Dynamic Models of Systemic Risk and Contagion

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Abstract

Modern financial systems are complicated networks of interconnected financial institutions and default of one of them may have serious consequences for others. The recent crises have shown that the complexity and interconnectedness are major factors of systemic risk which became a subject of intensive studies usually concentrated on static models. In this paper we develop a dynamic model based on the so-called structural approach where defaults are triggered by the exit of some stochastic process from a domain. In our case, this is a process defined by the evolution of bank's portfolios values. At the exit time a bank defaults and a cascade of defaults starts. We believe that the distribution of the exit time and the subsequent losses may serve as indicators allowing regulators to monitor the state of the system to take corrective actions to avoid the contagion in the financial system. We model the development of financial system as a random graph using the preferable attachment algorithm and provide results of numerical experiments on simulated data.

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1. Inroduction

In interbank market, systemic risk is a risk arising from a complexity of financial network and threatening the entire system by a potential financial crisis, resulting in high economical and social costs. Controlling financial stability and assessing systemic risk is a major concern of central banks and financial regulators. The rapid growth of financial innovation and integration as well as a complicated network of claims and obligations linking the balance sheets of banks raises the challenge for the analysis of systemic risk. This kind of risk is highly dynamic, slowly building up during periods of stability and rapidly rising during crises and spreading through the network. On the other hand, the interconnections of banks have a positive side since they enhance liquidity and increase the risk sharing among the financial institutions.

One of the aim of theoretical studies is to provide regulators comprehensive indicators allowing to monitor the risk of contagion, understood as a cascade of defaults that may lead to a serious consequences and even to the collapse of the whole economy. To the moment, there is a substantial progress in understanding various phenomena causing the contagion on the basis of modeling using random graphs. Network models became the mainstream of current researches in the field, see the recent book by T. Hurd [23] and the references wherein.

Recent crisis revealed that the systemic risk might take various forms. One of them is an interbank contagion process when, due to the interconnectedness of banks through interbank loans, the default of one bank leads to losses and subsequent chain of defaults of other banks. This kind of risk is usually combined with a risk related to a correlation between banks' portfolios that consists in the phenomena that a common shock, due to common asset holdings, affects many banks at once.

Bandt *et al.* (2009), [14] provided a categorization of systemic risks, distinguishing between those understood in a broad and in a narrow sense: contagion effects pose a systemic risk in the narrow sense while in the broad sense it is a common shock that affects many nodes and once. A similar idea is developed by Gai and Kapadia (2010), [20], who model two channels of contagion in financial system that can trigger further rounds of defaults: contagion due to the direct interbank claims and obligations and contagion due to common shocks on the asset side of the balance sheet, especially, when the market for key financial system assets is illiquid.

Deposits also could affect the financial system stability: a large sudden withdrawal of funds by depositors in panic could lead to a collapse of the system. However, in the present paper we do not consider this as one of the major sources of system risk. in fact, its impact can be minimized and controlled by the central bank intervention imposong an appropriate withdrawal limit.

A large part of the literature have focused on the analysis of the contagion effect due to the interbank market while only a few authors studied the impact of the correlated defaults which is of great importance related to the magnitude of correlation between the banks balance sheets, to the amount of external investments and to the appropriate assessment of the risk embedded in these external assets. Acharya and Yorulmazer (2008), [2] proved that banks are motivated to increase the correlation between their investments amplifying by such actions the risk of a common shock. In their analysis, Elsinger *et al.* (2006), [**?**] combine the two major sources of systemic risk and find that the correlation in investments is far more important than financial linkages.

We can also consider a subordinate source of risk due to the fire sale of external assets of defaulting banks which will lead to other banks default because of the price depreciation. This is why some banks have an interest to bailout other peers in order to minimize the default cost of the system and to prevent fire sale and the writing down of their own external assets.

While the interbank risk is concerned, Gai and Kapadia (2008) show that the risk of systemic crises is reduced with increasing connectivity while the amplitude of the systemic crises is increasing at the same time. Higher connectivity simply creates more channels of contact through which default could spread, increasing the potential or probability for contagion. However, in the financial system setup, greater connectivity allows counterparties risk sharing as exposures are distributed over a wider set of banks, especially, in periods of stability. In times of crisis, however, the same interconnections can amplify shocks that spread through the system.

Allen and Gale (2000), [3] demonstrated that the spread of contagion depends from the network structure of the financial system and strongly interconnected banking systems are less affected by the systemic risk. They also pointed out that the assumption that the agents have complete information on their environment is not realistic. Acharya and Bisin (2014), [1], compared over-the-counter (OTC) and centralized clearing markets in a general equilibrium model. They show that the intransparency of OTC markets is ex-ante inefficient and will lead to underpricing of counterparty risk.

The counterparty risk makes it clear that the network structure of financial system plays an important role when assessing systemic risk.

Empirical analyses of the interbank network structure exist for a number of countries. It shows that the interbank network has a scale free topology. This means that there are a few large banks with many interconnections and many small banks with a few connections. In contrast, other authors argue that the intransparency of real data makes the random network more valid to capture the hidden links. More formally, the terminology "scale free network" means that, at least, when the number of nodes increases to infinity the number k of connections ("in" or "out") attributed to each node decays as $k^{-\gamma}$, $\gamma > 1$.

Georg (2013) [21] proposed a dynamic model of cascading banking defaults: at each stage of the cascade, each bank collects all his exposures, pays all his liabilities, adjust the price of its external assets and, when remains solvent, it optimizes a portfolio of risky and risk-free assets and initiates other interconnections within the banking system.

On the other hand, Gai and Kapadia (2009) highlighted that in normal times, developed country banks are robust and minor variations in their default probabilities do not affect lending decisions on the interbank market. But in crises, as illustrated by the sudden failures of Lehman Brothers, contagion may spread rapidly with banks having little time to alter their behavior before they are affected. Thus, the almost static behavior of the system during crisis is best captured by the static model as also applied in our paper.

It seems that the majority of existing literature deals with "homogeneous" models, like Erdös–Renyi model where the network graph is generated by a matrix whose the non-diagonal entries are identically distributed independent Bernoulli random variables, see [20], or even models where all nodes has the same number of connections, [25]. Though such models are convenient for theoretical studies, they look to be too far from the reality and in the present paper we investigate the behavior of the systemic risk indicator using networks with a structure obtained by a preference attachment algorithm leading to a scale free network.

Under the Basel II accord, improving the quality of default models is the key risk-management priority. Many researchers have studied the loss or impact of the systemic risk once a crisis or shock is in place. However, there is a need to predict and prevent the defaults of banks before it happens. To the date, the major part of research papers concentrates on studies of static or stationary models. In this note we suggest an approach influenced by the structural model of defaultable securities, see [7]. Namely, we suppose that the cascade of default is triggered in a natural way when the value of a portfolio process of some bank falls below a certain level. Financial market react negatively to such an event. Prices of the external assets drop down and contagion propagates not only to interconnected banks but also via correlation. Assuming that the matrix of exposures as well as the vector of the investments into external assets is known, the regulators, having a model for the dynamic of the "reference portfolio" can compute, with moving time horizons, two ''alert indicators": the probability that the default happens during the planning period and the total losses incurred when the default happens. The total losses are the aggregation of the losses due to the external asset price depreciation (correlation) and the losses due to the interbank linkages (contagion). To simplify our calculation, we assume that there is a single external risky asset common to all banks in the system and the difference is only in the size of portfolios. A model where each bank has its own portfolio structure can be treated in a similar way. Our approach is rather flexible and can be combined with existing methods of reconstructing of the exposure matrix.

Thus, the main novelty of our approach, in contrast to the majority of existing studies concentrated on static or stationary models, is in developing a dynamic model of financial system before the crisis in combination with a static contagion model for the crisis. The model is described by a graph which nodes are banks (or other financial institutions). The directed graph structure arises from the matrix of liabilities/exposures. Each bank is characterized by a stylized balance sheet. On the asset side there are exposures (due to the interbank lending) and liquid assets, risky (stocks) and non-risky (cash). The liability side is composed by the received interbank loans and the net worth, the quantity, equating both sides of the balance sheet. The dynamic is introduced via random fluctuations of the value of the risky asset. Decreasing of its price means that the net worth is decreasing. We suppose that the risky asset is unique for all banks. One may think of this asset as a "benchmark (or reference) portfolio". Taking into account that banks try to mimic behavior of each other ("herding effect"), we believe that this assumption may suit to our highly stylized model but for practical applications it can be relaxed. Of course, there is a need to introduce dynamics in other parts of the balance sheet but we prefer avoid this in the paper.

The paper contains some numerical experiments. Unfortunately, the liability matrix of a financial system is not publicly available (with rare exceptions). By this reason we test applicability of our model on simulated data. In numerical experiments we use a construction of the scale-free network using a preferential attachment algorithm, see [4]. We populate the model by balance sheets and compute the alert indicators. Our experiments show that the alert indicators can be used as a tool to support regulator's decision to increase the stability of the financial system by withdrawal of the license of the bank having low reliability.

The structure of the article is as follows. In Section 2 we describe the general network approach to contagion. Section 3 gives the model description and the definition of the alert indicators. Section 4 is devoted to simulation results.

2. Network approach

2.1. General principles

The basic ideas are very simple and can be described as follows. The set $G = \{1, ..., N\}$ stands for the banking system involving N financial institutions described by an $N \times N$ matrix $L = (L^{ij})$ with non-negative entries vanishing on the diagonal ($L^{ii} = 0$) and a vector $C \in \mathbf{R}^N$ with non-negative components.

The entry L^{ij} represents the *liability* of the ith bank to the jth bank, i.e. the debts of i to j or, in other words, the total amount of credit provided by j to i. By the reciprocity, for the ith bank the value L^{ji} is its *exposure* to the bank j. By this reason, in the literature the model quite often is described by the matrix of the liabilities $X = (X^{ij})$, $X = L'$, where $'$ is used to denote the transpose. Let $B^{ij} = I_{\{L^{ij}>0\}}$. The matrix B (whose entries are zeros and units) defines the directed graph structure on the set of N points in a usual way (as is done in the theory of Markov chains): there is a flesh $i \to j$ if $B^{ij} > 0$, showing that the ith bank is indebted to the jth bank (attention: in some papers the direction of fleshes can be opposite). With this observation, one can use the standard terminology of the network theory and identify banks with the nodes of the (weighted) oriented graph.

The component C^i of the vector C represents the proper capital reserve the ith bank; it is *solvent* if the net worth

$$NW^i := \sum_{j \in G} L^{ji} - \sum_{j \in G} L^{ij} + C^i \geq 0. \tag{2.1}$$

If the above solvency condition does not hold, the bank *defaults*.

It is important to note that the definitions "exposure", "liability", "default" appeal to a common sense rather having a precise meaning. Their understanding varies from paper to paper. In practice, the balance sheet of a bank has a much more complicated structure. E.g., the exposure may include overnight credits as well as long term loans, the debts are of different seniority, and so on. The “standard" highly stylized balance sheet, i.e. the equality $Assets = Liabilities$ presented as a table, containg on the assets sides the interbank exposures (loans) and external assets (that can be split in liquid and illiquid, risky and non-risky) while on the liability side there are interbank borrowings, deposits and, to equate the both side, the net worth (called also capital reserve or equity) — in the case that the bank is solvent.

2.2. Defaults

In the literature, the typical descriptions of the contagion process and defaults “in cascade" can be found, e.g., in [23]. We present them in a succinct way as follows. Let us denote by $I_{out}(i)$ (respectively, by $I_{in}(i)$) the set of banks to which the bank i has a liability (respectively, an exposure). That is, $I_{out}(i)$ is the set of nodes terminal for the fleshes (arcs) outgoing from the node i while $I_{in}(i)$ is the set of nodes with fleshes ending at this node. We denote by $n_{out}(i)$ and $n_{in}(i)$ the cardinality of the corresponding sets, i.e. the numbers of outgoing and ingoing fleshes. Clearly, $n_{out}(i) = \sum_j B^{ij}$, $n_{in}(i) = \sum_j B^{ji}$.

The default of the bank i triggers the following procedure. The bank is excluded from the network. Debts are collected from debtors at liquidation. Creditors loose a fraction $1-R$ of their exposures to i, where the parameter R is referred to as *recovery rate*. Formally, one can think that the matrix L is replaced by the matrix $\bar{L}$ obtained by replacing the elements of the ith row and ith column by zeros. The transformed vector of capital reserves $\bar{C}$ has the components $\bar{C}^j = C^j + RL^{ij} - L^{ji}$, $j \neq i$, $\bar{C}^i = 0$. Put $D_0(i) := \{i\}$ and skip further the argument i here and in further definitions (depending also on R). For some j (different from i) the solvency condition

$$\sum_{k\in G\setminus D_0} \bar{L}^{kj} - \sum_{k\in G\setminus D_0} \bar{L}^{jk} + \bar{C}^j \geq 0, \tag{2.2}$$

which can be written also as

$$\sum_{k\in G} L^{kj} - \sum_{k\in G} L^{jk} + C^j - (1-R)L^{ij} \geq 0, \tag{2.3}$$

may fail. We denote by $D_1 := D_1(i)$ the set of such indices, corresponding to the first-order defaults in the cascade of the defaults triggered by the default of i. In the same way, the contagion is propagated further, to the set of banks $D_2 = D_2(i)$ which is a subset of indices j outside of the union D_0^1 of D_0 and D_1 and such that the solvency condition becomes negative:

$$\sum_{k\in G} L^{kj} - \sum_{k\in G} L^{jk} + C^j - (1-R)\sum_{k\in D_0^1} L^{kj} < 0.$$

Continue in the same way, for the set D_0^n, we put $D_0^{n+1} := D_0^n \cup D_{n+1}$ where D_{n+1} is the set of indices j in the complement of D_0^n such that

$$\sum_{k\in G}^{N} L^{kj} - \sum_{j\in G} L^{jk} + C^j - (1-R)\sum_{k\in D_0^n} L^{jk} < 0.$$

The process stops if $D_{n+1} = \emptyset$. One can consider the value

$$L(i) := (1-R)\sum_{n=0}^{N}\sum_{j\in D_{n+1}}\sum_{k\in D_0^n} L^{jk}$$

as the total losses incurred by the cascade of defaults triggered by the default of the ith bank.

It is not difficult to extend the above definitions to obtain expressions for losses triggered by simultaneous defaults of a group of banks.

2.3. Practical aspects and difficulties

On the first sight, the above formulae are of great help for the researchers in financial systemic risk providing them N functions of the recovery rate which allows to classify banks accordingly to their systemic importance. The described procedure also can be used to find the most vulnerable banks, sensitive to defaults of others. However, the practical implementation is not so straightforward. Indeed, in the majority of cases the exposure matrix X (having one million entries for a system with $N = 1000$) is not publicly available though a certain subset of its entries may be known. Usually, only the sums of elements along each line and each column can be extracted from the balance sheets. If only this information is available, one cannot recover the matrix L in a unique way: one needs to solve the system of $2N$ equations

$$\sum_{j\in G} L^{ji} = a^i, \qquad \sum_{j\in G} L^{ij} = b^i, \qquad 1 \le i, j \le N, \tag{2.4}$$

with $N^2 - N$ unknown $L^{ij} \geq 0$, $i \neq j$, and all $L^{ii} = 0$.

Obviously, the system (2.4) has the non-negative solution $x^{ij} = a^j b^i / \sum_i b^i$ (note that $\sum_i b^i = \sum_j a^j$). But this is not the needed solution since not all $x^{ii} = 0$. In the literature, see [27], it is recommended to take as the matrix X the solution of the entropy minimization problem:

$$\sum_{ij} \ln \frac{L^{ji}}{x^{ji}} \to \min,$$

under constraints (2.4), $L^{ij} \geq 0$ and $L^{ii} = 0$ for all i, j.

This approach is criticized since it leads to a matrix generating a complete graph and the overestimation of stability of financial system. On the other hand, in some cases, a part of the matrix L is known, e.g., the absence of connections between some nodes can be a plausible hypothesis. The entropy minimization method can be easily adapted to such cases leading to a rather realistic recovery of the exposure matrix.

2.4. Probabilistic modeling

Due to the lack of the information on the real structure of the financial system, there is an interest to generate numerically models which have, at least, basic features of such models.

Apparently, the liability matrix L and the reserve vector R are random and evolve as stochastic processes. Due to the high dimensionality of the problem their modeling is extremely complicated and simplifying assumptions are unavoidable. The majority of available studies consider static models or stationary models and start modeling with the description of the incidence matrix B.

The simplest model is based on the hypothesis that the non-diagonal elements of the incidence matrix B are independent identically distributed Bernoulli random variables, see, e.g., [29] where low-parameter models are suggested to evaluate the impact of various factors on the financial stability. In addition to N and $p = P(B^{ij} = 1)$, there are three more parameters: the total value of assets A, the value of external assets C, and the net worth as the percentage of the total value of assets γ. These parameters are used to generate the balance sheets. In our notations, the interbank exposures and liabilities for the ith bank are defined as follows:

$$a^i = (A - C)\frac{n_{in}(i)}{|B|}, \qquad b^i = (A - C)\frac{n_{out}(i)}{|B|},$$

where $|B| := \sum_{ij} B^{ij}$. The value of external assets of the bank are defined by the formula

$$C^i = (b^i - a^i)I_{\{a^i<b^i\}} + \frac{1}{N}\Big(C - \sum_j (b^j - a^j)I_{\{a^j<b^j\}}\Big).$$

If the second term is positive, then all banks in the system are solvent. Since a^j and b^j are random, one should have a sufficiently high ratio C/A (in [29] it was always taken greater then 0.3). The quantity $\gamma(a^i - b^i + C^i)$ models the net worth while $(1-\gamma)(a^i - b^i + C^i)$ stands for the consumer deposits.

3. Dynamic models and alert indicators

3.1. Structural model

The aim of the model is to provide regulators two functions on the current state of the system which can be used to calculate the alert indicators. The first one is the probability that the system will suffer a cascade of defaults before a specified time horizon. The second indicator is the total losses incurred by the cascade of defaults, if it happens.

We suppose that at time zero the regulators dispose the liability matrix L or its transpose the exposure matrix $X = L'$ (in reality, this information is public only in rare countries, like Brasil, but can be available to central banks) and the vector of capital reserves C which is decomposed into non-risky reserve c (say, Treasury bonds) and investments y into a single risky asset which can be interpreted as a market index or a market portfolio. In our very stylized model all these values are fixed up to the time horizon T. Of course, in reality the banks trade and portfolios are composed in many assets. Nevertheless, quite often banks mimic the behavior of each other and one may guess that a typical portfolio value has the same evolution as a certain reference portfolio. We describe its dynamics by a geometric Brownian motion:

$$dS_t/S_t = \mu dt + \sigma dW_t.$$

That is,

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t}.$$

At time zero all banks supposed to be solvent.

The default cascade will be triggered at the instant when one of the solvency conditions will be violated.

The solvency condition for the ith bank has the form:

$$V_t + y_0^i S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t} \geq 0. \tag{3.5}$$

where

$$V_t := b_t^i - a_t^i + c_t^i,$$
$$b_t^i := \sum_{j \in G} L_t^{ji}, \qquad a_t^i := \sum_{j \in G} L_t^{ij}.$$

Hypothesis: $V_t = V$ for all $t \in [0, T]$.

The above assumption allows us to provide the regulators some easily calculated indicators of the system stability. Without any doubts, in the present oversimplified form they can be criticized. For example, we assume that the interbank operations to a large extend are balanced by liquid assets. In favor of this are evidences that interbank lending is not the main activity of banks. We also assume a rigidity of the investment portfolio. Again, econometric studies confirm that banks have a tendency to follow similar behavior. The benchmark portfolio process may have various dynamics and various theoretical and statistical models can be used for its description.

Put

$$\lambda^i := \frac{1}{\sigma} \ln \frac{V^i}{y^i S_0}$$

with a convention that $\lambda^i := -\infty$, if $V^i \leq 0$. Let i_0 be the index corresponding to the largest of values of λ^i. We may assume, with very minor loss of generality, that all finite values of λ^i are different (the coincidence is not expected in the present context) and that λ^{i_0} is finite (otherwise there will be no defaults).

Let us introduce the stopping time

$$\tau := \inf\{t \geq 0 : \ w_t + \beta t \leq \lambda^{i_0}\}$$

where $\beta := \mu/\sigma - \sigma/2$. If $\tau \leq T$, the system will have a default and it happens with the node i_0; the price of the market portfolio at this date will be $S_0 e^{\sigma \lambda^{i_0}}$. The distribution of τ is the well-known inverse Gaussian distribution (see; e.g., [11]) and we have that

$$P(\tau \leq T) = \Phi(h_1(T)) + e^{2\beta\lambda^{i_0}} \Phi(h_2(T)),$$

where Φ is the standard Gaussian distribution function and

$$h_1(T) := \frac{\lambda^{i_0} - \beta T}{\sqrt{T}}, \qquad h_2(T) := \frac{\lambda^{i_0} + \beta T}{\sqrt{T}}.$$

The default of the bank i_0 generates a cascade of the defaults. It seems reasonable to suppose that the market reacts to such an event and the risky asset may loss a certain percentage of its value. With this assumption the set $D_1 = D_1(i_0)$ of first order defaults of the banks correspond to the indices j such that

$$\sum_{k \in G} L^{kj} - \sum_{j \in G} L^{jk} + c^j + \alpha y^j S_0 e^{\lambda^{i_0}} - (1-R)L^{ij_0} < 0, \tag{3.6}$$

$D_0^1(i_0) := D_0 \cup D_1$ etc. The parameter $\alpha \in]0,1]$ represents the default impact on the price of the reference portfolio.

The second alert indicator is the amount of total losses

$$L(i_0) := (1-R) \sum_{n=0}^{N} \sum_{j \in D_{n+1}} \sum_{k \in D_0^n} L^{jk}.$$

In the considered setting it can be augmented, e.g., by the losses of non-defaulted banks due to a depreciation of their portfolios:

$$\tilde{L}(i_0) := (1-\alpha) \sum_{j \in G \setminus D_0^N} y^j S_0 e^{\lambda^{i_0}}.$$

3.2. Discussion

The model introduced above has an advantage of its simplicity. It combines structural approach to defaultable securities with ideas of modern theory of financial networks. The alert indicators have a simple and comprehensive meaning. They can be easily computed at the monitoring dates t_m (when the new balance sheets are communicated) for the moving time horizons $t_m + T$. This allows regulator to see dangerous trends in the evolution of the system. It is worth noting that the model combines two channels of contagions: via the network as well as via the correlation due to common source of randomness.

Surely, the model is highly stylized. How serious are the weak points and how the model can be improved?

1. It is assumed that the investment in the single risky asset are static though in reality there is an intensive trading. For a fixed input there is only the bank triggering the default is uniquely determined.

To our mind, these objections should be examined carefully. Due to extreme complexity of financial systems (recall that they may contain hundreds of banks) and complexity of individual balance sheets, for more sophisticated models one can have an accumulation of various factors: misspecification errors, calibration errors, data aggregation errors etc. That is why simplifying hypotheses seems to be inevitable. It seems that we can accept that banks investment portfolios are close to the most performant one.

Of course, the predetermined bank triggering of the default cascade is not intuitive. However, as we know from the literature the matrix L is rarely known and should be reconstructed from the aggregated exposure of the banks. It is not difficult to implement a random reconstruction procedure, for each realized reconstruction one can compute conditional alert indicators, and take the average.

4. Numerical experiments

4.1. Network construction

We present here numerical simulations to offer further insight into the role of the external assets in contributing to a systemic risk in the financial system and to show an impact of parameters range to financial stability. Table 1 summarizes the baseline simulations parameters. The system comprises N banks. As in Eboli (2004), [15], we consider the banking system as a network of nodes, where each node represents a bank (or any other financial institution) and each link represents a directional lending relationship between two nodes (two banks). We believe that network reflects its "genetic" structure. The development of the system starts from relatively small kernel composed from a few banks. A newly created bank establishes relationships preferably with more "important" nodes of the network, namely, those that already have more connections than others. Also, well connecting bank, usually, has better chances not to be eliminated from the system (“too connected to fail).

As an example let us make a look on the development of banking system in Russia. In the USSR the number of banks were about a dozen. The first commercial bank was registered in August 1988 (Cooperative bank "Patent", Leningrad). Already to the end of 1989 the country had 43 commercial banks. Afterwards the number of banks in Russia evolved (approximately) as follows:

1991 – 1400, 1994 – 2300, 1997 – 2000, 2003 – 1300, 2011 – 1000.

In 2016 the total number is less than 700. To compare: the USA in 2014 had more than 6800 banks.

Of course, the evolution of the banking system is a rather complicated process but, statistically, it leads to a network having common features with other type of networks like internet connections. In particular, interbank networks have a scale free topology with a few large banks having many interconnections and many small banks with a few connections. By this reason, we generate our simulating financial system using the methodology introduced in Barabási and Albert (1999), [4]. We are based on the idea that a larger and more connected bank is usually more trusted and as a consequence, other system banks tend to deal with it rather than with a less connected one.

The algorithm starts with creating the initial network (the “seed") with a small amount of nodes n. The connections between them are taken randomly, and the maximal number of connections (“in" and “out") is limited by m as well as the total number M of connections (in our experiments $n = 10$, $m = 5$, $M = 20$). When drawing the network, we allow for the possibility that two banks can be linked to each other via both lending and borrowing links but at most one in the same direction is possible between the 2 banks.

Remark. In general, we expect that the structure of the initial network has a relatively small impact on the resulting network which may be in dozens or even hundred times larger than the “seed". So, the initial network involving a few nodes can be created in various way, e.g., as Erdös–Rényi network with the matrix $B = (b^{ij})$ where entries $b^{i,j}$, $i \neq j$, form a set of independent identically distributed Bernoulli random variables with $P(b^{i,j} = 1) = p$.

To generate a scale free network, assuming that a seed network of n banks is randomly generated as mentioned above, we proceed recursively. At each step we add $m' < m$ new nodes choosing each time a partner i between the existing nodes and selecting accordingly to the Connection Probability $P(i)$ defined as follows:

$$P(i) = \frac{\text{total number of connections of } i}{\text{total number of connections of the network}},$$

where i is an existing node in the network. Nodes will be added each time until we reach a network size limit equal to N. As shown on Figure 2, the distribution of a number of nodes in our model is in a conformity with that

Parameter	Description	value	Range of variation
n	Number of banks in the initial random graph	10	fixed
N	Total number of banks in the network	250	fixed
m	Maximum number of connection in the random network	5	fixed
m'	Maximum number of connection in the scale free network	3	fixed

Table 1: Summary of the parameters considered in the network construction of the model.

Parameter	Description	Value	Range of variation
A	Value of the total assets of the network	10,000,000,000	fixed
β	Proportion of external assets from the total assets of a bank	0.5	fixed
S_0	Initial price of risky asset	10,000	fixed
T	Time horizon	40	fixed

Table 2: Summary of the system benchmark parameters

one can expect from a typical scale free network topology where a few nodes have a high number of connections while the majority of nodes have a small number of connections. In particular, on the realization of the algorithm depicted on the scatterplot only 6 banks from 250 have each at least 25 connections while 185 have an most 5 connections.

For any realization of the random graph, we populate the individual banks' balance sheets in a manner consistent with bank level and aggregate balance sheet identities.

Amongst assets we distinguish external assets (investors' borrowing), denoted by C^i, and interbank assets (other banks borrowing), denoted by a^i. Thus, for the bank i, the asset part of the balance sheet can be decomposed as

$$\text{Total assets} = C^i + a^i, \qquad i = 1, \dots, N.$$

Moreover, the external assets can be of two types: risky r^i and riskless (cash) l^i, so that $C^i = r^i + l^i$. Introducing the parameter θ as a proportion of cash

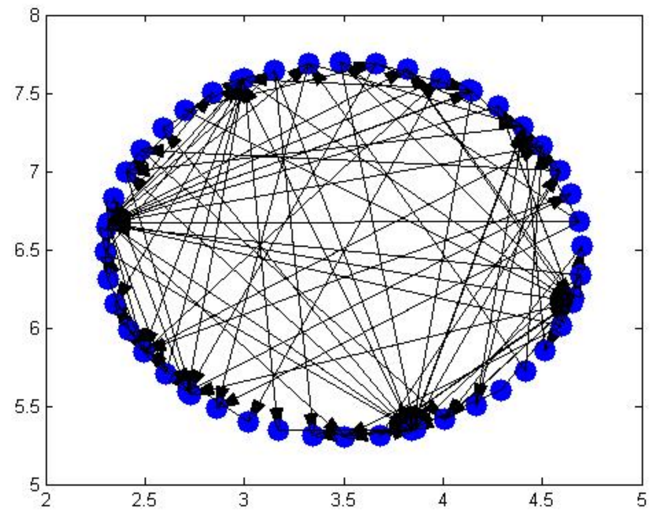


Figure 1: Illustration of a financial network generated by the algorithm and limited to 40 banks for clarity of the figure.

Parameter	Description	Value	Range of variation
α	Percentage of price degradation following the panic in the market during the crisis	1	0 to 1
R	Recovery rate	0.5	0 to 1
θ	Percentage of riskless asset from total external asset	0.4	0 to 1
σ	Volatility of the risky asset price	0.4	0 to 1
μ	Drift of the risky asset price	0.2	0 to 1

Table 3: Summary of the balance sheet parameters

holdings, the volume of the risky assets is $r^i = (1-\theta)C^i$ The liabilities of each bank are composed of the net worth of a bank, denoted by NW^i, and the interbank borrowing, denoted by b^i. Hence for the bank i, we have that

$$\text{Total liabilities} = NW^i + b^i, \qquad i = 1, \ldots, N.$$

An example of a bank balance sheet as generated by the simulator is also

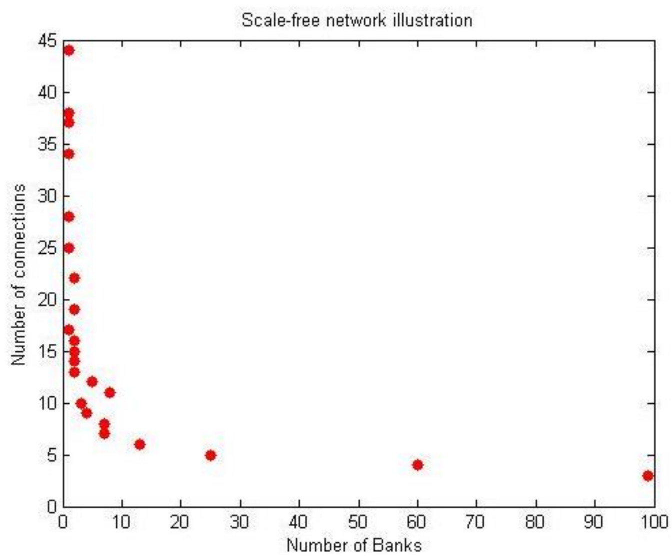


Figure 2: Representation of the scale free topology.

shown in Figure 3.

We construct balance sheets for individual banks in a sequence of steps. The entry parameter is the total value of all assets in the system denoted by A and the parameter β which defines the proportion of the external asset C representing the total loans made to ultimate investors and thus relating to the total size of the flow of funds from savers to investors through the banking system. That is, $\beta = C/A$. The aggregate assets of the whole banking industry can be written as $A = C+I$, where $I = (1-\beta)A$ represents the aggregate volume of interbank exposures, i.e. $I := \sum_{i,j} L^{ij}$.

Dividing the total interbank assets by the total number of nodes in the network we arrive at the level of each bank. So, weights of all links are equal banks borrow and lend by equal portions $w = I/|B|$. Though this looks not very realistic, we accept such a hypothesis to reduce the number of parameters. Hence, using w and the structure of the network, we can calculate for each bank the volume of its liabilities $a^i = n_{in}(i)w$ and exposures $b^i = n_{out}(i)w$. For any bank to be able to operate we require that the value of its external assets is not less than its net interbank borrowing, that is, we have: $C^i \geq b^i - a^i$. We fulfill this constraint by applying the following two-step algorithm.

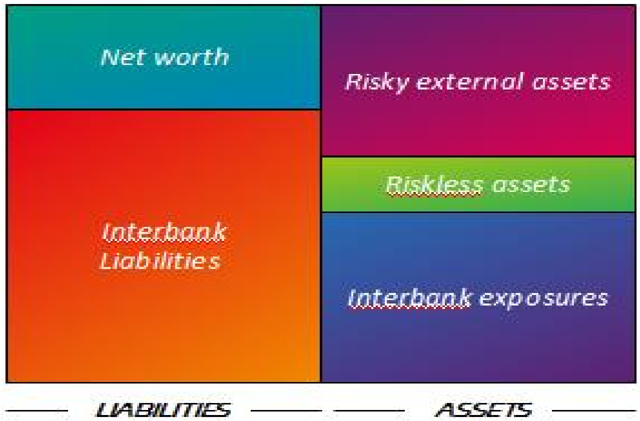


Figure 3: Representation of the balance sheet.

First, for each bank, we fill up the bank external assets part of the balance sheet is such a way that its external assets plus interbank lending will equalize its interbank borrowing. That is, we provide first the bank i the volume $\tilde{C}^i = (b_i - a_i)^-$ where $\tilde{C}^i$ is the fraction of the total volume C reserved for the external assets. At the second stage what is left in aggregated external assets is equally distributed among all banks. Note that the total of external assets is equal to C. Hence, in the second step we distribute $\bar{C} = C - \sum_{i=1}^{N} \tilde{C}^i$ equally among all N banks. Hence, we have $C^i = \tilde{C}^i + \bar{C}/N$. The constraint can become difficult to meet if the percentage of external asset is too low. Since the distribution of links is stochastic, some banks may be assigned interbank borrowing much larger than interbank lending. When the total amount of external assets is low, there may not be enough assets to go round to close all balance sheet gaps opened up in this way. To avoid this difficulty we make sure that the total volume of external assets is at least 30% of the total volume of all assets.

Although the model applies to fully heterogeneous banks, for the purpose of illustration and simplicity we consider one common risky asset for all banks (full correlation among the banks). Further studies can be conducted for portfolios composed of many different risky assets. Furthermore, we choose the risky asset evolving according to value of a reference portfolio whose dynamics follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

that is

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t)}.$$

Since $S_t = S_0$ at time $t = 0$, we can define y^i as the amount of the risky assets in the portfolio of each bank with $y^i = r^i/S_0$.

Hence, we complete the asset side of the bank balance sheet as well as interbank borrowing b on the liability side. The determination of the remaining component, the net worth (equity) NW on the liability side is relatively straightforward. The net worth is set as $NW = a + C - b$. This completes the construction of the banking system and of each constituent bank balance sheet.

Now we calculate the probability of the first default. Then we specify which bank will first default to check the price of the risky asset at the time of default S_τ where τ is the time of the first default (1st stopping time). This generates a loss in the asset price from S_0 to S_τ on each bank i which will suffer a loss of $y^i(S_0 - S_\tau)$. The sum of these losses is equal to $\sum_{i=1}^{N} y^i(S_0 - S_\tau)$ and is denoted as Corr_loss (correlation loss). In what follows we assume that the first default will affect the neighborhood and will trigger a cascade of default due to the loss transmission through the interbank connections. The sum of the total losses generated by the cascade of default is denoted as Con_loss and is equal to $(1-R)\sum_{n=0}^{N}\sum_{i\in D_{n+1}}\sum_{\in D_n}$. Having the probability as well as:

$$\text{Network total loss} = \text{Correlation loss} + \text{Contagion loss}$$

we can calculate

$$\text{Probable loss Indicator} = P \times \text{Total loss}.$$

In what follows, we vary the parameters in our experiments.

4.2. Experiment 1: The influence of the external assets volume and its riskless proportion on the probability and losses.

Given the balance sheet above we want to compute the probability of default when the proportion of the risky asset varies. For the sake of more profit and wealth, banks have the choice to invest in either risky assets, or riskless assets. It is known that the revenue in risky assets can be much larger than riskless ones. Thus, banks have more tendency to invest there while keeping an eye on the risks and their liabilities towards other banks making sure they can settle when needed. Therefore, we will check the probability of first default that can trigger a cascade of default in the system based on the percentage of the risky assets. Regulators may impose a minimum threshold on the level of riskless assets (θ) held by banks as well as define a certain proportion that a bank can invest in external assets. That is why we will consider both parameters in this experience to check how both of them can affect the vulnerability of the system and what is the optimal requirement on the riskless assets thresholds for each volume of external investments.

As for the following pictures, this experiment illustrates that with an increase of the riskless assets level, the probability of the first default decreases to become zero after a certain threshold of θ. We can also note that this threshold is smaller with an increasing volume of assets β.

For example, for $\beta = 0.55$, the probability of default is always less than 0.5 decreasing with θ and becomes null after $\theta = 0.6$. If the system is engaged with high level of external investments $\beta > 0.75$, the probability of default is very low with a very low rate of riskless assets $\theta = 0.55$. This means that the system is rather stable even if the level of risky assets is high. On the other hand, for a high level of external assets investments as shown in the figure below, the probability of first default remains low even with high level of risky assets.

In the following experiments, we will fix $\beta = 0.5$ as we assume that banks have the same probability to invest in external and internal assets.

4.3. Experiment 2: The influence of the volatility and drift of the risky asset price

Now we want to study how the volatility and drift of the asset price can also affect the probability taking into consideration also the level of the risky assets in banks balance sheet. The volatility reflects the risk of changing the portfolio price due to external and internal factors.

Assuming that the portfolio of risky asset price has a volatility that varies from 0.1 to 1, we can conclude that for the volatility less than 0.25 , the

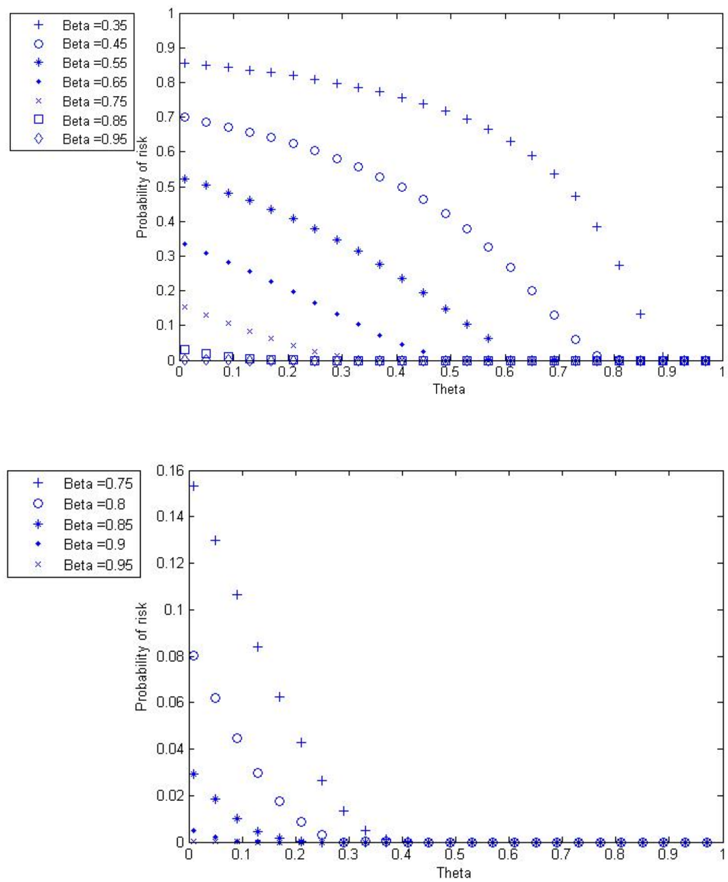


Figure 4: Probability of default with β percentage of external assets and θ proportion of riskless assets

probability of default is low for any θ level. When volatility increases above 0.25 , the probability of default increases with θ decreasing and volatility increasing. The pattern of every plot is changing with θ : having the plot concave for high θ and convex for low θ shows that the behavior of the volatility influences more the probability of default since for higher θ the increase in probability is faster than the lower one. So, summarizing, the volatility should be limited to a certain extend in order to save the network from a high probable default otherwise a high impact will be affecting the asset price which in its turn will trigger a higher indicator of default.

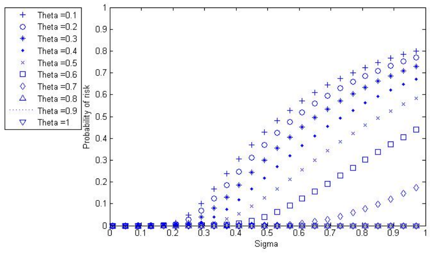


Figure 5: Probability of default with θ and volatility of the asset price.

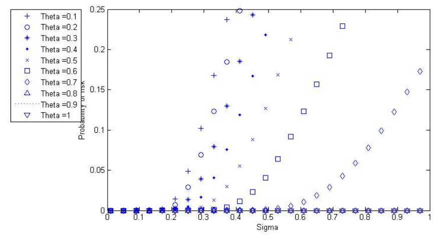


Figure 6: Zooming on probability of default with θ and volatility of the asset price.

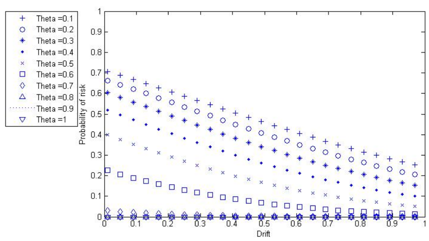


Figure 7: Probability of default with θ and the drift of the asset price.

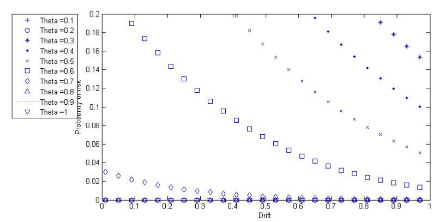


Figure 8: Zooming on probability of default with θ and the drift of the asset price.

Moreover, we also evaluate the effect of the asset price drift on the probability considering at the same time the level of the risky assets in Banks' balance sheet. We conclude that a portfolio price with high drift or high average of return will for sure lead to a more stable financial system. The optimal portfolio would be with high drift and low volatility.

Figure 7 shows for each level θ the variation of the probability relatively to the drift. We can see in particular that for $\theta = 0.5$ the level of drift required to assure a probability less than 0.2 is also 0.5 but if we have a $\theta = 0.6$ we can see that the required level of drift in this case is 0.05.

Figure 8 highlights the relation between the volatility and the drift affecting the probability of default. It is normal that the probability of default increases with increasing volatility and decreases with increasing drift. But we can also see from Figure 7 that when volatility is very high the influence of the increasing drift is reduced.

4.4. Experiment 3: the impact of recovery rate and the level of the riskless asset in the banks' balance sheet on the losses

In our model, the system loss is another parameter that influence the indicator as probability of first default can be low; however, when it happens the loss in the system can be huge. Therefore, we analyze the behavior of the network when recovery rate is changing knowing that recovery rate is one of the main influencer of the losses due to the default cascade. The figure below shows that the total loss in the system decreases when recovery rate is increasing due to the fact that defaulted banks need to settle their due payment. We can clearly observe that there is a linear relation between the recovery rate and contagion losses.

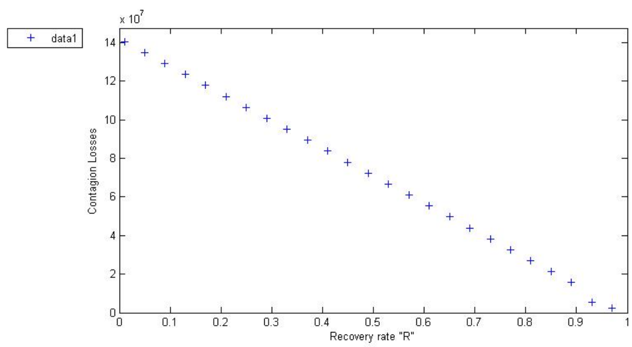


Figure 9: Contagion losses.

4.5. Experiment 4: the impact of the fire sale on the losses

We can also consider a subordinate source of risk due to the fire sale of external assets of defaulting banks which will lead to other banks default because of the price depreciation. In bad times in order to compensate certain losses, distressed banks tend to sell assets in a depressed price, a situation called asset fire sale. Because of the correlation between the banks' balance sheets having common assets, the decrease in the asset price affects all banks holding these assets and as a consequence a cascade of losses created by others banks too. In this experiment, we want to compute what will be the losses resulting from the "fire sale" mechanism that will respectively add to the total loss of the system. It is expected that the fire sales loss will increase with the drop rate of the price but it is worth to note that the increase is very sharp and fast.

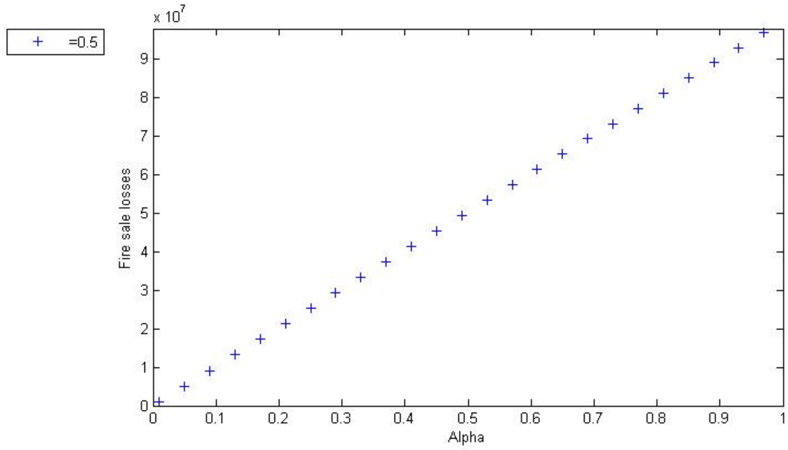


Figure 10: Fire sale losses.

4.6. Experiment 5: removing the weakest bank to make the system more resilient

The financial system is a number of financial institutions, in our paper considered as banks. Every bank has his investment strategy, priorities, relationship and connections as a consequence different influence, power, and risk level in the financial system. Banks balance sheets are populated on the quarterly basis and can be available to regulators at any time. Thus it is possible to understand the risk level of each bank to be defaulted. Since banks default can generate a default cascade, it is worth to verify if it is better to exclude the risky bank from the financial network. It is important to determine the weakest bank that can default anytime either due to a market price drop or liquidity shortage.

In this experiment, we create a scale free network of 250 nodes using the same methodology as above and determine the bank i, having the highest probability to default or, in other words, the weakest bank to default. We also calculate, on the basis of the above, the probable loss Indicator of this network. In some cases, we may have more than one bank defaulting simultaneously.

Once the bank is identified, we want to check if removing the bank from the network is healthier for the financial system. Of course, such an action has important consequences the owner, employees as well as debtors and creditors in this bank. We describe the modification on the balance sheet of this bank and all banks connected and in relation to this bank as follows:

- removed bank pays all his liabilities to all his creditors and as a consequence updates his creditors' balance sheets a_j by adding the amount of money bank i has landed from bank j to the riskless assets.

- removed bank collects all his exposures from all his debtors and as a consequence updates his debtors balance sheets b_j by removing the amount of money bank i has credited to bank j. Bank j is supposed to pay to bank i from their external assets (riskless assets and risky assets, so either bank j has to pay from his liquid reserve or sell some external assets to pay his due).

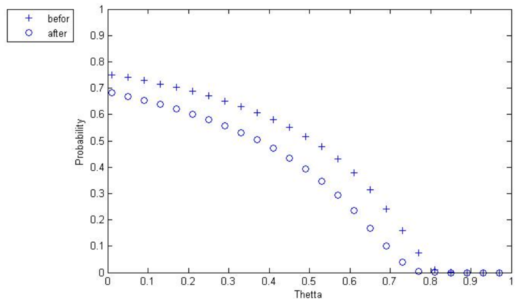
We conduct this experience and compute the probability, total loss and indicator having the vulnerable bank (banks) in the system and after removing it (them) from the network.

The figure below confirms that when removing the risky bank, the probability of default will decrease. This means that the system becomes more resilient to default.

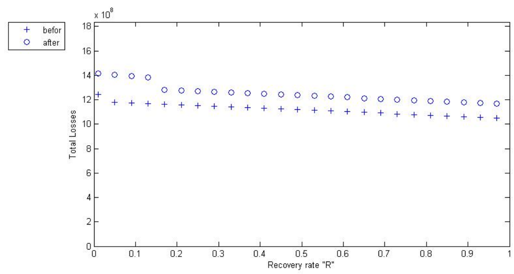
Now we check the total losses that may occur in the system due to a default in the above two mentioned cases and we note that, contrary to the probability, the total losses will be higher in the case where it has been decided to remove the vulnerable banks. The reason could be that the external assets prices that have decreased to S_τ causing the first default in the first round has to decrease more to trigger the default after removing the set of banks that could defaulted on S_τ. In this case, though the probability is lower, the increase in the total losses from before to after removing the banks is due to the fact that the risky assets price should drop more and, accordingly, the correlation losses increase. On the other hand, since the shock on the net worth of every bank becomes higher, we expect that the contagion losses are larger now. Having both correlations and contagion losses higher, this will lead to a high total loss as shown on the plot below. it is worthy to note that for small recovery rate the increase of the total loss after removing the weak banks is even higher.

5. Conclusions

In this paper we develop a financial network model of interbank interactions which incorporates a dynamical behavior of banks portfolios and



(a) Probability of default



(b) Total system loss

Figure 11: Comparison between before and after removing the weak bank(s).

combines it with cascade defaults. It is assumed that the portfolios contains, together with riskless asset, a unique risky asset, which can be interpreted as a market index or a benchmark portfolio and whose price evolution is described by a geometric Brownian motion. This part of modeling follows ideas of the so-called structural approach well known in the context of pricing defaultable securities. A crisis starts when the net worth of a bank hits zero triggering a cascade of defaults. The time to default and the total losses calculated for the "frozen" parameters of the balance sheets can constitute indicators of the "health" of financial system. They can be easily monitored, on a regular basis, by the regulators. Usually, the detailed structure of the system is available only to regulators but it is not public. By this reason, we complete our study by numerical experiments with simulated data. Due to complexity of financial systems this is a non-trivial problem. The network graph is build by a version of preferable attachment algorithm augmented by a procedure of simulations of balance sheets. Results of the experiments are presented by plots showing dependence of the indicators as functions of parameters. In particular, we present an experiment with removing the weakest bank from the system. We believe that developing our approach on the basis of practical data can provide regulator additional tools of monitoring vulnerability of banking system and measuring its stability.

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ДИНАМИЧЕСКИЕ МОДЕЛИ СИСТЕМНОГО РИСКА И ЗАРАЖЕНИЯ

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Резюме

Современные финансовые системы являются сложными сетями взаимосвязанных финансовых институтов (банков, хедж-фондов, страховых компаний и.т.д.) и дефолт одного из них может вызвать цепную реакцию дефолтов других институтов системы. После недавних финансовых кризисов важность системного риска вышла на первый план и теоретические исследования в этой области интенсифицировались. Большая часть известных в результатов относится к статическим моделям, которые посвящены процессам, происходящим в системе, когда каскад дефолтов уже начался. В данной работе мы предлагаем динамическую модель так называемого структурного типа, когда дефолт начинается в момент выхода некоторого стохастического процесса из области. Каскад инициируется в момент достижения критического уровня процессом, описывающим портфели банков. Мы полагаем, что вероятность выхода и суммарные издержки в результате каскада дефолтов, могут служить индикаторами, позволяющие регуляторам осуществлять мониторинг системы и предпринимать упреждающие коррекции для понижения системного риска. В работе проводится численное моделирование системы, которая строится на основе случайного графа, полученного при помощи алгоритма предпочтительного присоединения. Приводятся результаты численных экспериментов при различных значениях параметров.

Ключевые слова: системный риск, финансовые сети, заражение, дефолт.