ON UNIQUENESS OF CLEARING VECTORS REDUCING THE SYSTEMIC RISK

KHALIL EL BITAR, YURI KABANOV, AND RITA MOKBEL

Abstract. Clearing of financial system, i.e. of a network of interconnecting banks, is a procedure of simultaneous repaying debts to reduce their total volume. The vector whose components are repayments of each bank is called clearing vector. In simple models considered by Eisenberg and Noe (2001) and, independently, by Suzuki (2002), it was shown that the clearing to the minimal value of debts accordingly to natural rules can be formulated as a fixpoint problems. The existence of their solutions, i.e. of clearing vectors, is rather straightforward and can be obtained by a direct reference to the Knaster–Tarski or Brouwer theorems. The uniqueness of clearing vectors is a more delicate problem which was solved by Eisenberg and Noe using a graph structure of the financial network. We prove uniqueness results in two generalizations of the Eisenberg–Noe model: in the Elsinger model with seniority of liabilities and in the Amini–Filipovic–Minca type model with several types of liquid assets whose firing sale has a market impact.

1. Introduction

To explain the clearing problem we start with the simplest example of a financial system with two agents each having in a cash 10 dollars. The first agent gets from the second a credit 1M dollars, the second gets from the first a credit 1M and 1 dollar. Apparently, as a result both agents has a huge liabilities with respect to each other. Of course, the agents can be asked to reduce their liabilities by reimbursing credits partially (e.g., to the levels 0.5M and 0.5M+1 in liabilities and 10 dollars both in cash) or completely, with zero liabilities and cash reserves 11 and 9 dollars respectively. Intuitively, the situation where the liability are reduced (i.e. the system is cleared) seems to be less risky: if one of agent became bankrupt and only the percentage of the huge debt value can be reimbursed, the creditor's losses will be also huge. For complex financial systems involving large numbers of agents with chains of borrowing the clearing problem, that is the reduction of absolute values by reimbursement, looks much more complicated.

In the influential paper [3]published in 2001, Eisenberg and Noe suggested a clearing procedure in the model describing a financial system composed by N banks (under "banks" can be understood various financial institutions); a more general model was introduced independently at the same

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time by Suzuki, [6]. The assets of the bank are cash and interbank exposures which are, in turn, liabilities for its debtors. The clearing consists in simultaneous paying all debts. Each bank pays to its counterparties the debts *pro rata* of their relative volume using its cash reserve and money collected from the credited banks. The rule is: either all debts are payed in full or the zero level of the equity is attained and the bank defaults. The totals reimbursed by banks form an N-dimensional clearing vector. A remarkable feature is that this vector is a fixed point of a monotone mapping of a complete lattice into itself and its existence follows immediately from the Knaster–Tarski theorem, a beautiful and fairy simple result which proof needs only a few lines of arguments. The uniqueness of the clearing vector is a more delicate result involving the graph structure of the system.

The ideas of the Eisenberg–Noe paper happened to be very fruitful and their model was generalized in many directions having not only financial importance but posing interesting mathematical questions. One of them is the question on uniqueness of clearing vector or equilibrium on financial market.

Our first theorem provides a new sufficient condition for the Elsinger model of clearing with debts priority structure. This model is given by a set of liability matrices corresponding to each seniority. The idea of our approach is to use the largest clearing vector which always exists to construct a new liability matrix generating a graph structure with which one can work in a similar way as in the Eisenberg–Noe model. The second theorem deals with the uniqueness of equilibrium in a clearing model with several illiquid assets and a market impact. In the presence of several illiquid assets the banks are faced the choice of asset selling strategies. We use the proportional scheme of selling similar to that in the paper by Cont–Wagalath, [2], leaving game-theoretical versions for future studies. In the case of one illiquid asset our result is close to that of the study by Amini–Filipovic–Minca, [1], but our definition of the equilibrium is different (but equivalent).

The structure of the note is as follows. In the introductory Section 2 we discuss briefly the general principle and results in the framework of the Eisenberg–Noe model. To facilitate the comparison with further development we provide also short proofs. In Section 3 we prove a uniqueness of the clearing vector for the Elsinger model where senior liabilities must be reimbursed before juniors. The Section 4 contains the sufficient condition for the uniqueness of the equilibrium in the model where clearing requires selling of the illiquid assets with price impact. Economically speaking, it is oriented to the recovering of the market after fire sales. For the reader convenience we provide in the concluding section a short information about the Knaster–Tarski theorem adapted to our needs.

Notations. We denote by $\geq$ the partial ordering in $\mathbb{R}^n$ and its subsets induced by the cone $\mathbb{R}^n_+$. In other words, the inequality $y \geq x$ is understood componentwise. Also the symbols $x \wedge y$ and $x \vee y$ mean, respectively, the componentwise minimum and maximum, $x^+ := x \vee 0$, $x^- := (-x)^+$. The notation $[x, z]$ is used for the order interval, i.e. $[x, z] = \{y \in \mathbb{R}^n : \ x \leq y \leq z\}$. If $A \subseteq [x, z]$, then inf A is the unique element $\underline{y} \in [x, z]$ such that $\underline{y} \le y$ for all $y \in A$ and for any $\tilde{y}$ such that $\tilde{y} \le y$ for all $y \in A$ we have that $\tilde{y} \le \underline{y}$. That is, the component $\underline{y}^i = \inf\{y^i : \ y \in A\}$ for $i = 1, \dots, n$.

We use matrix notations where the vectors are columns, $'$ is the symbol of transpose, and $\mathbf{1}' := (1, \dots, 1)$ (the dimension of the vector is supposed to be clear from the context).

2. The Eisenberg–Noe model

In the paper [3], Eisenberg and Noe investigated the model describing a financial system composed of N banks (under “banks" can be understood various financial institutions). In the aggregate oversimplified form the balance sheet of the bank i can be split into to parts: assets and liabilities. The assets are of two types: interbank assets (exposures) $\tilde{X}^i$ and cash e^i. The liabilities are: interbank debts (liabilities) $\tilde{L}^i$ and the equity C^i (or proper capital reserve) equalizing the two sides of the balance sheet:

$$e^i + \tilde{X}^i = \tilde{L}^i + C^i.$$

All this values are assumed to be greater or equal to zero. The condition that $C^i \ge 0$ means that the bank is solvent.

More detailed balance sheet provides the information on the values of liabilities of the bank i to the bank j, namely, vectors $(L^{i1}, ..., L^{iN})'$ of and $(X^{i1}, ..., X^{iN})$ of exposures. With this we have $\tilde{X}^i = X^{i1} + ... + X^{iN}$ and $\tilde{L}^i = L^{i1} + ... + L^{iN}$.

The matrix $L = (L^{ij})$ with positive entries and zero diagonal defines the total interbank exposures. Since the value of the exposure of i to j is the value of the liability of j to i, we have that the transpose $L' = X$. So, the matrix L and the vector e gives a description of a financial system in this model.

Put

$$\Pi^{ij} := \frac{L^{ij}}{\tilde{L}^i} = \frac{L^{ij}}{\sum_j L^{ij}}, \quad \text{if } \ \tilde{L}^i \ne 0, \ \text{and } \Pi^{ij} := \delta^{ij} \text{ otherwise},$$

where the Kronecker symbol $\delta^{ij} = 0$ for $i \ne j$ and $\delta^{ii} = 1$. Then Π^{ij} describes the proportion of the value debtor i due to the creditor j of the total interbank debt of i; $\Pi = (\Pi^{ij})$ called relative liabilities matrix. Note that in this definition, to get a stochastic matrix Π, we deviate from in [3] where $\Pi^{ii} = 0$ when $L^i = 0$

In general, financial system may have a complicated structure with cyclical interdependences and banks may have large exposures within cycles. To reduce them one can impose a clearing mechanism satisfying several natural requirements: limited liability and proportionality. Formally, this leads to the concept of a *clearing payment vector* $p^* \in \prod_i [0, \tilde{L}^i]$ satisfying the following properties:

a. Limiting liability. For every i,

$$p_i^* \le e^i + \sum_j \Pi^{ji} p_j^*.$$

b. Absolute priority. For every i, either $p_i^* = \tilde{L}^i$, or

$$p_i^* = e^i + \sum_j \Pi^{ji} p_j^*.$$

One may think that the central clearing authority forces each bank to make a "fair" payment of debts in such a way that, having the total payment p_i^*, the bank i remains solvent and pays to j the fraction $p_i^* \Pi^{ij}$ in such a way that either its total debts are paid, or all the resources are exhausted.

Alternatively, the condition $a.$ and $b.$ can be written in the following way:

$$p^* = \min\left\{e + \Pi' p^*, \tilde{L}\right\}, \tag{2.1}$$

where the minimum is understood in the componentwise sense, i.e. accordingly to the partial ordering defined by the cone $\mathbf{R}_+^N$.

The main result of Eisenberg and Noe asserts that the set of clearing vectors is non-empty. Moreover, there are the minimal and the maximal clearing vectors, denoted here $\underline{p}$ and $\bar{p}$, respectively. This assertion follows immediately from the Knaster–Tarski fixed point theorem: the monotone mapping $f : p \mapsto (e + \Pi' p) \wedge \tilde{L}$ of a complete lattice $[0, \tilde{L}]$ into itself has the largest and the smallest fixed points, see Section 5 for details. The set $[0, \tilde{L}]$ is convex and compact and f is a continuous mapping. So, the existence of its fixed point follows also from the classical Brouwer theorem.

Using the obvious identity $(x - y)^+ = x - x \wedge y$ we can rewrite the equation (2.1) in the following equivalent form

$$(e + \Pi' p^* - \tilde{L})^+ = e + \Pi' p^* - p^* \tag{2.2}$$

where the left-hand side is the equity vector of the system after clearing.

An important but simple observation: *the equity (after clearing) does not depend on the clearing vector*. Indeed, P being a stochastic matrix, $\mathbf{1}'\Pi' = \mathbf{1}'$. Therefore, multiplying the above representation (2.2) from the left by $\mathbf{1}'$ we get that the sum of equities

$$\mathbf{1}'(e + \Pi' p^* - \tilde{L})^+ = \mathbf{1}' e$$

is equal to the sum of the initial cash reserves, that is invariant with respect to the choice of the clearing vector. On the other hand, by monotonicity, we have that

$$(e + \Pi' p^* - \tilde{L})^+ \le (e + \Pi' \bar{p} - \tilde{L})^+.$$

If the both side here are not equal, then $\mathbf{1}'(e + \Pi' p^* - \tilde{L})^+ < \mathbf{1}'(e + \Pi' \bar{p} - \tilde{L})^+$ in contradiction with the invariance of the total of equities.

Sufficient condition for the uniqueness of the clearing vector As in [3] we shall assume for simplicity that $\tilde{L}^i > 0$ for all i.

For a stochastic matrix Π, we say that $I \subseteq \{1, ..., N\}$ is a (Π-)*surplus set* if $\Pi^{ij} = 0$ for all $i \in I$, $j \in I^c$, and $\sum_{j \in I} e^j > 0$.

Recall that j is the creditor of i if $\Pi^{ij} > 0$ (i.e. $\Pi^{ij} > 0$); in this case we shall use, as in the theory of Markov chains or in the graph theory, the notation $i \to j$.

We denote by $o(i)$ *the orbit of* i that is the set of all j for which there is a directed path $i \to i_1 \to i_2 \to ... \to j$, i.e. $o(i)$ is the set of all direct or indirect creditors of i.

Note that the orbit $o(i)$ with $\sum_{j \in I} e^j > 0$ is a surplus set. Indeed, if $\Pi^{jj'} > 0$ for some $j \in o(i)$, $j' \notin o(i)$, i.e. $j \to j'$, then there is a path $i \to i_1 \to i_2 \to ... \to j \to j'$.

Lemma 2.1. *Suppose that the market is cleared by a vector $p^* \in [0, \tilde{L}]$. Let I be a surplus set. Then at least one node of I has strictly positive equity value.*

In particular, any orbit $o(i)$ with $\sum_{j \in o(i)} e^j > 0$ has an element with strictly positive equity value.

Proof. Multiplying the identity (2.2) by $\mathbf{1}_I'$ and noticing that $(\mathbf{1}_I'\Pi')^i = 1$ for $i \in I$, we obtain that

$$\mathbf{1}_I'(e + \Pi' p^* - \tilde{L})^+ \geq \mathbf{1}_I' e > 0$$

implying the claim. □

A financial system is called *regular* if for every i the orbit $o(i)$ is a surplus set.

Theorem 2.2. *Suppose that the financial system is regular. Then $\underline{p} = \bar{p}$.*

Proof. Suppose that $\underline{p}$ and $\bar{p}$ are not equal, i.e. $\underline{p} \leq \bar{p}$ but for some i we have the strict inequality $\underline{p}^i < \bar{p}^i$. We denote C the vector of equities (it is common for all clearing vectors). By assumption the orbit $o(i)$ is a surplus set and, by Lemma 2.1, it contains an element m with the equity value $C^m > 0$. By definition of the orbit, there is a path $i \to i_1 \to ... \to m$ and we may assume without loss of generality that in this path m is the first node with strictly positive equity value.

First, we prove that we may consider only the case where the path consists of one step, i.e. $i \to m$. To this end, we check that $\underline{p}^{i_1} < \bar{p}^{i_1}$ if $i_1 \neq m$. In other words, the property that $\underline{p}^i \neq \bar{p}^i$ propagates along the path.

Suppose that $\bar{p}^{i_1} < \tilde{L}^{i_1}$. Then also $\underline{p}^{i_1} < \tilde{L}^{i_1}$. In such a case

$$\underline{p}^{i_1} = e^{i_1} + \sum_j \Pi^{ji_1} \underline{p}^j, \qquad \bar{p}^{i_1} = e^{i_1} + \sum_j \Pi^{ji_1} \bar{p}^j,$$

and we have that

$$\bar{p}^{i_1} - \underline{p}^{i_1} = \sum_j \Pi^{ji_1} (\bar{p}^j - \underline{p}^j) > 0$$

because $\Pi^{ii_1} > 0$. That is, $\underline{p}^{i_1} < \bar{p}^{i_1}$. This inequality also holds trivially, if $\bar{p}^{i_1} = \tilde{L}^{i_1}$ but $\underline{p}^{i_1} < \tilde{L}^{i_1}$. The remaining case where $\underline{p}^{i_1} = \bar{p}^{i_1} = \tilde{L}^{i_1}$ is excluded as we suppose that $C^{i_1} = 0$. Indeed, accordingly to (2.2), this leads to the equalities

$$e^{i_1} + \sum_j \Pi^{ji_1} \bar{p}^j - \tilde{L}^{i_1} = 0, \qquad e^{i_1} + \sum_j \Pi^{ji_1} \underline{p}^j - \tilde{L}^{i_1} = 0,$$

implying the identity

$$\sum_j \Pi^{ji_1}(\bar{p}^j - \underline{p}^j) = 0$$

which cannot be true since in the above sum the term corresponding to $j = i$ is strictly positive.

So, it is sufficient to consider only one-step case. Since $C^m > 0$ we have the representations

$$C^m = e^m + \sum_j \Pi^{jm}\underline{p}^j - \tilde{L}^m, \qquad C^m = e^m + \sum_j \Pi^{jm}\bar{p}^j - \tilde{L}^m.$$

As above, we again obtain the impossible equality

$$\sum_j \Pi^{jm}(\bar{p}^j - \underline{p}^j) = 0.$$

Therefore, the assumption $\underline{p}^i < \bar{p}^i$ leads to a contradiction. The uniqueness of clearing vector is proven. □

Remark 2.3. The above theorem reveals that the problem to find a clearing vector is ill-posed. Indeed, adding an infinitesimally small amount $\varepsilon > 0$ (say, one cent) to the initial endowments leads to a unique clearing vector. Similar effect will have small a increase in liabilities. One can think that the "true" liability matrix has all elements strictly positive and the in the model matrix zero elements appeared because liabilities are neglected. These phenomena are related to the ill-posedness of the spectral problem for stochastic matrices. Another question is which clearing vector is natural.

The above proof is rather straightforward and uses graph-theoretical language. One can get another one appealing to the contraction property of the mapping $f : p \mapsto (e + \Pi' p) \wedge \tilde{L}$ defined on the set $[0, \tilde{L}]$ equipped with l_1-distance $|p - \tilde{p}|_1$.

Proposition 2.4. *For every $p, \tilde{p} \in [0, \tilde{L}]$*

$$|f(p) - f(\tilde{p})|_1 \le |\Pi'(p - \tilde{p})|_1 \le |p - \tilde{p}|_1. \tag{2.3}$$

Moreover, the first relation above is the equality if and only if the union of subsets $A := \{i : \ (\Pi' p)^i = (\Pi' \tilde{p})^i\}$ and $B := \{i : \ (\Pi' p)^i, (\Pi' \tilde{p})^i \le \tilde{L}^i - e^i\}$ is the set of indices $\{1, \dots, N\}$.

Proof. Using the elementary inequality $|a \wedge c - b \wedge c| \le |a - b|$ which holds as the equality if and only if when $a = b$ or $a, b \le c$ we obtain that $|f(p) - f(\tilde{p})|_1 \le |\Pi' p - \Pi' \tilde{p}|_1$ where the equality holds if and only if for every i we have $(\Pi' p)^i = (\Pi' \tilde{p})^i$ or $(\Pi' p)^i, (\Pi' \tilde{p})^i \le \tilde{L}^i - e^i$. Since $|\Pi' y|_1 \le |\Pi'|_1 |y|_1$ and $|\Pi'|_1 = 1$, we have the claim. □

Let us consider the case where the matrix Π is irreducible. Suppose that $\mathbf{1}' e > 0$ and p and $\tilde{p}$ are two different fixed points of the mapping f. According to above proposition

$$\sum_{j \in B} \Pi^{ji}(p^j - \tilde{p}^j) = p^i - \tilde{p}^i, \quad i \in B.$$

This means that the non-zero vector with the coordinates $p^i - \tilde{p}^i$, $i \in B$, is a left eigenvector of the matrix $(\Pi^{ij})_{i,j\in B}$ corresponding to unit eigenvalue. This is possible only if the latter matrix coincides with Π. Thus, $p = f(p) = e + \Pi' p$. Since $\mathbf{1}'\Pi' p = \mathbf{1}' p$ we get that $\mathbf{1}' e = 0$ which is a contradiction. Using the decomposition of the matrix Π on the irreducible component, we get that the clearing vector is unique if for any irreducible component there is a node with strictly positive initial endowment.

3. The Elsinger model

We consider a version of the Elsinger model where the interbank debts may be senior and junior. In this model the system of N banks is described by the vector of cash reserves and by M matrices $L_1 = (L_1^{ij})$, ..., $L_M = (L_M^{ij})$ representing the hierarchy of liabilities with decreasing seniority. That is, the element L_1^{ij} represents the debt of the bank i to the bank j of the highest seniority etc., $\sum_j L_S^{ij}$ is the total of debts of the bank i of the seniority S.

The relative liabilities are defined by the matrix Π_S with

$$\Pi_S^{ij} = \frac{L_S^{ij}}{\tilde{L}_S^i} = \frac{L_S^{ij}}{\sum_j L_S^{ij}}.$$

The clearing procedure requires the complete reimbursement of the debts starting from the highest priority and, for each seniority level, the distribution is proportional to the volume of debts of this seniority. For the bank i we denote by p_S^i the value distributed to cover the debts of the seniority S. So, the clearing can be described by the set of vectors p_S, $S = 1, \dots, M$, which can be considered as a "long" vector from $(\mathbb{R}^N)^M$ satisfying the system of equations

$$\begin{aligned} p_1^i &= \min\Big\{e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j, \tilde{L}_1^i\Big\}, \\ p_S^i &= \min\Big\{\Big(e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j - \sum_{r<S} \tilde{L}_r^i\Big)^+, \tilde{L}_S^i\Big\}, \qquad 1 < S \le M. \end{aligned}$$

In a vector form these equations can be written as follows:

$$p_S = \Big(e + \sum_S \Pi_S' p_S - \sum_{r<S} \tilde{L}_r\Big)^+ \wedge \tilde{L}_S, \qquad S = 1, ..., M. \tag{3.4}$$

It is clear that, for the partial ordering in $(\mathbb{R}^N)^M$ induced by the cone $(\mathbb{R}_+^N)^M$, the function

$$(p_1, ..., p_M) \mapsto \left(\Big(e + \sum_S \Pi_S' p_S^*\Big)^+ \wedge \tilde{L}_1, ..., \Big(e + \sum_S \Pi_S' p_S^* - \sum_{r<M} \tilde{L}_r\Big)^+ \wedge L_M\Big\}\right)$$

is a monotone mapping of the order interval $[0, \tilde{L}_1] \times ... \times [0, \tilde{L}_M] \subset (\mathbb{R}^N)^M$ into itself. Thus, according to the Knaster–Tarski theorem the set of fixed points of this mapping, i.e. the solutions of the equation (3.4), is non-empty and has the maximal and the minimal elements.

In the case of liabilities of different seniority after clearing by the vector $p \in (\mathbb{R}^N)^M$ the equity vector $C \in \mathbb{R}^N$ has the form:

$$C = \Big(e + \sum_S \Pi'_S p_S - \sum_S \tilde{L}_S\Big)^+.$$

Lemma 3.1. *The equity vector does not depend on the clearing vector.*

Proof. Note that

$$\Big(e + \sum_S \Pi'_S p_S\Big) \wedge \sum_S \tilde{L}^i_S = \sum_S p_S.$$

Therefore,

$$\Big(e + \sum_S \Pi'_S p_S - \sum_S \tilde{L}_S\Big)^+ = e + \sum_S \Pi'_S p_S - \sum_S p_S.$$

With this identity the reasoning is analogous to that with a single seniority class. $\square$

The aim of this section is to provide a sufficient condition for the uniqueness of clearing vector using a specific graph structure induced by the matrices Π_S.

For a given clearing vector p we define the *default index* d^i of the node i as the smallest r such that

$$\bar{p}^i_r = e^i + \sum_S \sum_j \Pi^{ji}_S \bar{p}^j_S - \sum_{r'<r} \tilde{L}^i_{r'}.$$

In another words, d^i is the lowest seniority for which the bank equity after clearing is equal to zero. Define the matrix $\Delta = \Delta(p)$ by putting $\Delta^{ij} = 1$ if $\Pi^{ij}_{d(i)} > 0$, and $\Delta^{ij} = 0$ otherwise. We use the notation $i \rightsquigarrow j$ if $\Delta^{ij} = 1$ and denote by $O(i)$ *the* Δ-*orbit of* i, that is the set of all j for which there is a directed path $i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow ... \rightsquigarrow j$.

Theorem 3.2. *Suppose that for the clearing vector $\bar{p}$ any Δ-orbit is a surplus set. Then the clearing vector is unique.*

Proof. By definition, the default index

$$d^i := \min\Big\{r:\ \bar{p}^i_r = e^i + \sum_S \sum_j \Pi^{ji}_S \bar{p}^j_S - \sum_{r'<r} \tilde{L}^i_{r'}\Big\}.$$

It follows that $\bar{p}^i_r = 0$, hence, $\underline{p}^i_r = 0$ for every $r > d^i$. Suppose that $\underline{p}^i_{d^i} < \bar{p}^i_{d^i}$ and consider a path

$$i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow ... \rightsquigarrow m$$

ending up at the node with strictly positive equity value.

First, we show that at least for one seniority $\underline{p}^{i_1}_S < \bar{p}^{i_1}_S$.

Let $r' := d^{i_1}$. By definition we have: $\bar{p}^{i_1}_r = \tilde{L}^{i_1}_r$, $r \le r'$, and $\bar{p}^{i_1}_r = \underline{p}^{i_1}_r = 0$, $r > r'$. The claim holds, if $\underline{p}^{i_1}_r < \tilde{L}^{i_1}_r$ for some $r < r'$. Thus, it remains to consider only the case where $\underline{p}^{i_1}_r = \bar{p}^{i_1}_r = \tilde{L}^{i_1}_r$

for all $r < r'$ and prove that $\underline{p}^{i_1}_{r'} < \bar{p}^{i_1}_{r'}$. We have the alternative: either $\underline{p}^{i_1}_{r'} < \bar{p}^{i_1}_{r'} \le \tilde{L}^{i_1}_r$ (what we need), or $\underline{p}^{i_1}_{r'} = \bar{p}^{i_1}_{r'} \le \tilde{L}^{i_1}_r$. The second case is impossible, since the equalities

$$\begin{aligned}\bar{p}^{i_1}_{r'} &= e^{i_1} + \sum_S \sum_j \Pi^{ji_1}_S \bar{p}^j_S - \sum_{r<r'} \tilde{L}^{i_1}_r,\\ \underline{p}^{i_1}_{r'} &= e^{i_1} + \sum_S \sum_j \Pi^{ji_1}_S \underline{p}^j_S - \sum_{r<r'} \tilde{L}^{i_1}_r.\end{aligned}$$

imply that

$$\bar{p}^{i_1}_{r'} - \underline{p}^{i_1}_{r'} = \sum_S \sum_j \Pi^{ji_1}_S (\bar{p}^j_S - \underline{p}^j_S) \ge \Pi^{ii_1}_{d^i} (\bar{p}^i_{d^i} - \underline{p}^i_{d^i}) > 0.$$

This is contradiction.

The above argument reduces the problem to the case $i \rightsquigarrow m$ and the node m has a strictly positive equity. The equity C^m does not depend on the clearing vector. Therefore,

$$\begin{aligned}C^m &= e^m + \sum_S \sum_j \Pi^{jm}_S \bar{p}^j_S - \sum_S \tilde{L}^m_S,\\ C^m &= e^m + \sum_S \sum_j \Pi^{jm}_S \underline{p}^j_S - \sum_S \tilde{L}^m_S.\end{aligned}$$

It follows that

$$0 = \sum_S \sum_j \Pi^{jm}_S (\bar{p}^j_S - \underline{p}^j_S) \ge \Pi^{im}_{d^i} (\bar{p}^i_{d^i} - \underline{p}^i_{d^i}) > 0.$$

This contradiction shows that $\underline{p} = \bar{p}$.

3.1. **Example 1.** Let us consider the system consisting of 3 nodes with the initial cash endowments given by the vector $e = (0.1, 0, 0)$ and the liability and the "distribution" matrices corresponding senior and junior debts:

$$L_S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad L_J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi_S = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Pi_J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this model the vectors of total liabilities corresponding to the senior and junior debts are, respectively, $\tilde{L}_S = (1, 2, 2)$ and $\tilde{L}_J = (0, 2, 0)$.

The equations for clearing vectors are:

$$\begin{aligned}p^1_S &= (0.1 + 0.5\, p^2_S) \wedge 1,\\ p^2_S &= (p^1_S + p^3_S) \wedge 2,\\ p^3_S &= (0.5\, p^2_S + p^2_J) \wedge 2,\\ p^1_J &= 0,\\ p^2_J &= (p^1_S + p^3_S - 2)^+ \wedge 2,\\ p^3_J &= 0.\end{aligned}$$

It is not difficult to check that there are infinite set of clearing vectors. Namely, we have that $p_S = (1, 2, 1+t)$, $p_J = (0, t, 0)$ where $t \in [0, 1]$. The minimal clearing vector corresponds to $t = 0$, the maximal corresponds to $t = 1$.

3.2. **Example 2.** The vector of cash endowments and the matrix of the senior debts is the same as in the Example 1. The junior debts matrix L_J and the corresponding distribution matrix Π_J are now:

$$L_J = \begin{pmatrix} 0 & 0 & 0 \\ 0.4 & 0 & 1.6 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_J = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.8 \\ 0 & 0 & 0 \end{pmatrix}.$$

We are looking for positive solutions of the following equations:

$$\begin{aligned} p_S^1 &= (0.1 + 0.5\, p_S^2 + 0.2\, p_J^2) \wedge 1, \\ p_S^2 &= (p_S^1 + p_S^3) \wedge 2, \\ p_S^3 &= (0.5\, p_S^2 + 0.8\, p_J^2) \wedge 2, \\ p_J^1 &= 0, \\ p_J^2 &= (p_S^1 + p_S^3 - 2)^+ \wedge 2, \\ p_J^3 &= 0. \end{aligned}$$

Note that $p_S^1 \le 1$, $p_S^2 \le 2$, hence, $p_J^2 \le 1$ and the 3rd equation is linear:

$$p_S^3 = 0.5\, p_S^2 + 0.8\, p_J^2. \tag{3.5}$$

Substituting into the 2nd equation this expression for p_S^3 and the expression for p_S^1 from the 1st equation we get that

$$p_S^2 = ((0.1 + 0.5\, p_S^2 + 0.2\, p_J^2) \wedge 1 + 0.5\, p_S^2 + 0.8\, p_J^2) \wedge 2$$

The inequality $p_S^1 < 1$ is impossible since in this case $0.1 + 0.5\, p_S^2 + 0.2\, p_J^2 < 1$, implying that

$$p_S^2 = (0.1 + p_S^2 + p_J^2) \wedge 2.$$

For positive values of unknown variables the last equality may hold only if $p_S^2 = 2$ but then the 1st equation tells us that $p_S^1 = 1$.

Thus, we determined that $p_S^1 = 1$.

Combining the 2nd equation with (3.5) we obtain the equality

$$p_S^2 = (1 + 0.5\, p_S^2 + 0.8\, p_J^2) \wedge 2$$

implying that $p_S^2 = 2$.

Available information allows us to reduce the 5th equation a simple one of the form $p_J^2 = 0.8(p_J^2)^+ \wedge 2$ having the unique solution $p_J^2 = 0$.

Summarizing, we get that $p_S = (1, 2, 1)$, $p_J = (0, 0, 0)$.

Comment. In the first example the bank 1 has met all liabilities and finished with a positive equity, the bank 2 has payed the senior liabilities but defaulted on the junior debts, the bank 3 has defaulted already at the senior debts; the bank 2 has no junior liabilities with the bank 1. So, the Δ-orbit of the banks 2 and 3 are not surplus sets and there are infinite many clearing vectors. In the second example the bank 2 has a junior debt to bank 1, all Δ-orbits are surplus sets and the clearing vector is unique.

4. Models with illiquid assets and a price impact

Let us consider the clearing problem without seniority structure where the bank i owns not only cash e^i but also K illiquid assets, in quantities $y^{i1}, \dots y^{iK}$ represented in the model by the row i of the matrix $Y = (y^{im})$, $i \le N$, $m \le K$. The nominal prices per unit of illiquid assets are strictly positive numbers $Q^1, ..., Q^K$. The clearing might require their partial sale influencing the market price. If the bank sells $u^{im} \in [0, y^{im}]$ units of the m-th assets for the price q_m, its total increase in cash is

$$(Uq)^i = \sum_{m=1}^{K} u^{im} q^m.$$

The price formation is modeled by the inverse demand function $F_0 : \mathbb{R}^K \to \mathbb{R}^K$ assumed to be continuous and monotone decreasing ($F_0(z) \le F_0(x)$ when $z \ge x$ in the sense of partial ordering defined by $\mathbb{R}^K_+$) and such that $F_0(0) = Q$ and $F_0^m(Y'\mathbf{1}) > 0$ for $m = 1, \dots K$. The first condition means that in the absence of supply the prices are just the nominal prices while the second one shows even in the case of total sale the prices of illiquid assets remain strictly positive.

The clearing rules: each bank pays debts in accordance to the matrix of relative liabilities and sell illiquid assets if it has insufficient amount of cash. The result of clearing should be: all debts of the bank are covered or its equity falls down to zero.

In the case of several illiquid assets there is a problem how the banks chose their strategies of selling. In principle, one can imagine the situation that they have full freedom and, acting in the noncooperative way, drop down the market of illiquid assets because of an excessive supply. It seems reasonable that the central authority may impose extra rules on selling illiquid assets. We suppose that this is done by prescribing that the bank i must sell all assets in the same proportion α^i:

$$\alpha^i(q) = \frac{\left(\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j\right)^+}{\sum_k y^{ik} q^k} \wedge 1, \quad i = 1, \dots, N. \tag{4.6}$$

This formula means that for a fixed market price the bank does not sell illiquid assets if its cash reserve together with collected debts covers the liabilities. In the another extreme case where

$$\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j \ge \sum_k y^{ik} q^k = (Yq)^i$$

all illiquid assets have to be sold and the bank defaults. In the intermediate case the bank sells a share $\alpha^i \in]0,1[$ of the mth asset adding to its cash an extra amount

$$\frac{\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j}{\sum_k y^{ik} q^k} y^{im} q_m.$$

The total increase in cash allows to cover the liabilities.

Under such a rule the ith bank sells u^{im} units of the mth asset where

$$u^{im} := u^{im}(p,q) := \frac{y^{im}\left(\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j\right)^+}{\sum_k y^{ik} q^k} \wedge y^{im}.$$

The total supply of the illiquid assets is given by the vector $\mathbf{1}'U(p,q)$ where $U(p,q)$ is the matrix with entries given by the above formula.

Define the equilibrium vector $(p^*,q^*) \in [0,\tilde{L}] \times [F_0(1Y), Q]$ as the solution of the system of $N+K$ equations written in the matrix form as

$$p = (e + U(p,q)q + \Pi' p) \wedge \tilde{L}, \tag{4.7}$$

$$q = F_0(U'(p,q)\mathbf{1}). \tag{4.8}$$

The existence of the equilibrium is easy. Indeed, we check that

$$U'(p,q)\mathbf{1} \geq U'(\tilde{p},\tilde{q})\mathbf{1}, \qquad U(p,q)q + \Pi' p \leq U(\tilde{p},\tilde{q})\tilde{q} + \Pi'\tilde{p}$$

when $(\tilde{p},\tilde{q}) \geq (p,q)$. Denoting $F(p,q)$ the right-hand side of the first equation we obtain that $(p,q) \mapsto (F(p,q), F_0(U'(p,q))\mathbf{1})$ is a monotone mapping of the order interval $[0,\tilde{L}] \times [F_0(1Y), Q]$ into itself. Accordingly to Knaster–Tarski theorem the set of its fixed points is nonempty and contains the minimal and maximal elements $(\underline{p}^*, \underline{q}^*)$ and $(\bar{p}^*, \bar{q}^*)$.

For a fixed q the function $p \to F(p,q)$ is monotone. Thus, by the Knaster–Tarski theorem the set of solutions of the equation (4.7) is nonempty and contains, in particular, the maximal element $\bar{p}(q)$.

For any fixed $q \in [F_0(Y), Q]$ the largest solution $\bar{p} = \bar{p}(q)$ of (4.7) is given by formula:

$$\bar{p} = \sup\{p \in [0,\tilde{L}] : \; p \leq (e + U(p,q)q + \Pi' p) \wedge \tilde{L}\}$$

implying that $q \mapsto \bar{p}(q)$ is an increasing (and continuous) function on $[F_0(Y), Q]$. It follows that the supply function

$$q \mapsto \zeta(q) := U'(\bar{p}(q), q)\mathbf{1}$$

is decreasing and, therefore, the $q \mapsto F_0(\zeta(q))$ is an increasing (and continuous) mapping of the interval $[F_0(Y), Q]$ into itself and, therefore, it has the minimal and maximal fixed points we shall denote q_1 and q_2.

Lemma 4.1. *Suppose that the scalar function $x \to x'F_0(x)$ is strictly increasing on $[F_0(Y), Q]$. Then the solution of the equation $q = F_0(\zeta(q))$ is unique, i.e. $q_1 = q_2$.*

Proof. Arguing by contradiction, suppose that $q_1 \neq q_2$. Since $q_1 \leq q_2$ and $\zeta(.)$ is decreasing, $\zeta(q_1) \geq \zeta(q_2)$. Moreover, $\zeta(q_1) \neq \zeta(q_2)$ as the values of F_0 at these points are q_1 and q_2. The assumed strict monotonicity implies that

$$\zeta'(q_1)F_0(\zeta(q_1)) > \zeta'(q_2)F_0(\zeta(q_2)).$$

It follows that

$$\zeta'(q_1)q_1 > \zeta'(q_2)q_2.$$

To get a contradiction it is sufficient to show that

$$\Delta := \zeta'(q_2)q_2 - \zeta'(q_1)q_1 \geq 0.$$

Let $\bar{p}_k := \bar{p}(q_k)$ and let

$$D_k := \{i : \ (\tilde{L} - e - \Pi'\bar{p}(q_k))^i \geq (Yq_k)^i\},$$

i.e. D_k is the set of banks that are forced to sell all their illiquid assets for the price q_k, $k = 1, 2$. Since $\bar{p}(.)$ is increasing, $D_2 \subseteq D_1$. With the notation $\mathbf{1}'_A$ for the row-vector representing the indicator function of the subset $A \subseteq \{1, \dots, N\}$, we have, taking into account that $a^+ = a + a^-$, that

$$\zeta'(q_k)q_k \quad = \quad \mathbf{1}'_{D_k}Yq_k + \mathbf{1}'_{D_k^c}(\tilde{L} - e - \Pi'\bar{p}_k) + \mathbf{1}'_{D_k^c}(\tilde{L} - e - \Pi'\bar{p}_k)^-.$$

This formula leads to the representation

$$\begin{aligned}\Delta \quad = \quad & \mathbf{1}'_{D_2}Y(q_2 - q_1) - \mathbf{1}'_{D_1 \setminus D_2}Yq_1 - \mathbf{1}'_{D_1^c}\Pi'(\bar{p}_2 - \bar{p}_1) + \mathbf{1}'_{D_2^c \setminus D_1^c}(\tilde{L} - e - \Pi'\bar{p}_2) \\ & + \mathbf{1}'_{D_1^c}((\tilde{L} - e - \Pi'\bar{p}_2)^- - (\tilde{L} - e - \Pi'\bar{p}_1)^-) + \mathbf{1}'_{D_2^c \setminus D_1^c}(\tilde{L} - e - \Pi'\bar{p}_2)^-.\end{aligned}$$

Since the function $x \to x^-$ (on $\mathbf{R}^N$) is positive and decreasing, the last two terms in the right-hand side are positive. Regrouping the third and the forth terms we get that

$$\Delta \geq \mathbf{1}'_{D_2}Y(q_2 - q_1) - \mathbf{1}'_{D_1 \setminus D_2}q_1Y - \mathbf{1}'_{D_2^c}\Pi'(\bar{p}_2 - \bar{p}_1) + \mathbf{1}'_{D_1 \setminus D_2}(\tilde{L} - e - \Pi'\bar{p}_1). \tag{4.9}$$

From the equation (4.7) it follows that

$$\begin{aligned}\mathbf{1}'\Pi'(\bar{p}_2 - \bar{p}_1) \quad & = \quad \mathbf{1}'(\bar{p}_2 - \bar{p}_1) = \mathbf{1}'_{D_1}(\bar{p}_2 - \bar{p}_1) \\ & = \quad \mathbf{1}'_{D_2}(q_2u(\bar{p}_2, q_2) - q_1u(\bar{p}_1, q_1) + \Pi'(\bar{p}_2 - \bar{p}_1)) \\ & \qquad + \mathbf{1}'_{D_1 \setminus D_2}(\tilde{L} - (e + q_1u(\bar{p}_1, q_1) + \Pi'\bar{p}_1)).\end{aligned}$$

implying that

$$\begin{aligned}\mathbf{1}'_{D_2^c}\Pi'(\bar{p}_2 - \bar{p}_1) \quad = \quad & \mathbf{1}'_{D_2}(U(\bar{p}_2, q_2)q_2 - U(\bar{p}_1, q_1)q_1) - \mathbf{1}'_{D_1 \setminus D_2}U(\bar{p}_1, q_1)q_1 \\ & + \mathbf{1}'_{D_1 \setminus D_2}(\tilde{L} - e - \Pi'\bar{p}_1).\end{aligned}$$

Substituting this expression in (4.9), we have

$$\begin{aligned}\Delta \quad \geq \quad & \mathbf{1}'_{D_2}Y(q_2 - q_1) - \mathbf{1}'_{D_1 \setminus D_2}Yq_1 \\ & - \mathbf{1}'_{D_2}(U(\bar{p}_2, q_2)q_2 - U(\bar{p}_1, q_1)q_1) + \mathbf{1}'_{D_1 \setminus D_2}q_1u(\bar{p}_1, q_1) = 0\end{aligned}$$

since the cash increment $(U(\bar{p}_2, q_2)q_2)^i = (Yq)^i$ for the bank $i \in D_2$ and $(U(\bar{p}_1, q_1)q_1)^i = (Yq_1)^i$ for $i \in D_1 \supseteq D_2$. $\Box$

Theorem 4.2. *Suppose that the scalar function $x \to x' F_0(x)$ is strictly increasing on $[F_0(Y), Q]$. Then there is q^* such that the set of solutions of the system (4.7), (4.8) is contained in the interval with the extremities $(\underline{p}(q^*), q^*)$ and $(\bar{p}(q^*), q^*)$. In particular, if for each q the solution of (4.7) is a unique, then the solution of the system is also unique.*

Proof. Let Γ be the set of q for which (p, q) is a solution of the system (4.7), (4.8). If $q^* \in \Gamma$, then $(\bar{p}(q^*), q^*)$ is the solution of (4.7), (4.8). Accordingly to the above lemma the point q^* is uniquely defined. This implies the result. $\Box$

Note that the uniqueness of the solution of (4.7) is guarantied if for each i the orbit of i contains an element with positive cash reserve.

Remark. In the paper [1] it was considered a model coinciding with studied above in the case of a single illiquid asset. The difference is that in the cited paper the equilibrium is defined as a vector (p, q) satisfying the system of equations

$$p = \big(e + qy + \Pi' p\big)^+ \wedge \tilde{L}, \tag{4.10}$$

$$q = F_0(\mathbf{1}'((q^{-1}(\tilde{L} - e - \Pi' p)^+) \wedge y)). \tag{4.11}$$

To our opinion, the definition of the equilibrium given by the system (4.7), (4.8), which is in the one liquid asset case has the form

$$p = \big(e + (\tilde{L} - e - \Pi' p)^+ \wedge (qy) + \Pi' p\big) \wedge \tilde{L}, \tag{4.12}$$

$$q = F_0(\mathbf{1}'((q^{-1}(\tilde{L} - e - \Pi' p)^+) \wedge y)), \tag{4.13}$$

is more natural. In fact, the right-hand sides of (4.10) and (4.12) as functions $R_1(p, q)$ and $R_2(p, q)$ defined on $[0, \tilde{L}] \times [F_0(1Y), Q]$ coincide. To see this, fix i and consider the three possible cases.

1. Let $e^i + qy + (\Pi' p)^i \leq \tilde{L}^i$. Then the expressions for $R_1^i(p, q)$ and $R_2^i(p, q)$ have the same form $e^i + qy + (\Pi' p)^i$.

2. Let $e^i + qy + (\Pi' p)^i > \tilde{L}^i$ and $\tilde{L}^i - e^i - (\Pi' p)^i \geq 0$. Then the values $R_1^i(p, q)$ and $R_2^i(p, q)$ are equal to $\tilde{L}^i$.

3. Let $e^i + qy + (\Pi' p)^i > \tilde{L}^i$ and $\tilde{L}^i - e^i - (\Pi' p)^i < 0$. Then the value of $R_1^i(p, q)$ is $\tilde{L}^i$ and the value of $R_1^2(p, q)$ is $(e^i + (\Pi' p)^i) \wedge \tilde{L}^i = \tilde{L}^i$.

5. Appendix. Knaster–Tarski fixpoint theorem

Let X be a set with a partial ordering $\geq$ and let A be its nonempty subset. By definition, $\sup A$ is an element $\bar{x}$ such that $\bar{x} \geq x$ for all $x \in A$ and if $\bar{x}'$ is such that $\bar{x}' \geq x$ for all $x \in A$ then $\bar{x}' \geq \bar{x}$. The definition of $\inf A$ follows the same pattern but with the dual ordering $\leq$. A partially ordered set X is *complete lattice* if for any its nonempty subset A there exist $\inf A$ and $\sup A$.

Theorem 5.1. *Let X be a complete lattice and let $f : X \mapsto X$ be an order-preserving mapping, $L := \{x : \ f(x) \leq x\}$, $U := \{x : \ f(x) \geq x\}$. The set $L \cap U$ of fixed points of f is non-empty and has the smallest and the largest fixed points which are, respectively, $\underline{x} := \inf L$ and $\bar{x} := \sup U$.*

Proof. Note that $L \neq \emptyset$ since it contains the element $\sup X$. Take arbitrary $x \in L$. Then $\underline{x} \leq x$ implying that $f(\underline{x}) \leq f(x) \leq x$. Thus, $f(\underline{x}) \leq \underline{x}$ as $\underline{x}$ is $\inf L$. So, $\underline{x} \in L$. Since $f(L) \subseteq L$, also $f(\underline{x}) \in L$, hence, $\underline{x} \leq f(\underline{x})$, i.e. $\underline{x} = f(\underline{x})$. All fixed points belong to L and, therefore, $\underline{x}$ is the smallest one.

The proof of the statement for the largest fixed point is analogous. □

Corollary 5.2. *Let $f(.;y)$ be an order-preserving mapping of a complete lattice $(X, \geq)$ into itself, depending on the parameter y from a partially ordered set $(Y, \succeq)$. Suppose that $f(., y)$ is increasing in y, that is $f(x, y') \geq f(x, y)$ for all $x \in X$ when $y' \succeq y$. Then the smallest and largest fixed points are also increasing in y.*

Proof. The claim is obvious because the sets $L(y) := \{x : \ f(x, y) \leq x\}$ are decreasing and the sets $U(y) := \{x : \ f(x, y) \geq x\}$ are increasing in y, see [5].

These general results are applied to the order intervals $[a, b] \subset \mathbb{R}^d$ with the ordering induced by $\mathbb{R}^d_+$.

References

[1] Amini H., Filipovic D., Minca A. (2014) To fully net or not to net: adverse effects of partial multilateral netting. *Swiss Finance Institute Research Paper Series*, N. 14-63. Forthcoming in *Operations Research.*

[2] Cont R., Wagalath L. (2015) Fire sale forensics: measuring endogenous risk. *Mathematical Finance*, doi: 10.1111/mafi.12071.

[3] Eisenberg L., Noe T.H. (2001) Systemic risk in financial systems. *Management Science*, 47(2):236–249.

[4] Elsinger H. (2009) Financial networks, cross holdings, and limited liability. Working paper from Oesterreichische Nationalbank.

[5] Milgrom J., Roberts J. (1994) Comparing equilibria. *American Econ. Rev.*, 84:441–454.

[6] Suzuki T. (2002) Valuing corporate debt: the effect of cross-holdings of stock and debt. J. Operations Research Society of Japan 45(2), 123-144.

[7] Tarski A. (1955) A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics,* 5(2):285–309.

О ЕДИНСТВЕННОСtИ КЛИРИНГОВЫХ ВЕКТОРОВ РЕДУЦИРУЮЩИХ СИСТЕМНЫЙ РИСК

Юрий Кабанов, Рита Мокбель, Халил Эль Битар

Резюме

В финансовых системах, т.е. в сети взаимосвязанных банков, процедура взаимозачёта или клиринга состоит в одновременной выплате задолженностей с целью уменьшения общего их суммы в системе. Вектор, компоненты которого есть суммарные выплаты каждого банка системы, называется клиринговым вектором. В простых моделях, предложенных Айзенбергом и Ноэ (2001) и независимо, Судзуки (2002) было показано, что полный клиринг описывается вектором, который является неподвижной точкой некоторого отображения. Существование клирингового вектора может быть получено прямой ссылкой на теоремы о неподвижной точке Кнастера–Тарского или Брауэра. Вопрос о его единственности является более деликатным. Айзенберг и Ноэ получили достаточное условие единственности в терминах графа связей финансовой системы. В настоящей работе мы доказываем единственность для двух блее общих моделей: модели Эльсингера с приоритетами долгов и модели типа Амини–Филиповича–Минки, в которой банки имеют неликвидные активы, продажа которых влияет на их рыночную цену.

Ключевые слова: системный риск, финасовые сети, клиринг, теорема Кнастера-Тарского, модель Айзеберга-Ноэ, приоритет финансовых обязательств, влияние на ценообразование.

Khalil El Bitar: Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray,, 25030 Besançon, cedex, France

E-mail address: khalilbitar_aw@hotmail.com

Yuri Kabanov: Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray,, 25030 Besançon, cedex, France, and Institute of Informatics Problems, Federal Research Center “Computer Science and Control” of Russian Academy of Sciences, Moscow, Russia, and National Researche University "MPEI", Moscow, Russia

E-mail address: Youri.Kabanov@univ-fcomte.fr

Rita Mokbel: Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray,, 25030 Besançon, cedex, France

E-mail address: ritamokbel@hotmail.com