EXTENDED STOCHASTIC INTEGRALS[1]

YU.M. KABANOV, A.V. SKOROKHOD

In this lecture notes we present a construction of the stochastic integral generalizing the Ito integral and allowing to integrate random functions without the non-anticipativaty assumption.

An important role in our theory will play the multiple stochastic integrals introduced by Ito ([1]–[3]).

Extended stochastic integrals for the Wiener process were introduced by Hitsuda ([4], [5]) and, independently, by A.V. Skorokhod for the Gaussian processes, [6]. In [6] conditions of integrability were explored in details and the notions of stochastic derivative and stochastic integral operator were introduced. Results of Sections 5 and 6 are taken from [5]. Extended Poisson integrals were studied by Yu.M. Kabanov ([7], [8]).

Another approach to constructions of stochastic integral was developed in the papers by Yu.L. Daletski and S.N. Paramonova ([9]-[11]).

1. Multiple stochastic integral

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the basic probability space and let $(E, \mathcal{E}, m)$ be a measure space. The measure m is the structure function of some stochastic measure μ with independent values. This means that to each element of the ring of sets $\mathcal{E}_0 = \{A : \ A \in \mathcal{E},\ m(A) < \infty\}$ corresponds a random variable $\mu(A) \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}\mu(A) = 0$ such that the following conditions are satisfied:

1. If $A_n \in \mathcal{E}_0$, $n = 0, 1, \dots$, $A_0 = \cup_{n=1}^{\infty} A_n$, A_n for $n \geq 1$ are disjoint, then

$$\mu(A_0) = l.i.m. \sum_{n=1}^{\infty} \mu(A_n).$$

2. $\mathbf{E}\mu(A)\mu(B) = m(A \cap B)$.

3. If $A \cap B = \emptyset$, then the random variables $\mu(A)$ and $\mu(B)$ are independent.

We shall assume also that m is continuous in the Ito sense:

4. For any $\varepsilon > 0$ and $A \in \mathcal{E}_0$ there exist disjoint sets $A_1, \dots, A_n \in \mathcal{E}_0$ such that $m(A_i) < \varepsilon$ and $A = \cup_{i=1}^{\infty} A_i$.

[1]This is an English translation of the lecture notes in *Proceedings of the School-Seminar on the Theory of Random Processes. Druskininkai, November 25-30, 1974.* Part I. Vilnius, 1975, 123–167.

Example 1. Let $E = \mathbb{R}^1_+$, $\mathcal{E}$ be the Borel σ-algebra $\mathcal{B}^1$, and m be the Lebesgue measure Λ. The random variables $\mu(A)$ are real-valued and form a Gaussian family. It is well-known that in this case the stochastic measure can be constructed from a Wiener process w by putting $\mu(A) = \int \chi_A(t)dw_t$ where χ_A is the indicator function of the set A.

Example 2. The space is the same as in the previous example but the random variables $\mu(A)$ form a complex-valued Gaussian family. Now $\mu(A) = \int \chi_A(t)dz_t$ where $z_t = \frac{1}{\sqrt{2}}(w_t+v_t)$ is a complex-valued Wiener process (here w_t and v_t are independent real-valued Wiener processes).

Example 3. Let $E = \mathbb{R}^1_+ \times \mathbb{R}^r$, $\mathcal{E} = \mathcal{B}^1 \times \mathcal{B}^r$, $m = \Lambda \times \Pi$ where Π is the canonical Lévy measure of some homogeneous stochastically continuous process $(X_t)_{t\in\mathbb{R}^1_+}$ with independent increments, $\mu = q$ where q the centered Poisson random measure defined from X in the following way:

if $A = (t_1, t_2] \times B$ and $p(A) = \sum_{t_1<s\le t_2} \chi_B(X_s - X_{s_-})$, then

$$q(A) = p(A) - \mathbf{E}p(A) = p(A) - (t_2 - t_1)\Pi(B).$$

We define for μ the multiple stochastic integrals (MSI) starting from the class $L^2_0(E^n, m^n)$ of special step functions consisting from all linear combinations of functions having the form

$$f(x_1, \dots, x_n) = \chi_{A_{i_1}}(x_1) \dots \chi_{A_{i_n}}(x_n), \tag{1.1}$$

where $A_{i_k} \cap A_{i_j} = \emptyset$ for $k \neq j$, $A_i \in \mathcal{E}_0$.

For the functions of the form (1.1) MSI $I_n(f) = I_n(f(x_1, \dots, x_n))$ is defined as follows:

$$I_n(f) = \int \dots \int f(x_1, \dots, x_n)\mu(dx_1) \dots \mu(dx_n) = \mu(A_{i_1}) \dots \mu(A_{i_n}).$$

This definition is extended to all special step function by linearity.

By convention, $I_0(c) = c$ (c is a constant). It is easy to check that MSI defined for the functions from $L^2_0(E^n, m^n)$ has the following properties:

1) $I_n(c_1 f + c_2 g) = c_1 I_n(f) + c_2 I_n(g)$.

2) $I_n(\tilde f) = I_n(f)$ where $\tilde f$ is the symmetrization of the function f, i.e.

$$\tilde f(x_1, \dots, x_n) = \frac{1}{n!} \sum_{(\pi)} f(x_{\pi(1)}, \dots, x_{\pi(n)}),$$

the sum is taken over all permutations of the set $\{1, \dots, n\}$.

3) $\mathbf{E}I_n(f)I_n(g) = \delta_{nk} n! (\tilde f, \tilde g)_n$, $n = 0, 1, \dots,$ $(.,.)_n$ is the scalar product in $L^2(E^n, m^n)$.

The condition **4.** (the continuity of measure m), ensuring that $L^2_0(E^n, m^n)$ is dense in $L^2(E^n, m^n)$, and the inequality $||\tilde{f}||_n \leq ||f||_n$ allows us to extend MSI to the whole space with preservation of the properties 1)-3). Moreover, we shall have also the property

4) If $||f_n - f||_n \to 0$, then $l.i.m.\ I_n(f_n) = I_n(f)$.

Let us consider the space $\mathcal{H} = \oplus \sum_{n=0}^{\infty} \mathcal{H}_n$ where $\mathcal{H}_n = \{\eta : \ \eta = I_n(f),\ f \in L^2(E^n, m^n)\}$ are subspaces of $L^2(\Omega)$ orthogonal to each other.

The space $\mathcal{H}$ is sufficiently reach. In the Examples 1 and 3 it coincides with the space $L^2(\mu)$ of (classes of) square integrable random variables measurable with respect to the σ-algebra $\mathcal{G} = \sigma\{\mu(A),\ A \in \mathcal{E}_0\}$, see [1], [3]. In the Example 2 this is not the case: $\mu(A) \notin \mathcal{H}$, see Section 5.

2. Extended stochastic integral

Let $f(x, \omega)$ be a random function satisfying the following conditions:

a) $f(x, \omega)$ is measurable with respect to both variables,

b) $f(x, \omega) \in \mathcal{H}$ for every $x \in E$,

c) $\mathbf{E} \int_E |f(x, \omega)|^2 m(dx) < \infty$.

Such functions form the Hilbert space $\mathcal{G}^2$ with the scalar product

$$\langle f, g \rangle = \mathbf{E} \int_E f(x, \omega) g(x, \omega) m(dx);$$

the corresponding norm we shall denote $|||.|||$.

Any function $f \in \mathcal{G}^2$ admits the representation

$$f(x, \omega) = \sum_{n=0}^{\infty} I_n(f_n(x, x_1, \ldots, x_n)) \tag{2.1}$$

where the functions $f_n(x, x_1, \ldots, x_n)$ can be taken symmetric in variables $x_1, \ldots, x_n$. In this case

$$|||f|||^2 = \sum_{n=0}^{\infty} n! \int_E ||f_n(x, .)||_n^2 m(dx) = \sum_{n=0}^{\infty} n! ||f_n||_{n+1}^2. \tag{2.2}$$

Definition. Let $\mathcal{D}(J)$ be the set of functions $f \in \mathcal{G}^2$ admitting the representation (2.1) and such that

$$\sum_{n=0}^{\infty} (n+1)! ||\tilde{\tilde{f}}_n(x, x_1, \ldots, x_n)||_{n+1}^2 < \infty,$$

where $\tilde{\tilde{f}}_n(x, x_1, \ldots, x_n)$ is the symmetrization of f_n over all variables.

Extended stochastic integral (ESI) is the operator J acting from $\mathcal{D}(J)$ into $L^2(\Omega)$ and such that

$$J(f) = \sum_{n=0}^{\infty} I_{n+1}(f_n(x, x_1, \dots, x_n)). \tag{2.3}$$

Using the properties 1)-4) of MSI it is not difficult to prove the following statement.

Theorem 2.1 *ESI is a closed linear operator mapping the subspace $\mathcal{D}(J)$ dense in $\mathcal{G}^2$ onto $\mathcal{H}$ and such that*

$$\mathbf{E}J(f) = 0; \qquad \mathbf{E}|J(f)|^2 = \sum_{n=0}^{\infty}(n+1)!||\tilde{\tilde{f}}_n||_{n+1}^2.$$

3. Stochastic derivative

Let $H \in \mathcal{H}$. Then, by definition,

$$\eta = \sum_{n=0}^{\infty} I_n(f_n(x_1, \dots, x_n)) \tag{3.1}$$

where f_n belongs to the space $\tilde{L}^2(E^n, m^n)$ of symmetric square integrable functions,

$$\mathbf{E}|\eta|^2 = \sum_{n=0}^{\infty} n!||f_n||_n^2. \tag{3.2}$$

Let denote by $\mathcal{D}(D)$ the subset of elements $\eta \in \mathcal{H}$ for which

$$\sum_{n=1}^{\infty} n^2(n-1)!||f_n||_n^2 < \infty.$$

Clearly, $\mathcal{D}(D)$ is dense in $\mathcal{H}$.

Definition 3.1 *Stochastic derivative* is the operator D mapping $\mathcal{D}(D)$ into $\mathcal{G}^2$ and given by the equality

$$(D\eta)(x_1, \omega) = \sum_{n=1}^{\infty} nI_{n-1}(f_n(x_1, x_2 \dots, x_n)) \tag{3.3}$$

where in the right-hand side above x_1 is the free variable and I_{n-1} acts on $x_2, \dots, x_n$.

Obviously,

$$|||D\eta|||^2 = \sum_{n=1}^{\infty} n^2(n-1)!||f_n||_n^2. \tag{3.4}$$

Theorem 2.1 *The operator D is a closed linear operator.*

Proof. Let $\eta^{(k)} = \sum_{n=0}^{\infty} I_n(f_n^{(k)}) \in \mathcal{D}(D)$, $\eta = \sum_{n=0}^{\infty} I_n(f_n)$, $\eta^{(k)} \to \eta$ in $L^2(\Omega)$ as $k \to \infty$, and $|||D\eta^{(k)} - D\eta^{(j)}||| \to 0$ as $j,k \to \infty$. Let us check that $\eta \in \mathcal{D}(D)$ and $|||D\eta^{(k)} - D\eta||| \to 0$ as $k \to \infty$.

Let us consider the random variables

$$\eta_N^{(k)} = \sum_{n=0}^{N} I_n(f_n^{(k)}), \qquad \eta = \sum_{n=0}^{N} I_n(f_n).$$

Since

$$\big| |||D\eta^{(k)}||| - |||D\eta^{(j)}||| \big| \le |||D\eta^{(k)} - D\eta^{(j)}||| \to 0,$$

$|||D\eta^{(k)}||| \le c < \infty$ implying that $|||D\eta_N^{(k)}||| \le c$. The stochastic derivative $D\eta_N$ exists and

$$|||D\eta_N||| \le |||D\eta_N^{(k)} - D\eta_N||| + |||D\eta_N^{(k)}||| \le |||D\eta_N^{(k)} - D\eta_N||| + c.$$

But

$$|||D\eta^{(k)} - D\eta^{(p)}|||^2 = \sum_{n=0}^{N} n^2(n-1)!||f_n^{(k)}) - f_n^{(p)})||_n^2 \le \sum_{n=1}^{\infty} n^2(n-1)!||f_n^{(k)}) - f_n^{(p)})||_n^2 \le \varepsilon < 1$$

for sufficiently large k and p. It follows that $|||D\eta_N^{(k)} - D\eta_N||| < 1$. Thus,

$$\sum_{n=1}^{\infty} n^2(n-1)!||f_n||_n^2 = \lim_{N\to\infty} |||D\eta_N|||^2 \le (c+1)^2 < \infty,$$

$\eta \in \mathcal{D}(D)$ and $|||D\eta^{(k)} - D\eta|||^2 \le \varepsilon$. □

Let us extend the notion of stochastic derivatives to elements of the space $\mathcal{G}^2$.

Definition 3.1 Let $f(x,\omega) \in \mathcal{G}^2$ admits the decomposition (2.1) with

$$\sum_{n=1}^{\infty} n^2(n-1)!||f_n||_{n+1}^2 < \infty.$$

We shall call *stochastic derivative* of the element f the random operator D_f, acting in $L^2(E,m)$, such that

$$(D_f)_h(y) = \int_E \left[\sum_{n=1}^{\infty} n\overline{I_{n-1}(f_n(x,y,x_2,x_2\ldots,x_n))} \right] h(x)m(dx) = D(h,f)_1. \tag{3.5}$$

Note that D_f is the Hilbert–Schmidt operator and its Hilbert–Schmidt norm $\sigma(D_f)$ has a finite second moment:

$$\mathbf{E}\sigma^2(D_f) = \sum_{n=1}^{\infty} n^2(n-1)!||f_n||_{n+1}^2. \tag{3.6}$$

Theorem 3.1 *Suppose that $f \in \mathcal{G}^2$ and the stochastic derivative of f is defined. Then $f \in D(J)$ and*

$$\mathbf{E}|J(f)|^2 = |||f|||^2 + \mathbf{E}\,\mathrm{Sp}\,\overline{D_f}D_f, \tag{3.7}$$

where $\overline{D_f}$ is the integral operator with the kernel conjugate to the kernel of operator D_f.
Proof. By definition, f admits the representation (2.1). Then

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)!||\tilde{\tilde{f}}_n||^2_{n+1} &= \sum_{n=0}^{\infty}(n+1)!\int\dots\int\frac{1}{(n+1)^2}\Big|f_n(x,x_1,\dots,x_n)\\ &\quad+\sum_{i=1}^{n}f_n(x_i,x_1,\dots,x,\dots,x_n)\Big|^2 m(dx)m(dx_1)\dots m(dx_n)\\ &= \sum_{n=0}^{\infty}n!||f_n||^2_{n+1}+\sum_{n=1}^{\infty}n^2(n-1)!\int\dots\int f_n(x,y,x_2,\dots,x_n)\\ &\quad\times\overline{f_n(y,x,x_2,\dots,x_n)}m(dx)m(dy)m(dx_2)\dots m(dx_n)\\ &= |||f|||^2+\mathbf{E}\,\mathrm{Sp}\,\overline{D_f}D_f.\end{aligned}$$

The Cauchy–Bunyakovski inequality gives that

$$\mathbf{E}\,\mathrm{Sp}\,\overline{D_f}D_f \le \sqrt{\mathbf{E}\sigma^2(\overline{D_f})}\sqrt{\mathbf{E}\sigma^2(D_f)} < \infty.$$

Hence, $f \in \mathcal{D}(J)$ and the equality (3.7) holds. □

4. Extended stochastic integral with respect to a Wiener process

Let $\mu(A) = \int_{\mathbb{R}^1_+}\chi_A(t)dw_t$. In this case one get additional results for the objects introduced in Sections 1–3. For $\mu(dt)$ we shall use the notation dw_t.

Theorem 4.1 [1]. *Let $f(t_1,\dots,t_n) \in L^2(\mathbb{R}^n_+,\Lambda^n)$, $g(t) \in L^2(\mathbb{R}^1_+,\Lambda)$. Then*

$$I_{n+1}(f\otimes g) = I_n(f)I_1(g) - \sum_{j=1}^{n}I_{n-1}(f\underset{(j)}{\times}g),$$

where $(f\otimes g)(t,t_1,\dots,t_n) = f(t_1,\dots,t_n)g(t)$,

$$(f\underset{(j)}{\times}g)(t_1,\dots,\hat{t}_j,\dots t_n) = \int_{\mathbb{R}^1_+}f(t_1,\dots,t_j,\dots,t_n)dt_j.$$

Now we define the Hermite polynomials:

$$H_n(t,x) = \frac{(-t)^n}{n!}\exp\Big\{\frac{x^2}{2t}\Big\}\frac{\partial^n}{\partial x^n}\exp\Big\{-\frac{x^2}{2t}\Big\},\quad n\ge 0,$$

and note that the satisfy the following identity:

$$\sum_{n=0}^{\infty} \gamma^n H_n(t,x) = \exp\Big\{\gamma x - \frac{\sigma^2 t}{2}\Big\}. \tag{4.1}$$

Using this formula it is easy to get the following properties:

1) $(n+1)H_{n+1}(t,x) = xH_n(t,x) - tH_{n-1}(t,x), \quad n \geq 1, \quad H_0 = 1,\ H_1 = x;$

2) $\frac{\partial H_n(t,x)}{\partial x} = H_{n-1}(t,x);$

3) $\frac{1}{2}\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial H_n}{\partial t} = 0.$

From the property 1) and Theorem 4.1 it follows

Theorem 4.2 [1]. *Let* $\varphi_i \in L^2(\mathbb{R}^1_+, \Lambda)$, $(\varphi_i, \varphi_j) = 0$, $i \neq j$. *Then*

$$I_{n_1+n_2+\cdots+n_k}(\varphi_1^{\otimes n_1} \otimes \varphi_2^{\otimes n_2} \otimes \cdots \otimes \varphi_k^{\otimes n_k}) = n_1! H_{n_1}\Big(\int_0^\infty \varphi_1^2(t)dt, \int_0^\infty \varphi_1(t)dw_t\Big)$$
$$\times n_2! H_{n_2}\Big(\int_0^\infty \varphi_2^2(t)dt, \int_0^\infty \varphi_2(t)dw_t\Big) \dots n_k! H_{n_k}\Big(\int_0^\infty \varphi_k^2(t)dt, \int_0^\infty \varphi_k(t)dw_t\Big).$$

On the other hand, the multiple Wiener integrals can be expressed via iterations of the Ito integrals.

Theorem 4.3 [1]. *Let* $f \in L^2(\mathbb{R}^n_+, \Lambda^n)$. *Then*

$$I_n(f(t_1,\dots,t_n)) = n! \int_0^\infty \Big(\int_0^{t_n} \Big(\dots \Big(\int_0^{t_2} \tilde{f}(t_1,\dots,t_n)dw_{t_1}\Big)dw_{t_2}\Big)\dots\Big)dw_{t_n}.$$

Remark. Since $L^2(w) = \mathcal{H}$, Theorem 4.3 implies that any r.v. $\eta \in L^2(w)$ is a sum of a constant and a stochastic integral Ito: $\eta = \mathcal{M}\eta + \int_0^\infty \varphi(s,\omega)dw_s$. This is the so-called "representation theorem" playing an important role in the nonlinear filtering theory.

Let us show that for the non-anticipating integrants ESI coincides with the Ito integral. Indeed, let $f(t,\omega) = c(\omega)\chi_{(s_1,s_2]}(t) \in \mathcal{D}(J)$. Then

$$f(t,\omega) = \sum_{n=0}^{\infty} I_n(\chi_{(s_1,s_2]}(t) f_n(t_1,\dots,t_n))$$

where f_n vanishes when at least one of the variables t_j is greater than s_1. By definition of ESI

$$J(f) = \sum_{n=0}^{\infty} I_{n+1}(\chi_{(s_1,s_2]}(t) f_n(t_1,\dots,t_n)) = (w_{s_2} - w_{s_1}) \sum_{n=0}^{\infty} I_n(f_n(t_1,\dots,t_n)) = c(\omega)(w_{s_2} - w_{s_1}).$$

The last expression coincides with the Ito integral for the function $f(t, \omega)$. In the above calculations we use the fact that the supports of f_n and χ are disjoint. Moreover, for the non-anticipations step functions

$$\mathbf{E}|J(f)|^2 = \sum_{n=0}^{\infty}(n+1)!||\tilde{\tilde{f}}_n||^2_{n+1} = \sum_{n=0}^{\infty} n! \int ||\tilde{f}_n(t)||^2_n dt = \mathbf{E}\int f^2(t,\omega)dt.$$

Using a passage to the limit we get that $J(f)$ is an extension of the Ito integral.

In the case of Gaussian processes multiple integrals can be constructed not starting from functions but prom polylinear functionals. We explain this construction by example.

Let w_t be a Wiener process, $t \in [0,1]$. For each element $f \in L^2([0,1],\Lambda)$ the Wiener integral $\int f(t)dw_t$ is defined, $\mathbf{E}\int f(t)dw_t = 0$, and if $g \in L^2([0,1],\Lambda)$, then $\mathbf{E}\int f(t)dw_t \int g(t)dw_t = (f,g)_1$.

It is well known that in the space $L^2([0,1],\Lambda)$ where is no random element $\dot{w}$ such that $\int f(t)dw_t = (f,\dot{w})_1 = \int f(t)\dot{w}dt$. Indeed, suppose that such an element exists. Then for an orthonormal basis $\{e_j\}_1^\infty$ in $L^2([0,1],\Lambda)$, the sequence $\{(e_j,\dot{w})\}_1^\infty$ is a sequence of independent Gaussian random variables with zero mean and unit variance. By the strong law of large numbers

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}(e_j,\dot{w})_1^2 = 1 \quad P\text{-a.s.}$$

On the other hand, since $\dot{w} = \sum_{j=1}^\infty (e_j,\dot{w})_1 e_j$, we have that $\sum_{j=1}^\infty (e_j,\dot{w})_1^2 = ||\dot{w}||_1^2 < \infty$ leading to a contradiction.

However, one can extend the space $L^2([0,1])$ with a help of some nuclear operator S and the obtained extension $\mathbb{X}_-$ will contain an element $\dot{w}$ such that the scalar product $(f,\dot{w})_1$ will have sense for $f \in L^2([0,1])$ and will be equal to $\int f(t)dw_t$.

Note that the formula

$$A_f(g_1,\dots,g_n) = \int\dots\int f(t_1,\dots,t_n)g_1(t_1)\dots g_n(t_n)m(dt_1)\dots m(dt_n), \tag{4.2}$$

where $f \in \tilde{L}^2(E^n,m^n)$, defines an n-linear symmetric continuous functional on $L^2(E,m)$ and, vice versa, each functional of this type can be obtained by (4.2) with some function $f \in \tilde{L}^2(E^n,m^n)$. To every n-linear symmetric functional A_{f_n} one can associate the random variable $I_n(f_n)$. This can be done without using the representation (4.2). Indeed, define MSI $A_n(\dot{w},\dots,\dot{w})$ by induction. For $n=1$ the continuous linear functional $A_1(g)$ is the scalar product (g,f). We associate with $A_1(g)$ the r.v. $A_1(\dot{w}) = (f,\dot{w})_1$. Suppose that MSI are defined for $k \le n$. If $\{e_j\}_1^\infty$ is an orthonormal basis, then

$$A_n(\dot{w},\dots,\dot{w}) = \sum_{j=1}^{\infty}\left[A_n(\dot{w},\dots,\dot{w},e_j)(e_j,\dot{w})_1 - (n-1)A_n(\dot{w},\dots,\dot{w},e_j,e_j)\right].$$

For a functional $A_n(g_1,\dots,g_n) = A_{n-1}(g_1,\dots,g_{n-1})(f,g_n)$ we define MSI by the equality

$$A_n(\dot{w},\dots,\dot{w}) = A_{n-1}(\dot{w},\dots,\dot{w})(f,\dot{w})_1 - (n-1)A_{n-1}(\dot{w},\dots,\dot{w},f).$$

Clearly, this definition leads to the multiple integrals and $\dot{w}$ can be replaced by any generalized Gaussian process, ([6]).

Now let us study the relation between stochastic derivative and differentiation of functionals in the Hilbert space.

Let $f(x)$ be a continuous function on $L^2([0,1])$ which can be continuously extended on $\mathbb{X}_-$. This extension we also will denote f. Since $\dot{w}$ is a usual r.v. with values in $\mathbb{X}_-$, the value $f(\dot{w})$ is well-defined. If $f_n(x)$ is such that $f_n(\dot{w})$ is defined and $\lim_{n,k\infty} \mathbf{E}|f_n(\dot{w}) - f_k(\dot{w})| \to 0$, we put $f(\dot{w}) := l.i.m.f_n(\dot{w})$. The set of functions which admits the argument $\dot{w}$ is sufficiently large. In particular, $\dot{w}$ can be plugged into polynomials. The substitution operation is different from formation of MSI. Indeed, if we have the n-linear functional $A_n(g_1,\dots,g_n) = (f,g_1)_1\dots(f,g_n)_1$ to which corresponds the polynomial $A_n(g,\dots,g_n) = (f,g)_1^n$, we get as the result of substitution $(f,\dot{w})_1^n$. The MSI constructed for the functional $A_n(g_1,\dots,g_n)$ is equal to $n!H_n(||f||_1^2,(f,\dot{w})_1)$, where H_n is the Hermite polynomial.

Theorem 4.3 *Let $f(x)$ be a polynomial. Then*

$$f'(x)\|_{x=\dot{w}} = D[f(\dot{w})].$$

Proof. It is sufficient to consider the case where $f(x) = (z,x)_1^n = \sum_{i=0}^n c_i H_i((z,x)_1)$, c_i are constants, $H_i((z,x)_1) = H_i(||z||_1^2,(f,x)_1)$ are the Hermite polynomials. Let us calculate the derivative:

$$(f'(x),y)_1 = n(z,x)_1^{n-1}(z,y)_1 = \sum_{i=1}^n c_i H_{i-1}((z,x)_1)(z,y)_1.$$

Thus, $(f'(x)|_{x=\dot{w}},y)_1 = \sum_{i=1}^n c_i H_{i-1}((z,\dot{w})_1)(z,y)_1$. On the other hand,

$$\begin{aligned}(D[f(\dot{w})],y)_1 &= \Big(D\sum_{i=0}^n c_i H_i((z,\dot{w})_1),y\Big)_1 = \sum_{i=0}^n \frac{1}{i!}c_i(DI_i(z^{\otimes i}),y)_1 \\ &= \sum_{i=1}^n \frac{1}{(i-1)!}c_i I_{i-1}(z^{\otimes(i-1)})(z,y)_1 = \sum_{i=1}^n c_i H_{i-1}((z,\dot{w})_1)(z,y)_1.\end{aligned}$$

Therefore, the stochastic derivative is the closure in L^2 of the ordinary derivatives with the element $\dot{w}$ plugged in. $\square$.

Let $\{e_j\}_1^\infty$ be an orthonormal basis in the space $L^2([0,1],\Lambda)$. We denote by $\mathcal{D}_0(J)$ the linear subspace in $\mathcal{D}(J)$ spanned by the functions $I_n(f_n(t,t_1,\dots,t_n))$ such that in the Fourier expansion $f_n(t,t_1,\dots,t_n) = \sum c_{ii_1\dots i_n}e_i(t)e_{i_1}\dots e_{i_n}$ the coefficients are such that for all $i_2,\dots,i_n$

$$\sum_i c_{iii_2\dots i_n} = 0. \tag{4.3}$$

Theorem 4.3 *The subspace $\mathcal{D}_0(J)$ is dense in $\mathcal{D}_(J)$ and for all $f(t,x)\in\mathcal{D}_0(J)$ there is the equality $J(f) = (f,\dot{w}).$, where the scalar product is understood as the convergent in $L^2(\Omega)$ series $\sum_i(f,e_i)_1(e_i,\dot{w})_1$.*

Proof. Let $f(t,\omega) = I_n(f_n(t,t_1,\dots,t_n)) \in \mathcal{D}(J)$ where the function $f_n(t,t_1,\dots,t_n)$ is symmetric in variables $t_1,\dots,t_n$. Then

$$f_n(t,t_1,\dots,t_n) = \sum c_{ii_1\dots i_n} e_i(t) e_{i_1}\dots e_{i_n}. \tag{4.4}$$

where the coefficients $c_{ii_1\dots i_n}$ are symmetric in $i_1,\dots,i_n$ and

$$|||f_n|||^2 = n!\sum c^2_{ii_1\dots i_n} < \infty. \tag{4.5}$$

As an approximating function we take $f^{(k)}(t,\omega) = I_n(f_n^{(k)}(t,t_1,\dots,t_n))$ where $f_n^{(k)}$ is defined by the Fourier coefficients

$$c^{(k)}_{ii_1\dots i_n} = \begin{cases} c_{ii_1\dots i_n}, & \text{if } i \le k \text{ or } \max_{1\le j\le n} i_j \le k, \\ \frac{-1}{N_k-k}\sum_{1\le i\le n} c_{ii_1\dots i_n}, & \text{if } k \le i \le N_k \text{ and there is } j \text{ such that } i_j = i, \\ 0, & \text{otherwise.} \end{cases}$$

The value N_k we choose later. It is clear that the coefficients $c^{(k)}_{ii_1\dots i_n}$ are symmetric in $i_1,\dots,i_n$ and the equality (4.3) and the inequality (4.5) hold. Obviously,

$$\frac{1}{n!}|||f_n - f_n^{(k)}|||^2 = \sum |c_{i_0i_1\dots i_n} - c^{(k)}_{i_0i_1\dots i_n}|^2 \le 2\sum_{i_0\vee\dots\vee i_n>k} |c_{i_0i_1\dots i_n}|^2 + 2\sum_{i_0\vee\dots\vee i_n>k} |c^{(k)}_{i_0i_1\dots i_n}|^2 = 2\Sigma_1 + 2\Sigma_2.$$

The first sum converges to zero as $k\to\infty$. The second sum can be estimated as follows:

$$\begin{aligned} \Sigma_2 &\le n \sum_{i_2,\dots,i_n}\sum_{i_1=i=k+1}^{N_k}\sum_{i=1}^{k} |c_{iii_2\dots i_n}|^2 + \sum_{i_0\vee\dots\vee i_n>k} |c_{i_0i_1\dots i_n}|^2 \\ &\le \frac{nk}{N_k-k}\sum_{i_2,\dots,i_n}\sum_{i=1}^{k} |c_{iii_2\dots i_n}|^2 + \Sigma_1 \le \frac{nk}{N_k-k}\sum |c_{iii_2\dots i_n}|^2 + \Sigma_1. \end{aligned}$$

If N_k are such that $N_k/k\to\infty$ as $k\to\infty$, then our estimates imply that $|||f_n - f_n^{(k)}|||\to 0$ as $k\to\infty$ and, therefore, $\mathcal{D}_0(J)$ is danse in $\mathcal{G}^2$.

Suppose now that $f(t,\omega) = I_n(f_n(t,t_1,\dots,t_n)) \in \mathcal{D}_0(J)$, that is the equality (4.3) hold. Then

$$\begin{aligned} J(f) &= I_{n+1}\Big(\sum c_{ii_1\dots i_n} e_i\otimes e_{i_1}\otimes\dots\otimes e_{i_n}\Big) \\ &= \sum_{i_1,\dots,i_n}\sum_i I_n(c_{ii_1\dots i_n} e_{i_1}\otimes\dots\otimes e_{i_n}) I_1(e_i) - nI_{n-1}\Big(\sum c_{ii_1\dots i_n}\delta_{i_1 i} e_{i_1}\otimes\dots\otimes e_{i_n}\Big) \\ &= \sum_i (f,e_i)_1(e_i,\dot w)_1. \end{aligned}$$

This completes the proof. □

Let $f(t,\omega) = I_n\big(\sum c_i\chi_{\Delta_i}(t) f_n^{(i)}(t_1,\dots,t_n)\big)$, where $f_n^{(i)}(t_1,\dots,t_n) \in \tilde L^2([0,1]^n,\Lambda^n)$. If

$$\int\sum c_i\chi_{\Delta_i}(t) f_n^{(i)}(t_1,\dots,t_n)dt = 0, \tag{4.6}$$

then ESI can be defined in a usual way via integral sums. If the equality (4.6) does not hold, then one need to regularize the integral sum ([9]).

Now we describe the following alternative approach to define ESI. Let $f(x)$ be a differentiable functional on $L^2([0,1],\Lambda)$ such that $f(\dot{w})$ and $f'(\dot{w})$ are defined. It is easy to check starting from polynomials that if $\mathbf{E}f(\dot{w})^2 < \infty$ and $\mathbf{E}||f(\dot{w})||_1^2 < \infty$, then

$$\mathbf{E}f(\dot{w})(\phi,\dot{w})_1 = \mathbf{E}(f'(\dot{w}),\phi)_1. \tag{4.7}$$

The formula (4.7) implies the equality

$$\mathbf{E}f(\dot{w})(\phi,\dot{w})_1(\phi,\dot{w})_1 = (\phi,\psi)_1\mathbf{E}f(\dot{w}) + \mathbf{E}[f''(\dot{w})(\phi,\psi)]. \tag{4.8}$$

The formula below for computing the correlation is a corollary of (4.8)[1]:

$$\mathbf{E}\big[f_1(\dot{w})(\phi,\dot{w})_1 - (f_1'(\dot{w}),\phi)_1\big]\big[f_2(\dot{w})(\psi,\dot{w})_1 - (f_2'(\dot{w}),\psi)_1\big] = (\phi,\psi)_1\mathbf{E}f_1(\dot{w})f_2(\dot{w}) \tag{4.9}$$
$$+\mathbf{E}(f_1'(\dot{w}),\psi)_1(f_2'(\dot{w}),\phi)_1.$$

Let now $f(t,\omega) = \sum_{k=1}^n \chi_{\Delta_k}(t)f_k(\dot{w})$, where $f_k(x)$ are smooth functionals. Put

$$J(f) = \sum_{k=1}^n \big[f_k(\dot{w})(\chi_{\Delta_k},\dot{w})_1 - (f_k'(\dot{w}),\chi_{\Delta_k})_1\big]. \tag{4.10}$$

The equality (4.10) defines the so-called "regularized" integral. Clearly, $\mathbf{E}J(f) = 0$. It is easy to show, using (4.9) that

$$\mathbf{E}[J(f)]^2 = |||f|||^2 + \mathbf{E}\mathrm{Sp}\, D_f D_f. \tag{4.11}$$

The formula (4.11) allows us to extend the definition to a wider class of functions. It is clear, that for the step functions the "regularized" integral given by (4.10) coincides with ESI.

This approach to definition of ESI allows generalizations in various directions ([9] – [11]).

5. Holomorphic stochastic integrals

In this section we consider the random measure of Example 2. So, $m(A) = \int \chi_A(t)dz_t$, i.e. $\mu(dt) = dz_t$. The space $\mathcal{H}$ consist of all random variables η admitting the representation

$$\eta = \sum_{n=0}^\infty I_n(f_n(t_1,\dots,t_n)). \tag{5.1}$$

Though $\mathcal{H} \neq L^2(\mu)$, one can introduce the multiple integrals

$$I_{pd}(f_{pq}) = \int\dots\int f_{pq}(t_1,\dots,t_p,s_1,\dots,s_q)dz_{t_1}\dots dz_{t_p}\bar{d}z_{s_1}\dots\bar{d}z_{s_q}$$

such that any r.v. $\zeta \in L^2(\mu)$ will be represented as $\zeta = \sum_{p,q=0}^\infty I_{pd}(f_{pq})$ ([2]). We are interested in the integrals $I_{p,0} = I_p$ which we shall call holomorphic.

In the construction of the HMSI there is no need to use special step functions.

[1] This formula was established in [9]

Note that for a complex-valued Gaussian r.v. ξ with $\mathbf{E}\xi = 0$ and $\mathbf{E}(\mathrm{Re}\,)^2 = \mathbf{E}(\mathrm{Im}\,)^2$ we have the following formula for the moments:

$$\mathbf{E}\xi^n\bar{\xi}^k = \delta_{nk}(\mathbf{E}|\xi|^2)^k, \quad n,k = 1;2;\ldots. \tag{5.2}$$

Define HMSI for all step functions in the usual way. Using the formula (5.2) it is not difficult to get for HMSI with step integrants $f \in L^2(\mathbb{R}^n_+, \Lambda^n)$ and $g \in L^2(\mathbb{R}^k_+, \Lambda^k)$ the equality

$$\mathbf{E}I_n(f)\overline{I_k(g)} = \delta_{nk}n!(\tilde{f},\tilde{g})_n \tag{5.3}$$

which allows us to extend the definition on the whole space $L^2(\mathbb{R}^n_+, \Lambda^n)$. This integral coincides with defined earlier.

The introduced HMSI has the following remarkable property (cf. Theorem 4.1 and 7.1):

$$I_n(f)I_k(g) = I_{n+k}(f \otimes g). \tag{5.4}$$

We shall denote $\int_S^T f(u,\omega)dz_u$ the ESI $J(\chi_{[S,T]}(u)f(u,\omega))$ and write that $f \in \mathcal{D}(J,[S,T])$ if the function $\chi_{[S,T]}(u)f(u,\omega) \in \mathcal{D}(J)$.

The equality (5.4) allows to establish the following two statements.

Lemma 5.1. $n\int_S^T z_u^{n-1}dz_u = z_T^n - z_S^n$.

Lemma 5.2. *If $f(u,\omega) \in \mathcal{D}(J)$, $c(\omega)f(u,\omega) \in \mathcal{D}(J)$, then*

$$\int c(\omega)f(u,\omega)dz_u = c(\omega)\int f(u,\omega)dz_u.$$

Theorem 5.1. *Let $f(t,z,\omega) = \sum_{n=0}^\infty a(t,\omega)z^n$ belongs to $\mathcal{H}$ for each (t,z) and continuously differentiable in (t,z) (i.e. $f(t,z,\omega)$ is analytic in z). Suppose that $a_n(t,\omega)$ and $\partial a_n(t,\omega)/\partial t$ also belong to $\mathcal{H}$ and the following conditions are fulfilled:*

(a) $f(t,,z_t,\omega) = \sum_{n=0}^\infty a(t,\omega)z_t^n$ (this series converges in $L^2(\omega)$);

(b) $\frac{\partial a_n}{\partial t}(t,\omega)z_u^n$ for every u as a function of variable t belongs to $\mathcal{D}(J,[S,T])$, continuous in $L^2(\Omega)$ as a function of (t,u), and the series $\sum_{n=0}^\infty \frac{\partial a_n}{\partial t}(t,\omega)z_t^n$ converges in $\mathcal{G}^2$;

(c) $a_n(t,\omega)nz_u^{n-1}$ for every $t \in [S,T]$ as a function of u belongs to $\mathcal{D}(J,[S,T])$, is continuous in $L^2(\Omega)$ as a function of (t,u), and the series $\sum_{n=1}^\infty a(t,\omega)nz_t^{n-1} = \frac{\partial f}{\partial z}(t,z_t,\omega)$ converges in $\mathcal{G}^2([S,T])$.

Then $\frac{\partial f}{\partial z}(t,z_t,\omega) \in \mathcal{D}(J,[S,T])$ and the following equality holds:

$$f(T,z_T,\omega) - f(S,z_S,\omega) = \int_S^T \frac{\partial f}{\partial z}(t,z_t,\omega)dz_t + \int_S^T \frac{\partial f}{\partial t}(t,z_t,\omega)dt.$$

Proof. Let $f(t,,z,\omega) = a(t,\omega)z^n$. Then

$$\begin{aligned}
a(T,\omega)z_T^n - a(S,\omega)z_S^n &= \sum_{j=1}^k \Big\{a(t^{(j)},\omega)z^n_{t^{(j)}} - a(t^{(j-1)},\omega)z^n_{t^{(j-1)}}\Big\} \\
&= \sum_{j=1}^k \Big\{a(t^{(j)},\omega)(z^n_{t^{(j)}} - z^n_{t^{(j-1)}} + \big[a(t^{(j)},\omega) - a(t^{(j-1)},\omega)\big] z^n_{t^{(j-1)}}\Big\} \\
&= \sum_{j=1}^k \Bigg\{\int_{t^{(j-1)}} t^{(j)} a(t^{(j)},\omega) n z_t^{n-1} + \int_{t^{(j-1)}}^{t^{(j)}} \frac{\partial a}{\partial t}(t,\omega) z^n_{t^{(j-1)}} dt\Bigg\}
\end{aligned}$$

for every partition $S = t^{(0)} \leq t^{(1)} \leq \dots \leq t^{(k)} = T$ of the interval $[S,T]$. The condition (a), (b), (c) ensures the possibility to take the limits. □

It is obvious that HMSI and MSI with respect to the real-valued Wiener process w_t are related in the following way:

$$\int \dots \int f(t_1,\dots,t_n) dw_{t_1} \dots dw_{t_n} = (\sqrt{2})^n \mathbf{E}\Big[\int \dots \int f(t_1,\dots,t_n) dz_{t_1} \dots dz_{t_n}\Big| \mathcal{F}^w\Big]. \tag{5.5}$$

This relation implies the following

Lemma 5.2. *If $f(s,\omega) \in \mathcal{D}(J_z,[S,T])$, then $\mathbf{E}[f(s,\omega)|\mathcal{F}^w] \in \mathcal{D}(J_w,[S,T])$ and*

$$\mathbf{E}\Big[\int_S^T f(s,\omega) dz_s | \mathcal{F}^w\Big] = \frac{1}{\sqrt{2}} \int_S^T \mathbf{E}[f(s,\omega)|\mathcal{F}^w] dw_s. \tag{5.6}$$

Remark. If the definition of HMSI is given first (without using the special step functions), then the equality (5.5) can be used to define the multiple stochastic integrals with respect to w_t.

6. A generalized Ito formula for ESI with respect to the Wiener process

We use the abbreviations $(\vec{t},\vec{x}) = (t_1,\dots,t_n,x_1,\dots,x_n)$ and $(\vec{t},w_{\vec{t}}) = (t_1,\dots,t_n,w_{t_1},\dots,w_{t_n})$.

Theorem 6.1. *Let $f(\vec{t},\vec{x})$ be a real-valued (or complex-valued) function on $Q \times \mathbb{R}^n$ where*

$$Q = \{\vec{t}\colon\ 0 \leq t_1 < \dots < t_n\}.$$

If

$$\mathbf{E}\Bigg\{\big|f(\vec{t},w_{\vec{t}})\big|^2 + \sum_{j=1}^n \Big|\frac{\partial f}{\partial t_j}(\vec{t},w_{\vec{t}})\Big|^2 + \sum_{i,j=1}^n \Big|\frac{\partial f^2}{\partial x_i \partial x_j}(\vec{t},w_{\vec{t}})\Big|^2\Bigg\} < c, \tag{6.1}$$

$t_i \in [S_i,T_i]$, $(t_1,\dots,S_i,\dots,t_n)$, $(t_1,\dots,T_i,\dots,t_n)$ in Q, then $\frac{\partial f}{\partial x_i} \in \mathcal{D}(J,[S_i,T_i])$ and

$$\begin{aligned}
f(\vec{t},w_{\vec{t}})\Big|_{s_i=S_i}^{t_i=T_i} &= \int_{S_i}^{T_i} \frac{\partial f}{\partial x_i}(\vec{t},w_{\vec{t}}) dw_{t_i} \qquad (6.2)\\
&+ \int_{S_i}^{T_i} \Bigg\{\frac{1}{2}\frac{\partial f^2}{\partial x_i^2}(\vec{t},w_{\vec{t}}) + \sum_{j=i+1}^n \frac{\partial f^2}{\partial x_i \partial x_j}(\vec{t},w_{\vec{t}}) + \frac{\partial f}{\partial t_i}(\vec{t},w_{\vec{t}})\Bigg\} dt_i.
\end{aligned}$$

Proof. We establish first that (6.2) holds for $f(\vec{t},\vec{x}) = x_1^{k_1}\dots x_n^{k_n}$. Let us introduce the auxiliary function

$$g(\vec{t},\vec{z}) = \exp\left\{\sum_{j=1}^{n}\frac{1}{2}\lambda_j^2 t_j + \sum_{n\geq l>j\geq 1}\lambda_l\lambda_j t_j\right\}\exp\left\{\sqrt{2}\sum_{j=1}^{n}\lambda_j z_j\right\}$$

and applied to it Theorem 5.1:

$$g(\vec{t},z_{\vec{t}})\Big|_{s_i=S_i}^{t_i=T_i} = \sqrt{2}\int_{S_i}^{T_i} g(\vec{t},z_{\vec{t}})dz_{t_i} + \Big(\frac{1}{2}\lambda_i^2 + \sum_{n\geq j>i}\lambda_i\lambda_j\Big)\int_{S_i}^{T_i} g(\vec{t},z_{\vec{t}})dt_i. \tag{6.3}$$

Since $\mathbf{E}[g(\vec{t},z_{\vec{t}})|\mathcal{F}^w] = \exp\left\{\sum_{j=1}^{n}\lambda_j w_{t_j}\right\}$, we get, using Lemma 5.3, the equality

$$\exp\Big\{\sum_{j=1}^{n}\lambda_j w_{t_j}\Big\}\Big|_{s_i=S_i}^{t_i=T_i} = \lambda_i\int_{S_i}^{T_i}\exp\Big\{\sum_{j=1}^{n}\lambda_j w_{t_j}\Big\}dw_{t_i} + \Big(\frac{1}{2}\lambda_i^2 + \sum_{n\geq j>i}\lambda_i\lambda_j\Big)\int_{S_i}^{T_i}\exp\Big\{\sum_{j=1}^{n}\lambda_j w_{t_j}\Big\}dt_i. \tag{6.4}$$

To check that (6.2) holds for functions of the form $f(\vec{t},\vec{x}) = x_1^{k_1}\dots x_n^{k_n}$ it is sufficient to expand both sides of (6.4) in series in powers λ and equalize the coefficients at the same power. Hence,

$$\begin{aligned} w_{t_1}^{k_1}\dots w_{t_i}^{k_i}\dots w_{t_n}^{k_n}\Big|_{s_i=S_i}^{t_i=T_i} \quad = \quad & \int_{S_i}^{T_i} k_i w_{t_1}^{k_1}\dots w_{t_i}^{k_i-1}\dots w_{t_n}^{k_n}dw_{t_i} \\ & + \int_{S_i}^{T_i}\Big\{\frac{1}{2}k_i(k_i-1)w_{t_1}^{k_1}\dots w_{t_i}^{k_i-2}\dots w_{t_n}^{k_n} \\ & + \sum_{j=i+1}^{n} k_i k_j w_{t_1}^{k_1}\dots w_{t_i}^{k_i-1}\dots w_{t_i}^{k_j-1}\dots w_{t_n}^{k_n}\Big\}dt_i \end{aligned} \tag{6.5}$$

The next step consists in the verification of (6.2) for the functions of the form

$$f(\vec{t},\vec{x}) = a(\vec{t})x_1^{k_1}\dots x_n^{k_n}, \tag{6.6}$$

where $a(\vec{t}) \in C^1$.

For any partition of the interval $[S_i, T_i]$, $S_i = t^{(0)} < ... < t^{(l)} < ... < t^{(r)} = T_i$, we have with $h = \max|t^{(l)} - t^{(l-1)}|$, using the formula (6.5):

$$\begin{aligned}
a(\vec{t})w_{t_1}^{k_1}...w_{t_i}^{k_i}...w_{t_n}^{k_n}\Big|_{s_i=S_i}^{t_i=T_i} &= \sum_{l=1}^{r}\big\{a(t_1,...,t^{(l)},...,t_n) - a(t_1,...,t^{(l-1)},...,t_n)\big\}w_{t_1}^{k_1}...w_{t^{(l)}}^{k_i}...w_{t_n}^{k_n} \\
&+\sum_{l=1}^{r} a(t_1,...,t^{(l-1)},...,t_n)\big\{w_{t_1}^{k_1}...w_{t^{(l)}}^{k_i}...w_{t_n}^{k_n} - w_{t_1}^{k_1}...w_{t^{(l-1)}}^{k_i}...w_{t_n}^{k_n}\big\} \\
&= \sum_{l=1}^{r}\Big\{\frac{\partial a}{\partial t_i}a(t_1,...,t^{(l)},...,t_n) + o(h)\Big\}w_{t_1}^{k_1}...w_{t^{(l)}}^{k_i}...w_{t_n}^{k_n}(t^{(l)} - t^{(l-1)}) \\
&+\sum_{l=1}^{r} a(t_1,...,t^{(l)},...,t_n)\Bigg\{k_i\int_{t^{(l-1)}}^{t^{(l)}} w_{t_1}^{k_1}...w_{t_i}^{k_i-1}...w_{t_n}^{k_n}dw_{t_i} \\
&+\frac{1}{2}k_i(k_i-1)\int_{t^{(l-1)}}^{t^{(l)}}\Big\{w_{t_1}^{k_1}...w_{t_i}^{k_i-2}...w_{t_n}^{k_n} \\
&+\sum_{j=i+1}^{n} k_ik_jw_{t_1}^{k_1}...w_{t_i}^{k_i-1}...w_{t_j}^{k_j-1}...w_{t_n}^{k_n}\Big\}dt_i\Bigg\}.
\end{aligned}$$

Taking the limit in $L^2(\Omega)$ as $h \to \infty$ we get that (6.2) holds for function having form (6.3).

To complete the proof we need to approximate appropriately an arbitrary function from $C^{1,2}$ satisfying the conditions (6.1). To simplify calculations, we make the change of variables

$$\bar{t}_j = t_j - t_{j-1}, \quad \bar{x}_j = x_j - x_{j-1}, \quad j = 1,...,n, \quad x_0 = t_0 = 0. \tag{6.7}$$

Define the function $\bar{f}$ by the formula

$$\bar{f}(\vec{\bar{t}}, \vec{\bar{x}}) = \bar{f}(\bar{t}_1, \dots, \bar{t}_n, \bar{x}_1, \dots, \bar{x}_n) = f(t_1, \dots, t_n, x_1, \dots, x_n)$$

and introduce the notation $(\vec{\bar{t}},\ \vec{\bar{w}}_t) = (\bar{t}_1, \dots, \bar{t}_n, w_{t_1}, w_{t_2} - w_{t_1}, ..., w_{t_n} - w_{t_{n-1}})$. Then the formula (6.2) can be written as

$$\begin{aligned}
(\vec{\bar{t}},\ \vec{\bar{w}}_t)\Big|_{s_i=S_i}^{t_i=T_i} &= \int_{S_i}^{T_i}\Big(\frac{\partial\bar{f}}{\partial\bar{x}_i} - \frac{\partial\bar{f}}{\partial\bar{x}_{i+1}}\Big)(\vec{\bar{t}},\ \vec{\bar{w}}_t)dw_{t_i} \\
&+\int_{S_i}^{T_i}\Big\{\frac{1}{2}\Big(\frac{\partial^2\bar{f}}{\partial\bar{x}_i^2} - \frac{\partial^2\bar{f}}{\partial\bar{x}_{i+1}^2}\Big)(\vec{\bar{t}},\ \vec{\bar{w}}_t) + \Big(\frac{\partial\bar{f}}{\partial\bar{t}_i} - \frac{\partial\bar{f}}{\partial\bar{t}_{i+1}}(\vec{\bar{t}},\ \vec{\bar{w}}_t)\Big)\Big\}dt_i,
\end{aligned} \tag{6.8}$$

since

$$\frac{\partial f}{\partial x_i} = \frac{\partial\bar{f}}{\partial\bar{x}_i} - \frac{\partial\bar{f}}{\partial\bar{x}_{i+1}}, \qquad \frac{\partial f}{\partial t_i} = \frac{\partial\bar{f}}{\partial\bar{t}_i} - \frac{\partial\bar{f}}{\partial\bar{t}_{i+1}}, \qquad \frac{1}{2}\frac{\partial f^2}{\partial x_i^2} + \sum_{j=i+1}^{n}\frac{\partial f^2}{\partial x_i\partial x_j} = \frac{1}{2}\Big(\frac{\partial^2\bar{f}}{\partial\bar{x}_i^2} - \frac{\partial^2\bar{f}}{\partial\bar{x}_{i+1}^2}\Big).$$

The condition (6.1) in these notations is

$$\mathbf{E}\Bigg\{|\bar{f}(\vec{\bar{t}},\ \vec{\bar{w}}_t)|^2 + \sum_{j=1}^{n}\Big|\frac{\partial\bar{f}}{\partial\bar{t}_j}(\vec{\bar{t}},\ \vec{\bar{w}}_t)\Big|^2 + \sum_{i,j=1}^{n}\Big|\frac{\partial\bar{f}^2}{\partial\bar{x}_i\bar{x}_j}(\vec{\bar{t}},\ \vec{\bar{w}}_t)\Big|^2\Bigg\} < c. \tag{6.9}$$

Let us introduce a family of spaces $L^2(\mathbb{R}^n, \mathcal{N}_{\vec{\bar{t}}})$ where

$$\mathcal{N}_{\vec{\bar{t}}}(d\bar{x}) = ((2\pi)^n \bar{t}_1 ... \bar{t}_n)^{-1/2} \exp\Big\{ -\frac{1}{2}\Big(\frac{\bar{x}_1^2}{\bar{t}_1} + ... + \frac{\bar{x}_n^2}{\bar{t}_n}\Big)\Big\} d\bar{x}_1 ... d\bar{x}_n.$$

Lemma 6.2. *Let $\bar{f}(\vec{\bar{t}}, \vec{\bar{x}}) \in C^{1,2}$ is such that $\bar{f}$, $\frac{\partial \bar{f}}{\partial \bar{t}_j}$, $\frac{\partial \bar{f}}{\partial \bar{x}_j}$, and $\frac{\partial^2 \bar{f}}{\partial \bar{x}_j x_k}$ belong to $L^2(\mathbb{R}^n, \mathcal{N}_{\vec{\bar{t}}})$ as functions of $\vec{\bar{x}}$. Then*

$$\bar{f}(\vec{\bar{t}}, \vec{\bar{x}}) = \sum a_{k_1 ... k_n}(\vec{\bar{t}}) H_{k_1}(\bar{t}_1, \bar{x}_1) ... H_{k_n}(\bar{t}_n, \bar{x}_n), \tag{6.10}$$

where H_k are the Hermite polynomials, the coefficients $a_{k_1 ... k_n}(\bar{t}_1, ..., \bar{t}_n) \in C^1$ $(t_j > 0)$ and the series converges in $L^2(\mathbb{R}^n, \mathcal{N}_{\vec{\bar{t}}})$ and can be differentiated in $\bar{t}_j$ and $\bar{x}_j$, $j = 1, ..., n$. Note that for the partial sums

$$\bar{f}_M(\vec{\bar{t}}, \vec{\bar{x}}) = \sum_{k_1 + ... + k_n \le M} a_{k_1 ... k_n}(\vec{\bar{t}}) H_{k_1}(\bar{t}_1, \bar{x}_1) ... H_{k_n}(\bar{t}_n, \bar{x}_n) \tag{6.11}$$

the formula (6.8) is already proven and the convergence $\bar{f}_M(\vec{\bar{t}}, \vec{\bar{x}}) \to \bar{f}(\vec{\bar{t}}, \vec{\bar{x}})$ in $L^2(\mathbb{R}^n, \mathcal{N}_{\vec{\bar{t}}})$ means that $\bar{f}_M(\vec{\bar{t}}, \vec{\bar{w}}_t) \to \bar{f}(\vec{\bar{t}}, \vec{\bar{w}}_t)$ in $L^2(\Omega)$. It remains to apply Theorem 3.1 asserting that J is a closed operator. □

Theorem 6.1 can be proven without using holomorphic stochastic integrals. The crucial moment is the proof of the formula (6.8) for the sum of the form (6.11). It is sufficient to consider only a single term with the unit coefficient. Moreover, we can assume that $H_{k_j} = 1$ for $j \neq i,\ i+1$, because $H_{k_j}(t_j - t_{j-1}) = 1$ can be taken out from the sign of the ESI $\int_{S_i}^{T_i}$ for $j \neq i,\ i+1$.

So, we should verify the equality

$$\begin{aligned} g(\bar{t}_i, \bar{w}_{t_i}) h(\bar{t}_{i+1}, w_{t_{i+1}})\Big|_{s_i = S_i}^{t_i = T_i} &= \int_{S_i}^{T_i} \Big(\frac{\partial}{\partial \bar{x}_i} - \frac{\partial}{\partial \bar{x}_{i+1}}\Big) g(\bar{t}_i, \bar{w}_{t_i}) h(\bar{t}_{i+1}, w_{t_{i+1}}) dw_{t_i} \\ &+ \int_{S_i}^{T_i} \Big\{\frac{1}{2}\Big(\frac{\partial^2}{\partial \bar{x}_i^2} - \frac{\partial^2}{\partial \bar{x}_{i+1}^2}\Big) + \Big(\frac{\partial \bar{f}}{\partial \bar{t}_i} - \frac{\partial}{\partial \bar{t}_{i+1}}\Big) g(\bar{t}_i, \bar{w}_{t_i}) h(\bar{t}_{i+1}, w_{t_{i+1}})\Big\} dt_i, \end{aligned} \tag{6.12}$$

where $t_{i-1} < S_i < T_i < t_{i+1}$.

In the case where $h = 1$ (6.12) coincides with the usual Ito formula. If $g = 1$, $h = h(\bar{x}_{i+1})$, then the formula (6.12) has the form

$$h(w_{t_{i+1}} - w_{t_i})\Big|_{s_i = S_i}^{t_i = T_i} = -\int_{S_i}^{T_i} \frac{\partial h}{\partial \bar{x}_i}(w_{t_{i+1}} - w_{t_i}) dw_{t_i} - \frac{1}{2}\int_{S_i}^{T_i} \frac{\partial^2 h}{\partial \bar{x}_{i+1}^2}(w_{t_{i+1}} - w_{t_i}) dt_i. \tag{6.13}$$

Note that together with the Ito integral $\int_0^1 f(s, \omega) dw_s$, defined for the integrants "independent on the future", one can consider the integral for the integrants "independent on the past", i.e. for the random functions $f(s, \omega)$ which are measurable, for each s, with respect to the σ-algebra $\mathcal{F}^w_{\ge s} = \sigma\{w_u - w_v,\ u > v \ge s\}$. This integral is reduced to the classical Ito integral. Indeed, the process $w^*_t = w_1 - w_{1-t}$ is Wiener. The integral $\int_0^1 f(s, \omega) dw_s$ with the integrant "independent on the past" will coincide with $\int_0^1 f(1-s, \omega) dw^*_s$ where $f(1-s, \omega)$ is a function, independent on the future values of the process w^*_s. Thus, (6.13) holds. Now it is easy to get also (6.12). Suppose that

g and h do not depend on $\bar{t}_i$ and $\bar{t}_{i+1}$. Let u_j, $j = 0, ..., N$, define a partition of the interval $[S_i, T_i]$, $\delta = \max|u_{i+1} - u_j|$. Then the left-hand side of (6.12) is equal to

$$\sum_{j=0}^{N} g(w_{t_i} - w_{t_{i-1}})h(w_{t_{i+1}} - w_{t_i})\Big|_{t_i=u_j}^{t_i=u_{j+1}} = \sum_{j=0}^{N} g(w_{u_j} - w_{t_{i-1}})h(w_{t_{i+1}} - w_{t_i})\Big|_{t_i=u_j}^{t_i=u_{j+1}}$$
$$+ \sum_{j=0}^{N} g(w_{t_i} - w_{t_{i-1}})h(w_{t_{i+1}} - w_{u_{j+1}})\Big|_{t_i=u_j}^{t_i=u_{j+1}}$$
$$- \sum_{j=0}^{N} \left[- \int_{u_j}^{u_{j+1}} \frac{\partial h}{\partial \bar{x}_{i+1}}(w_{t_{i+1}} - w_{t_i})dw_{t_i} - \frac{1}{2}\int_{u_j}^{u_{j+1}} \frac{\partial h}{\partial^2 \bar{x}_{i+1}^2}(w_{t_{i+1}} - w_{t_i})dt_i \right] g(w_{u_j} - w_{t_{i-1}})$$
$$+ \sum_{j=0}^{N} \left[\int_{u_j}^{u_{j+1}} \frac{\partial g}{\partial \bar{x}_i}(w_{t_i} - w_{t_{i-1}})dw_{t_i} - \frac{1}{2}\int_{u_j}^{u_{j+1}} \frac{\partial^2 g}{\partial \bar{x}_i^2}(w_{t_i} - w_{t_{i-1}})dt_i \right] h(w_{t_{i+1}} - w_{u_{j+1}})$$

in virtue of (6.13) and the Ito formula. The random variables $g(w_{u_j} - w_{t_{i-1}})$ and $h(w_{t_{i+1}} - w_{u_{j+1}})$ can be written under the signs of integrals. We get (6.12) by letting $\delta \to 0$.

7. Extended Poisson integrals

For the random measure q of Example 3 one can also get similar results but looking rather different.

Let $f = f(x_1, ..., x_k)$, $g = g(x)$. We introduce the notation

$$(f \underset{(j)}{*} g)(x_1, ..., x_k) = f(x_1, ..., x_j, ..., x_k)g(x_j).$$

Theorem 7.1 *Let $f \in L^2(E^k, m^k)$, $g \in L^2(E, m)$. Then for the MSI with respect to the measure q we have the following formula:*

$$I_{k+1}(f \otimes g) = I_k(f)I_1(g) - \sum_{j=1}^{k} I_k\big(f \underset{(j)}{*} g\big) - \sum_{j=1}^{k} I_{k-1}\big(f \underset{(j)}{\times} g\big). \tag{7.1}$$

Proof. We consider the case where f and g are special step functions:

$$f(x_1, ..., x_k) = \sum a_{i_1...i_k}\chi_{A_{i_1}}(x_1)...\chi_{A_{i_k}}(x_k), \qquad g(x) = b_i\chi_{A_i}(x_1),$$

$A_i \in \mathcal{E}_0$, $i = 1, ..., n$, $a_{i_1,...,i_k} = 0$, if at least two of indices are equal. Put $M = \max|a_{i_1...i_k}|$, $N = \max|b_i|$. Suppose that M, N and $m(\sum A_i)$ are finite. The measure m has the continuity property and we may suppose that $m(A_i) < \varepsilon$ for all i whatever is $\varepsilon > 0$ given in advance.

Let us introduce the special step function

$$h_\varepsilon(x_1, ..., x_k, x) = \sum_{i \neq i_1,...,i_k} a_{i_1...i_k}b_i\chi_{A_{i_1}}(x_1)...\chi_{A_{i_k}}(x_k)\chi_{A_i}(x).$$

Then

$$\begin{aligned}
I_{k+1}(f\otimes g) &= \sum_{i\neq i_1,\dots,i_k} a_{i_1\dots i_k} b_i q(A_{i_1})\dots q(A_{i_k})q(A_i) + \sum_{j=1}^k \sum a_{i_1\dots i_k} b_{i_j} q(A_{i_1})\dots q^2(A_{i_j})\dots q(A_{i_k}) \\
&= I_{k+1}(h_\varepsilon) + \sum_{j=1}^k \sum a_{i_1\dots i_k} b_{i_j} q(A_{i_1})\dots q(A_{i_j})\dots q(A_{i_k}) \\
&\quad + \sum_{j=1}^k \sum a_{i_1,\dots,i_k} b_{i_j} q(A_{i_1})\dots m(A_{i_j})\dots q(A_{i_k}) \\
&\quad + \sum_{j=1}^k \sum a_{i_1,\dots,i_k} b_{i_j} q(A_{i_1})\dots\big[q^2(A_{i_j}) - p(A_{i_j})\big]\dots q(A_{i_k}) \\
&= I_{k+1}(h_\varepsilon) + \sum_{j=1}^k I_k\big(f \underset{(j)}{*} g\big) + \sum_{j=1}^k I_{k-1}\big(f \underset{(j)}{\times} g\big) + \sum_{j=1}^k R_j.
\end{aligned}$$

From the properties of multiple integrals we have:

$$\begin{aligned}
||I_{k+1}(h_\varepsilon) - I_{k+1}(f\otimes g)||^2_{L^2(\Omega)} &\leq (k+1)!||h_\varepsilon - f\otimes g||^2_{k+1} \\
&= (k+1)!\sum_{j=1}^k \sum a^2_{i_1\dots i_k} b^2_{i_j} m(A_{i_1})\dots m^2(A_{i_j})\dots m(A_{i_k}) \\
&\leq \varepsilon(k+1)!kM^2N^2\Big(\sum m(A_i)\Big)^k = \varepsilon\,\mathrm{const}.
\end{aligned}$$

A similar estimate hold for $||R_j||^2_{L^2(\Omega)}$:

$$\begin{aligned}
||R_j||^2_{L^2(\Omega)} &= \sum a^2_{i_1\dots i_k} b^2_{i_j} m(A_{i_1})\dots\mathbf{E}\big[q^2(A_{i_j}) - p(A_{i_j})\big]\dots m(A_{i_k}) \\
&= 2\sum a^2_{i_1\dots i_k} b^2_{i_j} m(A_{i_1})\dots m^2(A_{i_j})\dots m(A_{i_k}) \\
&\leq \varepsilon 2M^2N^2\Big(\sum m(A_i)\Big)^k = \varepsilon\,\mathrm{const}.
\end{aligned}$$

Since ε is arbitrary, the statement of the theorem holds for special step functions. Passage to the limit is easy. □

Theorem 7.2 *Let $f(x_1,\dots,x_n)\in L^2(E^n,m^n)$ where $E=\mathbb{R}^1_+\times\mathbb{R}^r$. Then*

$$I_n(f) = n!\int_0^\infty\int\left(\int_0^{t_n}\int\left(\dots\int_0^{t_2}\int \tilde f(t_1,u_1,\dots,t_n,u_n)q(dt_1,du_1)\right)\dots\right)q(dt_n,du_n).$$

The proof is similar to the proof of Theorem 4.3. Since $\mathcal{H}=L^2(q)=L^2(X)$ (see [3]), the above theorem implies

Theorem 7.3 (Predictable representation.) *Any random variable $\eta \in L^2(q)$ admits the representation*

$$\eta = \mathbf{E}\eta + \int_0^\infty \int \varphi(t,u,\omega) q(dt,du),$$

where $\varphi(t,u,\omega)$ is measurable in all variables and predictable for every $u \in \mathbb{R}^r$, uniquely determined, and

$$\int_0^\infty \int \varphi^2(t,u,\omega)\Pi(du)dt < \infty.$$

Let us consider the particular case where q corresponds to the centered Poisson process, i.e. the canonical Lévy measure is concentrated in a single point: $\Pi(A) = \lambda\chi_A(\{1\})$. In this case it is natural to consider multiple integrals depending on time variable. These MSI we shall call multiple integrals with respect to the Poisson process and denote $\int ... \int f(t_1,...,t_n)dx_{t_1}...dx_{t_n}$.

Define the Poisson–Charlier polynomials $G_n(t,x,\lambda)$ with help of their generating function

$$\Phi(z,t,x,\lambda) = \sum_{n=0} z^n G_n(t,x,\lambda) = (1+z)^{x+t\lambda}\exp\{-zt\lambda\}.$$

It is easy to check the following properties:

1. $\frac{\partial \Phi}{\partial z} = \frac{x - zt\lambda}{1+z}\Phi$.
2. $G_0(t,x) = 1$, $G_1(t,x) = x$, $\quad (n+1)G_{n+1}(t,x) = (x-n)G_n(t,x) - t\lambda G_{n-1}(t,x)$.
3. $G_n(t,x+1) - G_n(t,x) = G_{n-1}(t,x)$.
4. $G_{n-1}(t,x) - \frac{\partial G_n(t,x)}{\partial x} + \frac{1}{\lambda} + \frac{\partial G_n(t,x)}{\partial t} = 0$.

The following statements relates the multiple integrals with respect to the Poisson process and the Poisson–Charlier polynomials.

Lemma 7.1 *Let x_t be the centered Poisson process with parameter λ. Then*

$$G_n(t,x_t,\lambda) = \frac{1}{n!} I_n(\chi_{[0,t]}(t_1)...\chi_{[0,t]}(t_n)) = \int_0^t \left(\int_0^{t_1} \cdots \left(\int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} dx_{t_n} \right) dx_{t_{n-1}} \right) \cdots \right) dx_{t_1}.$$

The proof is by induction in n using the property 2. and Theorem 7.1.

Lemma 7.2 *Let x_t be the centered Poisson process, $G_n(t,x)$, $G_m(t,x)$ be the Poisson–Charlier polynomials, $0 \le t_0 \le S < T \le T_2$. Then*

$$\begin{aligned} &\int_S^T G_n(t_1 - t_0, x_{t_1} - x_{t_0}) G_m(t_2 - t_1, x_{t_2} - x_{t_1}) dx_{t_1} \\ &= \Big\{ G_{n+m+1}(t_1 - t_0, x_{t_1} - x_{t_0}) + G_{n+m}(t_1 - t_0, x_{t_1} - x_{t_0}) G_1(t_2 - t_1, x_{t_2} - x_{t_1}) \\ &\quad + \cdots + G_{n+1}(t_1 - t_0, x_{t_1} - x_{t_0}) G_m(t_2 - t_1, x_{t_2} - x_{t_1}) \Big\} \Big|_{t_1 = S}^{t_1 = T}. \end{aligned} \tag{7.2}$$

Proof. One can check the identity (7.2) by direct calculation but this way is rather tedious. Alternatively, one can proceed as follows. In virtue of Theorem 6.1 and properties of the Hermite polynomials the formula (7.2) holds if one replaces the Poisson process by the Wiener process and the Poisson–Charlier polynomials by the Hermite polynomials. This analog of (7.2) can be written as the equality of two multiple Wiener integrals of order $n+m+1$. But if the multiple Wiener integrals coincide on functions from $L^2(\mathbb{R}_+^{n+m+1}, \Lambda^{n+m+1})$, then the MSI with respect to the Poisson process are also equal. □

Let us introduce the following notations:

$$\begin{aligned} S_y^{(k)}(\vec{t},\vec{x}) &= f(t_1,...,t_n,x_1,...,x_k+y,x_{k_1},...,x_n), \\ \Delta_y^{(k)}(\vec{t},\vec{x}) &= S_y^{(k)}(\vec{t},\vec{x}) - (\vec{t},\vec{x}). \end{aligned}$$

Theorem 7.4 *Let x_t be the centered Poisson process. Suppose that the function $F(\vec{t},\vec{x})$ is continuous together with its derivatives $\frac{\partial F}{\partial t_j}$, $\frac{\partial F}{\partial x_j}$, $j=1,...,n$, in the domain $Q\times\mathbb{R}^n$ where the set $Q=\{\vec{t}\colon\ 0\le t_1<t_2<...<t_n\}$. If*

$$\mathbf{E}F^2(\vec{t},x_{t_1},...,x_{t_k}+\vartheta,...,x_{t_n}+\vartheta)<c, \qquad \vartheta\in\{0,1\}, \quad k=i,i+1,$$

$$\mathbf{E}\Big\{\Big[\frac{\partial F}{\partial t_j}(\vec{t},x_{\vec{t}})\Big]^2+\Big[\frac{\partial F}{\partial x_j}(\vec{t},x_{\vec{t}})\Big]^2\Big\}<c, \qquad j=1,...,n,$$

when $t_i\in[S_i,T_i]$ for all fixed $(t_1,...,S_i,...,t_n)$ and $(t_1,...,T_i,...,t_n)$ in Q, then

$$\Delta_1^{(i)}\prod_{j=i+1}^{n}S_1^{(j)}F(t_1,...,t_n,x_{t_1},...,x_{t_n})$$

as a function of t_i belongs to $\mathcal{D}(J,[S_i,T_i])$ and

$$\begin{aligned} F(\vec{t},x_{\vec{t}})\Big|_{t_i=S_i}^{t_i=T_i} &= \int_{S_i}^{T_i}\Delta_1^{(i)}\prod_{j=i+1}^{n}S_1^{(j)}F(\vec{t},x_{\vec{t}})dx_{t_i} \\ &+\int_{S_i}^{T_i}\Big(\Delta_1^{(i)}\prod_{j=i+1}^{n}S_1^{(j)}-\frac{\partial}{\partial x_i}F(\vec{t},x_{\vec{t}})\Big)\lambda dt_i+\int_{S_i}^{T_i}\frac{\partial T}{\partial t_i}F(\vec{t},x_{\vec{t}})dt_i. \end{aligned}$$

We skip the proof of this theorem noting only that the key point is using Lemma 7.2.

Under more restrictive assumptions one can prove a stronger version of the above theorem, for a larger class of processes.

Let $x_t=\int_{|u|\le q}uq(t,du)$ be a one-dimensional process and let $Q=\{\vec{t}\colon\ 0\le t_1<t_2<...<t_n\}$.

Theorem 7.5 *Let $F(\vec{t},\vec{x})$ be continuous together with its partial derivatives $\frac{\partial F}{\partial t_j}$, $\frac{\partial F}{\partial x_j}$, $\frac{\partial^2 F}{\partial x_j^2}$, $j=1,...,n$, in the domain $Q\times\mathbb{R}^n$ and, moreover, the following conditions are fulfilled:*

$$\mathbf{E}\Big\{[F(\vec{t},x_{\vec{t}})]^2+\Big[\frac{\partial F}{\partial t_j}(\vec{t},x_{\vec{t}})\Big]^2\Big\}<c, \tag{7.3}$$

$$\mathbf{E}\sup_{0\le\theta\le1}\left\{S^{(i)}_{\theta y}\prod_{j=i+1}^{n}S^{(j)}_{y}\Big(\Big|\frac{\partial F}{\partial x_i}\Big|+|F|\Big)(\vec{t},x_{\vec{t}})\right\}^2<c_1(y), \tag{7.4}$$

$$\mathbf{E}\sup_{0\le\theta\le1}\left\{S^{(i)}_{\theta y}\prod_{j=i+1}^{n}S^{(j)}_{y}\Big|\frac{\partial^2 F}{\partial x_i^2}\Big|(\vec{t},x_{\vec{t}})\right\}^2<c_1(y), \tag{7.5}$$

where $t_i\in[S_i,T_i]$, $(t_1,...,S_i,...,t_n)$ *and* $(t_1,...,T_i,...,t_n)$ *in* Q,

$$\int_{|y|\le1}c_k(y)y^2\Pi(tdy)<\infty,\quad k=1,2.$$

Then $\chi_{[-1,1]}(y)\Delta^{(i)}_y\prod_{j=i+1}^n S^{(j)}_y F(\vec{t},x_{\vec{t}})$ *as a function of* t_i *belongs to* $\mathcal{D}(J,[S_i,T_i])$ *and the following equality holds:*

$$\begin{aligned}F(\vec{t},x_{\vec{t}})\Big|_{t_i=S_i}^{t_i=T_i} &= \int_{S_i}^{T_i}\int_{|y|\le1}\Delta^{(i)}_y\prod_{j=i+1}^{n}S^{(j)}_y F(\vec{t},x_{\vec{t}})q(dt,dy) \\ &+\int_{S_i}^{T_i}\int_{|y|\le1}\left[\Delta^{(i)}_y\prod_{j=i+1}^{n}S^{(j)}_y-y\frac{\partial}{\partial x_i}\right]F(\vec{t},x_{\vec{t}})\Pi(dy)dt+\int_{S_i}^{T_i}\frac{\partial F}{\partial t_i}(\vec{t},x_{\vec{t}})dt_i.\end{aligned} \tag{7.6}$$

Proof. Suppose that $F(\vec{t},\vec{x})$ has a compact support. Make a change of variables (6.7) and note that the function $\bar{F}(\vec{\bar{t}},\vec{\bar{x}})$ has also a compact support. The formula (7.7) is transformed to the following form:

$$\begin{aligned}\bar{F}(\vec{\bar{t}},\vec{\bar{x}})\Big|_{t_i=S_i}^{t_i=T_i} &= \int_{S_i}^{T_i}\int_{|y|\le1}\Big[\bar{\Delta}^{(i)}_y-\bar{\Delta}^{(i+1)}_y\Big]\bar{F}(\vec{\bar{t}},\vec{\bar{x}})q(dt,dy) \\ &+\int_{S_i}^{T_i}\int_{|y|\le1}\Big[\bar{\Delta}^{(i)}_y-\bar{\Delta}^{(i+1)}_y+\Big(\frac{\partial}{\partial\bar{x}_i}-\frac{\partial}{\partial\bar{x}_{i+1}}\Big)\Big]\bar{F}(\vec{\bar{t}},\vec{\bar{x}})\Pi(dy)dt \\ &+\int_{S_i}^{T_i}\Big(\frac{\partial}{\partial\bar{t}_i}-\frac{\partial}{\partial\bar{t}_{i+1}}\Big)\bar{F}(\vec{\bar{t}},\vec{\bar{x}})dt_i.\end{aligned} \tag{7.7}$$

It is well-known that the function $\bar{F}(\vec{\bar{t}},\vec{\bar{x}})$ can be approximated uniformly with its derivatives by functions of the form $a(\bar{\vec{t}})g(\bar{x}_1)...g(\bar{x}_n)$ and we need to prove (7.8) only for such functions. On the other hand, the process $q^*(t,A)=q(1,A)-q(1-t,A)$ is a left-continuous modification of the Poisson process. This remark allows us to prove (7.8) by arguments similar to those used at the end of Section 6 to derive the generalized Ito formula with ESI with respect to the Wiener process. The conditions (7.3) – (7.6) ensure the passage to the limit. □

References

[1] K. Ito. Multiple Wiener integrals. *J. Math. Soc. Japan*, v. 3 (1951), 159–169.

[2] K. Ito. Complex multiple Wiener integrals. *Japanese Journal of Math.*, v. XXII (1952), 63–86.

[3] K. Ito. Spectral type of the shift transformation of differential processes with stationary increments. *TAMS*, v. 81 (1956), 253–263.

[4] M. Hitsuda. Formula for Brownian partial derivatives. *Abstracts of the Second Japan–Soviet Symposium on Probability Theory. Kyoto, August, 1972*, v. 2, 111–114.

[5] M. Hitsuda. Formula for Brownian partial derivatives. Preprint. [Appeared in: *Proceedings of Faculty of Integrated Arts and Sciences, Hiroshima Universit*, v. III-4 (1978), 1-15.]

[6] A.V. Skorokhod. On a generalization of a stochastic integral. In print. [Appeared in: *Probability Th. and Its. Appl.*, V. 20 (1975), 2, 223–238. English translation: v. 20 (1976), 2, 219–233.]

[7] Yu.M. Kabanov. A generalized Ito formula for an extended stochastic integral with respect to the Poisson random measure. (Russian). *Uspehi Mat. Nauk*, v. 29 (1974), 4(178), 167–168.

[8] Yu.M. Kabanov. On extended stochastic **integrals**. In print. [Appeared in: *Probability Th. and Its. Appl.*, v. 20 (1975), 4, 725–737. English translation: v. 20 (1976), 4, 710–722 (with an error in the title: "intervals" instead of "integrals").]

[9] Yu. L. Daletsky, S.N. Paramonova. Stochastic integrals with respect to a normally distributed additive set function. *Dokl. Akad. Nauk SSSR*; 208 (1973), 512–515. English translation: *Soviet Math. Dokl.*, v. 14 (1973), 9699.

[10] Yu. L. Daletsky, S.N. Paramonova. On a formula from the theory of Gaussian measures and on the estimation of stochastic integrals. *Probability Th. and Its. Appl.*, v. 29 (1974),4, 844–848. English translation: v. 19 (1975), 4, 812-817.

[11] S.N. Paramonova. Functionals on normally distributed functions. *Abstracts of the International Conference on Probability Theory and Mathematical Statistics. Vilnius, 1973.*

[12] I.I. Gikhman, A.V. Skorokhod. *Theory of Random Processes.* Moscow, "Nauka", 1971.

HISTORICAL COMMENT

This is the English translation of notes prepared in summer of 1974 when I was still a PhD student of Steklov Mathematical Institute supervised by A.N. Shiryaev. They were based on the papers and preprints listed above. Two lectures were presented by myself in November 1974 at the school-seminar in Druskininkai in front of members of the major seminars in probability of the Soviet Union lead by A.N. Shiryaev (Moscow), I.A. Ibragimov (Leningrad), A.V. Skorokhod (Kiev), B. Grigelionis (Vilnius) and others. This meeting happen to be an important event in the history of stochastic calculus. In particular, for the first time the difference between the notions of weak and strong solutions of SDEs were clearly understood and explained: in a few days afterwards Tsirel'son constructed his famous counterexample.

Regretfully, the proceedings of school-seminar were never translated into English and the aim of this translation is to shed the light to the origin of concepts of extended stochastic integral and the stochastic derivative playing an important role in the theory known as the Malliavin calculus. By strange coincidence, the story with other early publications on this subject was also unfortunate. At this period the short notes in *Uspehi Matematicheskih Nauk* were not translated. The translation of my paper in *Theory of Probability and Its Applications* contained a misleading error in the title. The preprint by Hitsuda circulating already in 1972 was published only in 1978 and in a rarely available journal. I could not find the issue containing his paper even in the Kyoto University Research Institute for Mathematical Sciences (RIMS) library and got the copy directly from the author. Clearly, Masuyuki Hitsuda has a priority in introducing the concept which happen to be so important. To my knowledge, Skorokhod had the book of abstracts (I saw it on his bookshelf) but not the Hitsuda preprint (he got it from my hands only in April 1974).

Compiling the text of lecture notes and using the manuscripts by Hitsuda, Skorokhod, and myself, I explained the Skorokhod setting (described in the language of generalized Gaussian processes) in terms of a more specific model considered by Hitsuda, that is, in the notation of the latter. The small changes were in considering the multiple integrals not with respect to the Wiener process but with respect to a "stochastic measure" (note the difference with the common notion of random measure: a "stochastic measure" not always be represented as a kernel). The chosen approach allowed to present in a unified way the theory which includes also the Poisson case. I also tried to clarify the concept of the Skorokhod stochastic derivative. Nowadays, it is clear that the latter is a kind of gradient requiring more smoothness than the Malliavin directional derivative. The principal aim of my own research was to extend to the Lévy processes the generalized Ito formula. The latter was obtained by Hitsuda in a rather ingenious way, from a formula for complex Wiener process (looking like the Newton–Leibnitz formula), by conditioning with respect to the real part. However, this idea does not work for the Lévy processes and I obtained the analog using the time reversal which gave an alternative proof also in the Wiener case.