingales. Usually in the literature, the process S^1 at the very beginning is chosen as the numéraire ("bond" or "bank account"), i.e. $S^1 \equiv 1$.

Example. Let us consider the (deterministic) one-period two-asset model with $(S^1_0, S^2_0) = (1,1)$ and $(S^1_1, S^2_2) = (1,2)$. Assume that the entries of Λ are equal to zero except $\lambda^{12} = \lambda$. The vectors $(1,1)$ and $(1,1+\lambda)$ are generators of K^*. Clearly, transfers at $T = 1$ cannot increase the value, so the only strategy to be inspected is with $\Delta B_0 = (-(1+\lambda), 1)$ (the transfers are $\Delta L^{12}_0 = 1$, $\Delta L^{21}_0 = 0$) and $\Delta B_1 = (0,0)$. So, $V^B_1 = (-(1+\lambda), 2)$. For $\lambda \in [0,1[$ we have $V^B_1 \in \operatorname{int} K$, i.e. B is a strict arbitrage opportunity, for $\lambda = 1$ the model satisfies $\mathbf{NA}^w_1$ condition but the strategy B is a weak arbitrage opportunity, and if $\lambda > 1$ the model enjoys $\mathbf{NA}^s_1$ property.

We can extend the model by assuming that at the second period S^2_2 takes values ε and $1/\varepsilon$, say, with probabilities $1/2$. For $\lambda = 1$ this model satisfies $\mathbf{NA}^s_2$ when the parameter $\varepsilon > 2$ (i.e. the price increment ΔS^2_2 takes a negative and a positive value). Thus, $\mathbf{NA}^s_T$ does not imply $\mathbf{NA}^s_t$ for $t < T$.

Proof of Theorem **??**. We start with the following

Lemma 3.3 *For every* $Z \in \mathcal{D}_T$ *and* $V = V^B$, $B \in \mathcal{B}$, *the process* $\widehat{Z}V$ *is a supermartingale.*

Proof. The claim follows because in the right-hand side of the easily verified identity

$$\widehat{Z}V = \widehat{V}_- \cdot Z + \widehat{Z} \cdot B$$

(following from (**??**) and the product formula) the first term is a martingale and the second is a decreasing process. □

As in the classic case, the implications $\mathbf{H}^s_T \Rightarrow \mathbf{NA}^s_T$ and $\mathbf{H}^w_T \Rightarrow \mathbf{NA}^w_T$ are easy.

If $V \in \mathcal{V}$ is such that $V_T \in L^0(K_T)$ then for every $Z \in \mathcal{D}_T$ we have $\widehat{Z}_T V_T \geq 0$. But, in virtue of the above lemma, $E\widehat{Z}_T V_T \leq 0$. Thus, $\widehat{Z}_T V_T = 0$.

In the case of $\mathbf{H}^s_T$ where $\widehat{Z}^{(T)}_T \in L^0(\operatorname{ri} K^*_T)$ we have in virtue of (**??**) that

$$\widehat{Z}^{(T)}_T V_T = \Theta^{*-1}_T \widehat{Z}^{(T)}_T \Theta_T V_T \geq \kappa |\Theta_T V_T|$$

for some positive constant κ. Thus, $V_T \in L^0(F_T)$ and the condition $\mathbf{NA}^s_T$ is fulfilled.

In the case of $\mathbf{H}^w_T$ the relation $\widehat{Z}^o_T V_T = 0$ implies that $V_T \in L^0(\partial K_T)$.

The converse implications are also easy: they are proved by a standard argument based on the finite-dimensional separation theorem.

Let $\mathbf{NA}^s_T$ holds. Then

$$\Theta_T A^0_T \cap L^0(\Theta_T K_T) = \{0\}.$$

Let $\{U_j,\ j \leq m\}$, be the set of all (non-trivial) generators of the polyhedral cone $L^0(\Theta_T K_T)$. Let $\eta_j \neq 0$ be an element of $L^0(\mathbf{R}^r)$ which separates U_j and $\Theta_T A^0_T$, i.e. the following inequality holds:

$$\sup_{\xi \in \Theta_T A^0_T} E\eta_j \xi < E\eta_j U_j. \tag{9}$$

Because the set $\Theta_T A^0_T$ contains the cone $-L^0(\tilde{K}_T)$, we have necessarily $\eta_j \in L^0(\tilde{K}^*_T)$. Moreover, $E\eta_j U_j > 0$. Put $\eta := n^{-1}\sum_j \eta_j$. Since $E\eta U^j > 0$ for all generators U_j, we have that $\eta \in L^0(\operatorname{int} \tilde{K}^*_T)$. Define the random variable $\widehat{Z}_T := \Theta^* \eta$. Recall that $\tilde{K}^*_T = \Theta^{*-1}_T K_T$. Thus, $\widehat{Z}_T \in L^0(\operatorname{ri} K^*_T)$. The process Z with

$$Z^i_t := E(\widehat{Z}^i_T S^i_T | \mathcal{F}_t) \tag{10}$$

has the properties required in $\mathbf{H}^s_T$. We need only to check that $\widehat{Z}_t \in L^0(K^*_t, \mathcal{F}_t)$ for all $t \leq T$. It follows from (**??**) that

$$\sup_{\xi \in A^0_T} E\widehat{Z}_T \xi < \infty \tag{11}$$

and we conclude by Lemma **??** given below.

Lemma 3.4 *Let Z be the process defined by (**??**) where $\widehat{Z}_T \in L^0(K^*_T)$ is such that (**??**) holds. Then $\widehat{Z}_t \in L^0(K^*_t, \mathcal{F}_t)$ for all $t \leq T$.*

Proof. Take arbitrary $\xi \in L^0(M_t, \mathcal{F}_t)$ with $\xi^i := \sum_j((1+\lambda^{ij})\alpha^{ij} - \alpha^{ji})$, where $\alpha^{ij} \geq 0$. For any fixed t and $\beta \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ we consider the buy-and-hold strategy L with $L^{ij} := \beta\alpha^{ij} I_{[t,T]}$. Obviously,

$$E\widehat{Z}_T V_T^{v,L} = EZ_T \widehat{V}_T^{v,L} = EZ_T \widehat{V}_0 + EZ_T \widehat{B}_t = Z_0 \widehat{V}_0 + EZ_t \widehat{B}_t = \widehat{Z}_0 V_0 - E\beta \widehat{Z}_t \xi.$$

As β is arbitrary, (**??**) implies that $\widehat{Z}_t \xi \geq 0$. But $Z^i \geq 0$ and hence $\widehat{Z}_s \zeta \geq 0$ for all $\zeta \in L^0(K_t, \mathcal{F}_t)$. □

Assume now that $\mathbf{NA}_T^w$ holds. Take a countable set $\{U_i : \ i \in \mathbf{N}\}$ dense in $L^0(K_T) \setminus L^0(\partial K_T)$ and including the random variables $e_1 I_{\{\omega_j\}}$, $j \leq N$. Separate each U_i from A_T^0 by a functional η_i with $E|\eta_i| = 1$ to obtain the inequality

$$\sup_{\xi \in A_T^0} E\eta_i \xi < E\eta_i U_i.$$

As above, we infer that $\eta_i \in L^0(K_T^*)$ and $E\eta_i U_i > 0$. Put $\widehat{Z}_T := \sum_i 2^{-i}\eta_i$. Then $\widehat{Z}_T \in L^0(K_T^*)$ and $E\widehat{Z}_T U_i > 0$ for all i. In particular, we have that $E\widehat{Z}_T e_1 I_{\{\omega_j\}} > 0$. Hence $\widehat{Z}_T^1 > 0$ a.s. □

4 Hedging theorem

Let C be a $\mathbf{R}^d$-valued $\mathcal{F}_T$-measurable random variable, interpreted as a *contingent claim* of values of corresponding assets.

Our aim now is to describe the set of all initial endowments starting from which one can "super-replicate" the contingent claim C by a terminal value of a certain self-financing portfolio.

The formal description of the convex set of *hedging endowments* (in values) is as follows:

$$\Gamma := \{v \in \mathbf{R}^d : \ \exists\ B \in \mathcal{B} \text{ such that } \ V_T^{v,B} \succeq C\}.$$

We introduce also the closed convex set

$$D := \{v \in \mathbf{R}^d : \ \sup_{Z \in \mathcal{D}} E(\widehat{Z}_T C - \widehat{Z}_0 v) \leq 0\} = \bigcap_{Z \in \mathcal{D}} \{v \in \mathbf{R}^d : \ \widehat{Z}_0 v \geq E\widehat{Z}_T C\}.$$

Theorem 4.1 *We have* $\Gamma = D$.

By Lemma **??** for every process $V \in \mathcal{V}$

$$\widehat{Z}_0 v \geq E\widehat{Z}_0 V_0 \geq E\widehat{Z}_T V_T \geq E\widehat{Z}_T C$$

and the "easy" inclusion $\Gamma \subseteq D$ holds.

Take now $v \notin \Gamma$. To show that $v \notin D$ we need to find $Z \in \mathcal{D}_T$ with $\widehat{Z}_0 v < E\widehat{Z}_T C$. Since $C \notin A_T^v$, by the separation theorem

$$\sup_{U \in A_T^v} E\widehat{Z}_T U < E\widehat{Z}_T C \tag{12}$$

for some $\widehat{Z}_T \in L^0(\mathbf{R}^d)$ with $\widehat{Z}_T^i \geq 0$ due to the inclusion $-L^0(\mathbf{R}^d_+) \subseteq A^0_T$. Define Z in accordance with (**??**). It follows from Lemma **??** (which holds also with A^v_T) that $Z \in \mathcal{D}_T$. Since $\widehat{Z}_0 v = E\widehat{Z}_T V_T^{v,0}$ and $V_T^{v,0} \in A^v_T$, (**??**) yields the desired inequality. □

5 Final comments

One can observe that the different matrices Λ of transaction costs coefficients may generate the same geometric structures of the problem (i.e., the solvency cone). Moreover, in multi-asset models sometimes is tacitly assumed that all exchanges are done through the money and only transaction costs coefficients for buying and selling assets (λ^{1i} and λ^{i1}) are specified. In the need, one can complete the matrix Λ by assigning sufficiently large values of the remaining transaction costs coefficients to make the direct transfers prohibitively expensive. As an alternative, a purely geometric approach seems to be useful. The model considered in this paper allows easily for the following generalization. Assume that the portfolio dynamics is given by the relation (**??**) (or (**??**)) where the adapted process B is such that its increments ΔB_t take values in the polyhedral cones $-M_t$ (eventually, depending on ω in a causal way). Defining the solvency cones $K_t := M_t + \mathbf{R}^d_+$, we can prove Theorems **??** and **??** in this purely geometric framework assuming only that $\operatorname{int} K_t \supset \mathbf{R}^d_+ \setminus \{0\}$. The mapping Θ in this context will be the projection on the quotient space $\mathbf{R}^d/F$. Such ramifications may have a certain importance also for comparing various models with transaction costs including those where baskets of currencies are exchanged.

At last, let us consider the model where $S^1 \equiv 1$, i.e. the first asset ("money") is the numéraire, and for all i and j

$$(1+\lambda^{i1})(1+\lambda^{1j}) \leq 1+\lambda^{ij}.$$

This means that the direct exchanges are not less expensive than those via money; they can be excluded at all (as it is usually done in stock market models). According to (**??**) the cone K^* consists of all $w \in \mathbf{R}^d_+$ satisfying the inequalities

$$\frac{1}{1+\lambda^{i1}} w^1 \leq w^i \leq (1+\lambda^{1i}) w^1, \qquad i > 1.$$

Indeed, it follows that for any pair i, j we have

$$w^j \leq (1+\lambda^{1j}) w^1 \leq (1+\lambda^{i1})(1+\lambda^{1j}) w^i \leq (1+\lambda^{ij}) w^i.$$

By Theorem **??** the condition $\mathbf{NA}^w_T$ holds if and only if there is a process $Z \in D$ with $Z^i_T > 0$. In particular, $\widehat{Z}^1 = Z^1$ is a martingale; we can always assume that $EZ^1_T = 1$ and define the probability $\tilde{P} = Z^1_T P$. The condition that $\widehat{Z}$ evolves in K^* reads as

$$\frac{1}{1+\lambda^{i1}} Z^1 \leq \frac{Z^i}{S^i} \leq (1+\lambda^{1i}) Z^1, \qquad i > 1,$$

Putting $\tilde{S}^i := Z^i/Z^1$ and introducing the selling and buying prices

$$\underline{S}^i := \frac{1}{1+\lambda^{i1}} S^i, \qquad \overline{S}^i := (1+\lambda^{1i}) S^i,$$

we conclude that $\mathbf{NA}_T^w$ holds if and only if there are a process $\tilde{S}$ and an equivalent probability measure $\tilde{P}$ such that $\tilde{S}$ a martingale with respect to $\tilde{P}$ and

$$\underline{S}^i \le \tilde{S}^i \le \overline{S}^i, \quad i > 1.$$

Thus, in this case our criteria for $\mathbf{NA}_T^w$ coincides with that suggested in [**?**].

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