Noname manuscript No.
(will be inserted by the editor)

Clearing in Financial Networks

Khalil El Bitar - Yuri Kabanov - Rita Mokbel

Received: date / Accepted: date

Abstract This paper is a survey of recent results on the clearing in financial systems. Mathematically, the principal questions of the reviewed studies are on the existence and uniqueness of solutions of specific nonlinear equations $x = f(x)$ where $f : \mathbf{R}^d \to \mathbf{R}^d$ is a mapping defined via stochastic and substochastic matrices. Some algorithms of calculations of fixed points are discussed.

Keywords Systemic Risk · Financial Networks · Clearing · Knaster–Tarski theorem · Seniority

Mathematics Subject Classification (2000) 90B10, 90B50

JEL Classification G21 · G33

1 Inroduction

To explain the clearing problem we start with the simplest example of a financial system with two agents each of them having in cash one dollar. The first agent got from the second a credit 1000 dollars. By circumstances, the second agent needs a cash and borrows from the first agent 900 dollars in credit. As a result, both agents have large liabilities with respect to each other. This means that in this system circulates money much larger than the proper capital of agents. Potentially, if one of them, by reasons not described in the model, fails to pay its debt, the creditor will suffer huge losses. That is why the regulators are motivated to force the agents to clear their positions by reimbursing the credits, fully or partially, diminishing in this way eventual consequences of defaults. For the complex financial systems involving large numbers of agents, with chains of borrowing, the clearing problem, that is the reduction of absolute values of credits by reimbursement, is more complicated.

Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray,
25030 Besançon, cedex, France, and
Steklov Mathematical Institute of RAS, Moscow, Russia
E-mail: ykabanov@univ-fcomte.fr

In the influential paper [5] published in 2001, Eisenberg and Noe suggested a clearing procedure in a simple static model describing a financial system composed of N banks (under "banks" one can understand various financial institutions). The assets of each bank are of two types: the cash (or any external asset) and the credits provided to other banks of the system, called exposures; they are, in turn, the liabilities for its debtors. The clearing means simultaneous repaying all debts. Each bank returns to its counterparties the debts *pro rata* of their relative volume using its cash reserve and obtained repayments of the credited banks. The rule is: for each bank either all debts are paid in full or the zero level of equity is attained; in the latter case the bank defaults. The totals reimbursed by the banks form an N-dimensional *clearing vector*. It was shown that clearing vectors must satisfy a nonlinear equation involving a stochastic matrix with rows formed by the fractions of the total liabilities of each bank to its creditors. The key observation is that this equation is a fixpoint problem for a monotone mapping f of a closed N-dimensional interval into itself. The existence of such points follows immediately from the Knaster–Tarski theorem, a beautiful and fairy simple result which proof needs only a few lines of arguments. The uniqueness of the clearing vector is a more delicate result. An elegant proof in [5] involves the graph structure of the system built in the same way as is usually done in the theory of Markov chains.

A more general model with crossholdings was developed by Suzuki, [16], independently and at the same time, but unfortunately, being published in a rarely available journal, until recently it was rarely cited.

The ideas of the Eisenberg–Noe paper happened to be very fruitful and the model and methods used to its analysis were generalized in many directions having not only financial importance but posing an interesting mathematical questions in more sophisticated situations: on the uniqueness of clearing vector, on efficient numerical methods of its calculation, and so on. Nowadays the clearing problem became an important chapter of the new mathematical discipline — theory of systemic risk.

The algorithm of calculating the largest clearing vector in a finite number of steps, extending one suggested in [5], was investigated in a model with default losses by Rogers and Luitgard Veraart, [13]. Mathematically, the model is a minor generalization but it is interesting from the point of view of applications: it provides a natural framework to investigate problems of merging and rescue (bailout) of defaulting banks.

In the deep study [7] Elsinger considered a model combining crossholdings, in the same spirit as in [16], with seniorities of liabilities. To avoid cumbersome formulae we present both generalization separately.

In the real world banks protect themselves from the default risk of their debtors using such financial instruments as credit default swaps (CDS), i.e. by buying from third parties contingent claims with pay-offs covering losses if the debtor is insolvent. Mathematically, this means that the liability matrices may depend on the clearing vector. Model of this type was analyzed by Fisher in [9] on the basis of ideas from [16].

A specific class of models deals with situations where banks, besides the cash and liabilities, may have one or more illiquid assets whose selling might influence the market prices, see [2] (one-asset model) and [6] (multi-asset model).

In this review we try to present in a unified way the essential mathematical content of the aforementioned papers with detailed proofs. In Section 2 we discuss briefly the general principles and results of the seminal Eisenberg–Noe paper and explain the adaptation of the classical Gauss elimination algorithm to non-linear fixpoint problems due to Sonin, [15].

Section 3 deals with the Rogers–Veraart model and contains the algorithm of computing the largest clearing vector in finite number of steps. In Section 4 we present results for the Suzuki–Elsinger model with cross-holdings. Section 5 contains results on the Elsinger model with seniority of debts. Existence and uniqueness of clearing vectors in the Fischer model are given in Section 6. Models with illiquid assets and price impact are discussed in Section 7 in the multi-asset setting generalizing the models of Amini–Filipovic–Minca [2] and aslo Feinstein [8]. For the reader convenience we conclude by Appendix with a short information about the Knaster–Tarski theorem adapted to our needs.

Notations. We denote by $\geq$ the partial ordering in $\mathbf{R}^n$ and its subsets induced by the cone $\mathbf{R}^n_+$. In other words, the inequality $y \geq x$ is understood componentwise. Also the symbols $x \wedge y$ and $x \vee y$ mean, respectively, the componentwise minimum and maximum, $x^+ := x \vee 0$, $x^- := (-x)^+$. The notation $[x, z]$ is used for the order interval, i.e. $[x, z] := \{y \in \mathbf{R}^n : \ x \leq y \leq z\}$. If $A \subseteq [x, z]$, then $\inf A$ is the unique element $\underline{y} \in [x, z]$ such that $\underline{y} \leq y$ for all $y \in A$ and for any $\tilde{y}$ such that $\tilde{y} \leq y$ for all $y \in A$ we have that $\tilde{y} \leq \underline{y}$. That is, the component $\underline{y}^i = \inf\{y^i : \ y \in A\}$ for $i \in \{1, \dots, n\}$.

We use the matrix notations where the vectors are columns, $'$ is the symbol of transpose, $\mathbf{1}' := (1, \dots, 1)$ (the dimension of the vector is supposed to be clear from the context). If $D \subset \{1, \dots, n\}$, then $\mathbf{1}_D$ is the vector with the ith component equal to 1 if $i \in D$ and 0 otherwise. The diagonal matrix $\Lambda_D := \operatorname{diag} \mathbf{1}_D$ in the matrix notations is a substitute for the indicator function when vectors from $\mathbf{R}^n$ are interpreted as a function on $\{1, \dots, n\}$. Symbols $|.|_1$ and $|.|_\infty$ denote l^1-norm and l^∞-norm, respectively.

2 The Eisenberg–Noe model

2.1 The model description and existence of clearing vectors

In the paper [5], Eisenberg and Noe investigated a financial system composed of N banks represented by the points of the set $\mathcal{N} := \{1, \dots, N\}$. The model is given by the pair (e, L) where $e \in \mathbf{R}^N_+$ is interpreted as a vector of cash reserves (or external assets) and the $N \times N$-matrix L with entries $l^{ij} \geq 0$ and the zero diagonal describes the interbank liabilities: l^{ij} is the value borrowed by the bank i from the bank j. The transpose L' is called the exposure matrix. The vector $l = L\mathbf{1}$ with $l^i = \sum_j l^{ij}$ describes the total interbank liabilities. Respectively, the vector $x := L'\mathbf{1}$ describes the total exposures of the banks.

The equity (or the bank value) is the quantity

$$c^i := (e^i + x^i - l^i)^+.$$

Note that this quantity provides a limited information on the situation: the value $c^i = 0$ does not show how deeply the bank i is in trouble.

Put

$$\pi^{ij} := \frac{l^{ij}}{l^i} = \frac{l^{ij}}{\sum_j l^{ij}}, \quad \text{if } \ l^i \neq 0, \ \text{and } \pi^{ij} := \delta^{ij} \text{ otherwise}, \tag{2.1}$$

where the Kronecker symbol $\delta^{ij} = 0$ for $i \neq j$ and $\delta^{ii} = 1$. Then π^{ij} describes the fraction of the value of the debtor i due to the creditor j of the total interbank debt of i; $\Pi = (\pi^{ij})$ is called relative liabilities matrix. The value $\pi^{ii} = 1$ means that the bank i has no interbank debts.

In general, a financial system may have a complicated structure, with cyclical interdependences and banks having large exposures within cycles. To reduce them, the authors of [5] suggested a clearing mechanism satisfying several natural requirements: limited liability, absolute priority, and proportionality. Formally, this leads to the concept of *clearing payment vector* $p \in [0, l] = \prod_i [0, l^i]$ satisfying the following properties:

a. Limiting liability. For every i,

$$p^i \le e^i + \sum_j \pi^{ji} p^j.$$

b. Absolute priority. For every i, either $p^i = l^i$, or

$$p^i = e^i + \sum_j \pi^{ji} p^j.$$

One may think that the central clearing authority forces each bank to make a "fair" repayment of debts in such a way that, having the total payment $p^i \le l^i$, the bank i pays to j the fraction $p^i \pi^{ij}$ in such a way that either its total debts are paid ($p^i = l^i$) or all the resources are exhausted ($p^i < l^i$).

Alternatively, the condition $a.$ and $b.$ can be written in the following way:

$$p = (e + \Pi' p) \wedge l \tag{2.2}$$

where the minimum is understood in the componentwise sense, i.e. accordingly to the partial ordering defined by the cone $\mathbf{R}^N_+$.

Remark. An attentive reader may observe a striking resemblance of (2.2) with the Bellman equation

$$v = (e + \Pi v) \vee l \tag{2.3}$$

whose minimal solution (the smallest $(1, -e)$-excessive majorant in the terminology of Shiryaev) is the payoff function

$$v(x) = \sup_\tau E_x \Big(l(X_\tau) + \sum_{s=0}^{\tau-1} e(X_s) \Big)$$

in the optimal stopping problem with cost of observations where X is the discrete-time Markov process with values in $\mathcal{N}$, transition matrix Π, and initial state x, see [14], Section 2.14.

The existence of p solving the equation (2.2) is obvious in view of the classical Brouwer fixpoint theorem: $f : p \mapsto (e + \Pi' p) \wedge l$ is a continuous mapping of the convex compact set $[0, l]$ into itself. The important contribution of Eisenberg and Noe is the observation that the set of clearing vectors is not only non-empty but it has the minimal and the maximal clearing vectors $\underline{p}$ and $\bar{p}$, respectively. This assertion follows immediately from the Knaster–Tarski fixpoint theorem (much simpler than the Brouwer one): the monotone mapping $f : p \mapsto (e + \Pi' p) \wedge l$ of the complete lattice $[0, l]$ into itself has the largest and the smallest fixed points, see Section 8 for details.

Using the identity $(x - y)^+ = x - x \wedge y$ we can rewrite the equation (2.2) in the following equivalent form:

$$(e + \Pi' p - l)^+ = e + \Pi' p - p, \tag{2.4}$$

where the left-hand side is the equity vector $c(p)$ of the system after clearing.

Lemma 2.1 *The equity after clearing does not depend on the clearing vector.*

Proof. As Π is a stochastic matrix, $\mathbf{1}'\Pi' = \mathbf{1}'$. Therefore, multiplying the above representation (2.4) from the left by $\mathbf{1}'$ we get that

$$\mathbf{1}'(e + \Pi' p - l)^+ = \mathbf{1}'e,$$

i.e., the sum of equities is equal to the sum of the initial cash reserves, whatever is the clearing vector. On the other hand, by monotonicity, we have that

$$(e + \Pi' p - l)^+ \le (e + \Pi' \bar{p} - l)^+.$$

If the both side here are not equal, then $\mathbf{1}'(e + \Pi' p - l)^+ < \mathbf{1}'(e + \Pi' \bar{p} - l)^+$ in contradiction with the invariance of the total of equities. □

Since $c(p)$ does not depend on p, we shall use c to denote the equity vector after clearing.

Remark. Unlike our definition (2.1), the authors of [5] used the convention $0/0 = 0$ when $l^i = 0$. In the proofs it is essential that the matrix Π is stochastic and they immediately imposed the additional assumption that all $l^i > 0$, excluding in this way the case where some banks may provide credits without borrowing within the system. Another specific feature of the Eisenberg–Noe is the assumption that the cash reserves $e^i \ge 0$ which also can be avoided. The negative value of e^i means that the bank has external debt and the amount $|e^i|$ appears in the liability side of the balance sheet.

2.2 Sufficient condition for the uniqueness of the clearing vector

As in the theory of Markov chains we associate with the stochastic matrix Π the structure of directed graph on $\mathcal{N}$ by relating with each pair i, j with $\pi^{ij} > 0$ the arrow $i \to j$ denoting, in the considered context, that j is the creditor of i if $i \ne j$. The arrow $i \to i$ corresponds to the case where $l^i = 0$ and means that the bank i has no liabilities to the banks in the system; in the theory of Markov chains such i is an absorbing state. In the language of graph theory the banks are the nodes of $\mathcal{N}$.

We denote by $o(i)$ *the orbit of* i defined as the set of all $j \in \mathcal{N} \setminus \{i\}$ for which there is a directed path $i \to i_1 \to i_2 \to \ldots \to j$. So, if $o(i)$ is not empty, it is the set of all direct or indirect creditors of i.

Note that $k \notin o(i)$ if and only if $\pi^{jk} = 0$ for all $j \in o(i)$.

Lemma 2.2 *Let the market be cleared by a vector $p \in [0, l]$. Let $I = o(i)$ be an orbit such that $\mathbf{1}'_I e > 0$. Then $\mathbf{1}'_I c > 0$. That is, if an orbit has a node with a strictly positive cash endowment, then this orbit after clearing has a node with a strictly positive equity value.*

Proof. Multiplying the identity (2.4) from the left by $\mathbf{1}'_I$ and noticing that $(\mathbf{1}'_I \Pi')^j = 1$ for $j \in I$, we obtain that

$$\mathbf{1}'_I c = \mathbf{1}'_I (e + \Pi' p - l)^+ = \mathbf{1}'_I (e + \Pi' p - p) \ge \mathbf{1}'_I e > 0$$

implying the claim. □

A financial system is called *regular* if $\mathbf{1}'_{o(i)} e > 0$ for every orbit $o(i) \ne \emptyset$.

Note that if $o(i)$ is empty, i.e. $l^i = 0$, then $\bar{p}^i = 0$. This property follows from the relation (2.4) whose ith component in this case is

$$e^i + (\Pi' \bar{p})^i = e^i + (\Pi' \bar{p})^i - \bar{p}^i.$$

Theorem 2.3 *Suppose that the financial system is regular. Then $\underline{p} = \bar{p}$.*

Proof. Suppose that $\underline{p}$ and $\bar{p}$ are not equal, i.e. $\underline{p} \le \bar{p}$ but for some i we have the strict inequality $\underline{p}^i < \bar{p}^i$ and, therefore, $o(i) \neq \emptyset$.

By the above lemma, there exists $m \in o(i)$ with the equity value $c^m > 0$. Let us consider a path $i \to i_1 \to \ldots \to m$ assuming without loss of generality m is the first node with strictly positive equity. If $m = i_1$, a contradiction is immediate: since $c^m > 0$, we have the representation

$$c^m = e^m + \sum_j \pi^{jm} \underline{p}^j - l^m, \qquad c^m = e^m + \sum_j \pi^{jm} \bar{p}^j - l^m$$

implying the impossible equality

$$\sum_j \pi^{jm} (\bar{p}^j - \underline{p}^j) = 0$$

(the sum includes is a strictly positive term corresponding to $j = i$).

Suppose that $m \neq i_1$. Thus, $c^{i_1} = 0$ and, in virtue of (2.4),

$$e^{i_1} + \sum_j \pi^{j i_1} \underline{p}^j - \underline{p}^{i_1} = 0, \qquad e^{i_1} + \sum_j \pi^{j i_1} \bar{p}^j - \bar{p}^{i_1} = 0,$$

Thus,

$$\bar{p}^{i_1} - \underline{p}^{i_1} = \sum_j \pi^{j i_1} (\bar{p}^j - \underline{p}^j) > 0.$$

The property that $\underline{p}^i < \bar{p}^i$ propagates along the path and we have an obvious reduction to the considered one-step case. □

Remark. The above theorem reveals that the problem to find a clearing vector is ill-posed. Indeed, adding an infinitesimally small amount $\varepsilon > 0$ (say, one cent) to the initial endowments leads to a unique clearing vector. Similar effect will have a small increase in liabilities.

The above proof is rather straightforward and uses graph-theoretical language. An alternative one, based on spectral properties of stochastic matrices, can be found in [1]. The theorem can be formulated in the equivalent form: if any connected component of the graph which is not a singleton contains a node with strictly positive cash endowment, then the clearing vector is unique.

2.3 The Gauss elimination algorithm

Let $J \neq \emptyset$ be a proper subset of $\mathcal{N}$. Changing the numbering we may assume without loss of generality that $J := \{1, \ldots, m\}$, $1 \le m < N$. We introduce the notations $\Pi_J := (\pi^{ij})_{i,j \in J}$, $\Pi_{J^c} = (\pi^{ij})_{i,j \in J^c}$, $e_D := (e^i)_{i \in J}$, etc. Supposing that p solves the equation $p = e + \Pi' p$, we rewrite the latter in the form

$$\begin{pmatrix} p_J \\ p_{J^c} \end{pmatrix} = \begin{pmatrix} e_J \\ e_{J^c} \end{pmatrix} + \begin{pmatrix} \Pi'_J & R' \\ T' & \Pi'_{J^c} \end{pmatrix} \begin{pmatrix} p_J \\ p_{J^c} \end{pmatrix}. \tag{2.5}$$

Thus, we have that

$$p_J = e_J + \Pi_J' p_J + R' p_{J^c}, \tag{2.6}$$
$$p_{J^c} = e_{J^c} + T' p_J + \Pi_{J^c}' p_{J^c}. \tag{2.7}$$

Suppose that the matrix $I_m - \Pi_J$ is invertible. Substituting in (2.7) the expression for p_J from (2.6) we obtain that the vector $p_1 := p_{J^c} \in \mathbf{R}^{N-m}$ solves the equation

$$p_1 = e_1 + \Pi_1' p_1, \tag{2.8}$$

where

$$e_1 := e_{J^c} + T'(I_m - \Pi_J')^{-1} e_J, \tag{2.9}$$
$$\Pi_1 := R(I_m - \Pi_J)^{-1} T + \Pi_{J^c}. \tag{2.10}$$

It is easily seen that Π_1 is a stochastic matrix (substochastic if we start with the substochastic matrix Π). The equation (2.8) is of the same type as the initial one but of a lower dimension and its solution, via (2.6), gives us the solution of the former. Of course, for $m = 1$ the reduction to a lower dimension described above is nothing but the Gauss elimination algorithm for solving linear equation in $\mathbf{R}^N$.

As was observed by Sonin in a different context, namely, of the Bellman equations arising in the optimal stopping theory, see [15] and references therein, the elimination algorithm can be modified to solve the fixpoint problem of the type $p = (e + \Pi' p) \wedge l$, even with an arbitrary positive matrix. Indeed, if the set of indices

$$D_0 := \{i \in \mathcal{N} : \ e^i + (\Pi' l)^i < l^i\} = \emptyset,$$

then the solution is $p = l$. If this set is non-empty, take its subset $J \neq \emptyset$. Without loss of generality we may assume that $J = \{1, \dots, m\}$. In an analogy with (2.6), (2.7) we get that

$$p_J = (e_J + \Pi_J' p_J + R' p_{J^c}) \wedge l_J, \tag{2.11}$$
$$p_{J^c} = (e_{J^c} + T' p_J + \Pi_{J^c}' p_{J^c}) \wedge l_{J^c}. \tag{2.12}$$

By definition of J the first equation is linear: it is the same as (2.6). Thus, if the matrix $I_m - \Pi_J$ is invertible, then $p_1 := p_{J^c}$ solves the equation of lower dimension

$$p_1 = (e_1 + \Pi_1' p_1) \wedge l_{J^c}, \tag{2.13}$$

with e_1 and Π_1 given by (2.9) and (2.10).

As in the classical Gauss algorithm, we take $J = \{1\}$. If $\pi^{11} \neq 1$, we can eliminate p^1 reducing the problem to the search of the vector $(p^2, \dots, p^N)$ satisfying the equation of the same type. If $\pi^{11} = 1$, then $e^1 = 0$ and $R' p_{J^c} = 0$. This means that p^1 is not determined by the first equation and it can be taken as a free parameter $p^1 \in [0, l^1]$. In particular, it has to be taken equal to l^1 in the case of search of the maximal solution. The problem is reduced to the same problem, in the dimension $N-1$, when R' has all components equal to zero, and even in the dimension $N-1-k$ when R' has k nonzero components. Indeed, for the latter the corresponding components of the solution must be equal to zero.

There are other procedures of calculations of the clearing vectors. The vector $\bar{p}$ can be obtained by the iterative procedure $p_n = f(p_{n-1})$, $n \geq 1$, starting from $p_0 = l$. Indeed, since f is monotone we get easily that $\bar{p} \leq p_{n+1} \leq p_n$; the decreasing bounded sequence p_n has a limit point $p_\infty \in [\bar{p}, l]$. The continuity of f implies that $p_\infty = f(p_\infty)$. Since $\bar{p}$ is the largest fixed point, $p_\infty = \bar{p}$. The same procedure but starting from the zero vector provides a sequence converging to $\underline{p}$. Of course, to reach the limit this procedure needs, in general, infinite number of iterations. Similarly to the Gauss elimination algorithm, the Fictitious Default algorithm suggested in [5] attains the clearing vector (supposed to be unique) at $N+1$ steps at most. In the next section we describe this algorithm in a slightly more general framework. It is interesting that an analogous algorithm was discovered independently in the theory of optimal stopping, see [12].

3 The Rogers–Veraart model

3.1 Default losses and GA algorithm

In the Rogers–Veraart model, [13], which is an extension of the Eisenberg–Noe model, the clearing vectors are solutions of the following non-linear equation:

$$p = (I - \Lambda(p))l + \Lambda(p)(\alpha e + \beta \Pi' p), \tag{3.1}$$

where $e \in \mathbf{R}^N_+$ and $\Lambda(p) := \operatorname{diag} \mathbf{1}_D$ with $D := \{i \in \mathcal{N}\colon\ e^i + (\Pi' p)^i < l^i\}$. The parameters $\alpha, \beta \in]0,1]$ express the default losses: one can think that if the ith bank fails, the amount $(1-\alpha)e^i + (1-\beta)(\Pi' p)^i$ is used to cover the liquidation expenditures. Thus, the model is given by the quadruple (e, L, α, β). The Eisenberg–Noe model corresponds to the case $\alpha = \beta = 1$.

Let us denote by $f(p)$ the right-hand side of (3.1). Apparently, $p \mapsto f(p)$ is an increasing function of $[0, l]$ into $[0, l]$. Thus, the Knaster–Tarski theorem ensures that there are the smallest $\underline{p}$ and largest $\bar{p}$ clearing vectors. Also, f is continuous from above, i.e. $f(p_n) \to f(p_\infty)$ when a decreasing sequence of p_n converges to p_∞, and the procedure described at the end of previous section leads to $\bar{p}$ when it starts from l. If $\alpha < 1$ or $\beta < 1$, then the continuity of the function $p \mapsto f(p)$ in general fails and we cannot guaranty the convergence for the iterations starting from the zero vector.

Now we describe a procedure called in [13] the Greatest Clearing Vector Algorithm (GA). It is the recursively defined sequence $p_n \in \mathbf{R}^N_+$, $n \geq 0$, with the initial value $p_0 := l$ and the general term

$$p_{n+1} := (I - \Lambda_n)l + \Lambda_n \hat{p}_{n+1}, \qquad n \geq 0, \tag{3.2}$$

where $\Lambda_n := \operatorname{diag} \mathbf{1}_{D_n}$, $D_n := \{i \in \mathcal{N}\colon\ e^i + (\Pi' p_n)^i < l^i\}$, and $\hat{p}_{n+1}$ is the maximal solution from those laying in $[0, \Lambda_n p_n]$ of the linear equation $p = f_n(p)$,

$$f_n(p) := \Lambda_n\big(\alpha e + \beta \Pi'(I - \Lambda_n)l + \beta \Pi' \Lambda_n p\big). \tag{3.3}$$

Let us check, using the monotonicity of $f_n(p)$ on $\mathbf{R}^N_+$, that this sequence is well-defined and decreasing. To this aim observe that

$$f_0(\Lambda_0 l) = \Lambda_0(\alpha e + \beta \Pi' l) \leq \Lambda_0(e + \Pi' l) \leq \Lambda_0 l.$$

So, the increasing function f_0 maps the set $[0, \Lambda_0 p_0]$ into itself. By the Knaster–Tarski theorem the required maximal fixed point $\hat{p}_1$ exists. Obviously, $p_1 \leq l = p_0$.

Suppose that $p_k \in [0,l]$ for $k \le n$ are already determined (so, $\hat p_k$ are also determined) and form a decreasing sequence. Thus, $D_n \supseteq D_{n-1}$. Put $\Delta_n := \operatorname{diag} \mathbf{1}_{D_n \setminus D_{n-1}}$. Taking into account that $\Delta_n l = \Delta_n p_n$, $\Lambda_{n-1} p_n = \Lambda_{n-1} \hat p_n$, and $p_n \le l$, we get that

$$\begin{aligned} f_n(\Lambda_n p_n) = {} & \Lambda_{n-1}\big(\alpha e + \beta \Pi'(I-\Lambda_{n-1})l + \beta\Pi'\Lambda_{n-1}p_n\big) \\ & + \Delta_n\big(\alpha e + \beta \Pi'(I-\Lambda_{n-1})l + \beta\Pi'\Lambda_{n-1}p_n\big) \le \Lambda_{n-1}\hat p_n + \Delta_n l = \Lambda_n p_n. \end{aligned}$$

Thus, the monotone function f_n maps the set $[0,\Lambda_n p_n]$ into itself and the existence of the required maximal element $\hat p_{n+1} \le \Lambda_n p_n$ is ensured by the Knaster–Tarski theorem. This means that the recursive sequence is well-defined. It follows from (3.2) that

$$p_{n+1} - p_n = -\Delta_n l + \Lambda_n \hat p_{n+1} - \Lambda_{n-1} p_n \le -\Delta_n l + \Lambda_n p_n - \Lambda_{n-1} p_n = \Delta_n(p_n - l) \le 0.$$

Proposition 3.1 *There exists $n_0 \le N+1$ such that $p_n = \bar p$ for all $n \ge n_0 - 1$.*

Proof. For the maximal solution $\bar p$ of (3.1) we have, trivially, $\bar p \le p_0 = l$. Suppose that we already established that $\bar p \le p_n$. By definition, $\Lambda_n p_{n+1} = \Lambda_n \hat p_{n+1} = \hat p_{n+1}$ where $\hat p_{n+1} \in [0, \Lambda_n p_n]$ is the maximal solution of the equation

$$\hat p_{n+1} = \Lambda_n\big(\alpha e + \beta\Pi'(I-\Lambda_n)l + \beta\Pi'\Lambda_n \hat p_{n+1}\big).$$

Due to the induction hypothesis $\Lambda(\bar p) \ge \Lambda_n$ and we obtain from (3.1) that

$$\Lambda_n \bar p = \Lambda_n(\alpha e + \beta\Pi'(I-\Lambda_n)\bar p + \beta\Pi'\Lambda_n\bar p).$$

Since $\bar p \le l$, Corollary 8.2 ensures that $\hat p_{n+1} \ge \Lambda_n \bar p$. Thus, $p^i_{n+1} \ge \bar p^i$ for $i \in D_n$. For $i \in D_n^c$ this inequality is obvious.

Let $n_0 := \min\{n \ge 0\colon\ D_{n+1} = D_n\}$. The subsets D_n are increasing and contain at most N elements. So, $n_0 \le N+1$. Since $\Lambda_{n_0+1} = \Lambda_{n_0}$, the vectors $\hat p_{n_0+1}$ and $\hat p_{n_0+2}$ are the maximal fixed points of the same function f_{n_0} considered, respectively, on the order intervals $[0, \Lambda_{n_0} p_{n_0}]$ and $[0, \Lambda_{n_0+1} p_{n_0+1}]$. It follows that $\hat p_{n_0+1} \ge \hat p_{n_0+2}$ and $p_{n_0+1} \ge p_{n_0+2}$. Hence, $p_{n_0+1} = p_{n_0+2}$ and $\Lambda_{n_0+1} = \Lambda_{n_0+2}$. Note that

$$\begin{aligned} f_{n_0+1}(\hat p_{n_0+2}) &= \Lambda_{n_0+1}(\alpha e + \beta\Pi'(I-\Lambda_{n_0+1})l + \beta\Pi'\Lambda_{n_0+1}\hat p_{n_0+2}) \\ &= \Lambda_{n_0+1}(\alpha e + \beta\Pi' p_{n_0+2}) \end{aligned}$$

It follows that

$$p_{n_0+1} = p_{n_0+2} = (I - \Lambda_{n_0+1})l + \Lambda_{n_0+1}(\alpha e + \beta\Pi' p_{n_0+1}),$$

i.e. p_{n_0+1} solves the equation (3.1). Since p_{n_0+1} dominates its maximal solution $\bar p$, it coincides with the latter. □

3.2 The Gauss elimination algorithm in the Rogers–Veraart model

The Gauss elimination algorithm can be adjusted to calculate clearing vectors in the model with default costs in a finite number of steps.

If $D_0 := \{i \in \mathcal{N}\colon\ e^i + (\Pi' l)^i < l^i\} = \emptyset$, then the maximal solution of (3.1) is l. If $D_0 \ne \emptyset$, we can assume without loss of generality that $e^1 + (\Pi' l)^1 < l^1$. Using the block representations of matrices as in (2.3) and taking $J = \{1\}$, we rewrite (3.1) in the form

$$p^1 = \alpha e^1 + \beta\pi^{11}p^1 + \beta R'\tilde{p}, \tag{3.4}$$
$$\tilde{p} = (I_{N-1} - \tilde{\Lambda})\tilde{l} + \tilde{\Lambda}(\alpha\tilde{e} + \beta(T'p^1 + \tilde{\Pi}'\tilde{p})), \tag{3.5}$$

where $\tilde{p}$, $\tilde{e}$, and $\tilde{l}$ are vectors obtained by deleting the first component of the vectors p, e, and l while $\tilde{\Pi}$ and $\tilde{\Lambda}$ are $(N-1)\times(N-1)$ matrices obtained from of Π and Λ by deleting the first row and the first column.

Let us consider first the case $\beta < 1$. Then $\beta\pi^{11} < 1$. Solving the equation (3.4) with respect to p^1 and substituting the obtained expression, namely,

$$p^1 = \alpha(1-\beta\pi^{11})^{-1}e^1 + \beta(1-\beta\pi^{11})^{-1}R'\tilde{p},$$

into (3.5), we get that

$$\tilde{p} = (I_{N-1} - \tilde{\Lambda})\tilde{l} + \tilde{\Lambda}(\alpha e_1 + \beta\Pi_1'\tilde{p}), \tag{3.6}$$

where

$$e_1 := \tilde{e} + \beta(1-\beta\pi^{11})^{-1}e^1T', \tag{3.7}$$
$$\Pi_1 := \tilde{\Pi} + \beta(1-\beta\pi^{11})^{-1}RT. \tag{3.8}$$

Also, $\tilde{\Lambda} = \mathbf{1}_{\tilde{D}}$ where $\tilde{D} := \{i \in \mathcal{N}\setminus\{1\}:\ \tilde{e}^i + (\Pi_1'\tilde{p})^i\}$. The matrix Π_1 is substochastic. The equation (3.6) is of the same type as (3.1) but in the dimension $N-1$. Successive descending to lower dimensions may stop either at the equation whose solution is a vector formed by components of l, or by a linear equation with a single unknown variable admitting a unique solution. It remains to compute p by successive ascending via the explicit formulae exactly as in the Gauss algorithm.

When $\beta = 1$ it may happen that $\pi^{11} = 1$. In such a case the equation (3.4) does not define p^1 and one can take as p^1 any value from the interval $[0, l^1]$. In particular, for the maximal solution the needed value is $p^1 = l^1$. Note also that $e^1 = 0$ and $R'p_{J^c} = 0$. Let the set $A := \{j \ge 2:\ \pi^{j1} > 0\}$ contains k elements (changing the numbering we can always assume that $A := \{2, \dots, k+1\}$). Then $p^j = 0$ for all $j \in A$. The initial problem is reduced to finding solutions of the equation

$$\tilde{p} = (I_{N-1-k} - \tilde{\Lambda})\tilde{l} + \tilde{\Lambda}(\alpha\tilde{e} + \beta\tilde{\Pi}'\tilde{p}), \tag{3.9}$$

where $\tilde{\Lambda}$ and $\tilde{\Pi}$ are $(N-1-k)\times(N-1-k)$ matrices obtained from $\tilde{\Lambda}$ and $\tilde{\Pi}$ by deleting the rows and columns numbered by elements of the set $\{1\}\cup A$, the vectors $\tilde{l}$ and $\tilde{e}$ are obtained from l and e by deleting the components numbered by the elements of the same set of indices.

3.3 Merging and rescue consortium

To describe the *merging* of a group of k banks it is convenient to put it, without loss of generality, at the end of the list. We assign to the new bank the index 0 leaving unchanged the indices of the $m = N - k$ remaining banks. Formal description is as follows. Let $M := \{m+1, \dots, N\}$ be a subset having at least 2 elements. The model

(e, L, α, β) is replaced by the model $(e_M, L_M, \alpha, \beta)$ where $e_M = (e^0, e^1, \dots, e^m)$ with $e^0 := \mathbf{1}_M e$ and the entries of $(m+1) \times (m+1)$ matrix $L_M = (l_M^{ij})$ are:

$$l^{00} = 0, \quad l_M^{0i} = \sum_{j \in M} l^{ji}, \quad l_M^{i0} = \sum_{j \in M} l^{ij}, \quad l_M^{ij} = l^{ij}, \ \ 1 \le i, j \le m.$$

A *stress test* of the system consists in replacing the cash vector e by a smaller one $\hat{e} > 0$. Suppose that $D_0 := \{i \in \mathcal{N}\colon \ \hat{e}^i + (\Pi' l)^i < l^i\} \neq \emptyset$. The *bailout cost* (of the stressed system) is defined as $\mathbf{1}'_{D_0}\delta$ where the vector $\delta := (l - \Pi' l - \hat{e}) \vee 0$ represents an extra cash needed by the banks to be solvent after paying in full their liabilities.

Let p^* be the largest clearing vector in the model $(\hat{e}, L, \alpha, \beta)$.

A set of banks $A \subseteq D_0^c$ has a *rescue incentive* if

$$\mathbf{1}'_A(e + \Pi' l - l) - \mathbf{1}'_{D_0}\delta > \mathbf{1}'_A(e + \Pi' p^* - p^*)^+,$$

i.e. the total value of the banks from A under assumption that they get back all credits in full minus the *bailout cost* dominates their total value in the case of clearing by p^*. The weaker inequality $\mathbf{1}'_A(e + \Pi' l - l) > \mathbf{1}'_{D_0}\delta$ means that the set A has *rescue ability*, that is have sufficient resources to cover the debts of the failing banks D_0 if the liabilities will be payed in full. Thus, a group of banks having incentive can create a *rescue consortium*.

If in the initial system all banks are solvent, its equity vector $c = e + \Pi' l - l$. In the stressed system cleared by a vector p^* the equity vector is

$$(\hat{e} + \Pi' p^* - l) I_{\{l \le p^*\}}.$$

Proposition 3.2 *Suppose that in the model $(e, L, 1, 1)$ every bank is solvent and let $\hat{e} \in [0, e]$ is such that at least one bank in the model $(\hat{e}, L, 1, 1)$ is insolvent. Then the latter system does not have rescue consortiums.*

4 The Suzuki–Elsinger model with crossholdings

4.1 Existence of equilibrium

Now we consider a version of the Suzuki–Elsinger model, [16], [7], with crossholdings defined by a substochastic matrix $\Theta = (\theta^{ij})$ where $\theta^{ij} \in [0, 1]$ is a share of the bank i held by the bank j.

In this model the clearing vector and the equity vectors are interdependent and the problem is formulated in the spirit of equilibrium problem, that is as a simultaneous search of both vectors satisfying an equation in $\mathbf{R}^{2N}$. The latter can be presented in several equivalent forms.

We assume in this section as the standing hypothesis that there is no group composed by banks owned completely by banks of this group, that is the condition:

H. There is no non-empty subset $A \subseteq \{1, \dots, N\}$ such that $\mathbf{1}'_A \Theta = \mathbf{1}'_A$.

In other words, there is no $A \neq \emptyset$ such that

$$\sum_{j \in A} \theta^{ij} = 1 \quad \forall\, i \in A,$$

that is Θ does not contain stochastic submatrices.

Lemma 4.1 *The condition* **H** *holds if and only if unit is not an eigenvalue of* Θ.

Proof. Suppose that **H** holds but $x \in \mathbf{R}^N \setminus \{0\}$ is such that $\Theta x = x$. Let us consider the set of indices $A := \{i \in \mathcal{N}:\ |x^i| = |x|_\infty\}$. Then for any $i \in A$

$$|x^i| = |(\Theta x)^i| \le \sum_j \theta^{ij}|x^j| \le |x^i|\Big(\sum_{j\in A}\theta^{ij} + \sum_{j\in A^c}\theta^{ij}\Big) \le |x^i|.$$

It follows that $\sum_{j\in A}\theta^{ij} = 1$ (otherwise $|x^i| < |x^i|$) contradicting **H**.

Conversely, suppose that $\Lambda_A\Theta\Lambda_A\mathbf{1} = \Lambda_A\mathbf{1}$ for some $A \ne \emptyset$. Then $x \mapsto \Theta x$ is a monotone mapping of the complete lattice $\mathbf{1}_A + [0, \mathbf{1}_{A^c}]$ into itself. The fixed point of this mapping, existing in virtue of the Knaster–Tarski theorem, is the (right) eigenvector of Θ corresponding to the unit eigenvalue. □

Clearly, the above lemma could be formulated as the equivalence of the condition **H** and the invertibility of the matrix $I - \Theta$ (or $I - \Theta'$).

For any substochastic matrix Θ the spectral radius $\rho(\Theta) \le |\Theta|_\infty \le 1$ and $\rho(\Theta)$ is its eigenvalue (see, e.g., Ths. 5.6.9 and 8.3.1 in [10]). Thus, the condition **H** holds if and only if $\rho(\Theta) < 1$. It is useful to recall that $\rho(\Theta)$ is the infimum of the matrix norms of Θ and so in our case

$$(I - \Theta')^{-1} = \sum_{n=1}^{\infty} \Theta'^n.$$

Due to this representation it is obvious that the mapping $x \mapsto (I-\Theta')^{-1}x$ is increasing.

Lemma 4.2 *Suppose that* **H** *holds. Let* $y \in \mathbf{R}^N$ *and let* $B := \{i \in \mathcal{N}:\ y^i < 0\} \ne \emptyset$. *Then* $\mathbf{1}'_B\Lambda_B(I - \Theta')\Lambda_B y < 0$.

Proof. Note that

$$\mathbf{1}'_B\Lambda_B(I-\Theta')\Lambda_B y = \sum_{i\in B} y^i - \sum_{i\in B}\sum_{j\in B}\theta^{ji}y^j = \sum_{i\in B} y^i - \sum_{j\in B} y^j \sum_{i\in B}\theta^{ji} \le 0$$

since the sum of elements in each row of a substochastic matrix is less or equal than unit. The inequality above holds as equality only if $\sum_{i\in B}\theta^{ji} = 1$ for every $j \in B$ but such a case contradicts the condition **H**. □

Lemma 4.3 *For every* $x \in \mathbf{R}^N$ *the equations*

$$v = (x + \Theta' v)^+, \tag{4.1}$$
$$w = x + \Theta' w^+ \tag{4.2}$$

have unique solutions $v = v(x) \in \mathbf{R}^N_+$ *and* $w = w(x) \in \mathbf{R}^N$.

The mappings $x \mapsto v(x)$ *and* $x \mapsto w(x)$ *are order preserving, positive homogeneous, convex, and satisfy the Lipschitz condition.*

Proof. Existence. If $x \in \mathbf{R}^N_+$, the solutions are given explicitly: $v = w = (I-\Theta')^{-1}x$. Let us denote the right-hand sides of (4.1) and (4.2) by $f_1(v;x)$ and $f_1(w;x)$, respectively. Then $f_k(.,x) \le f_k(.,x^+)$, $k = 1,2$, and both mappings $v \mapsto f_k(.,x)$ are increasing on $\mathbf{R}^N$. Note that

$$f_1((I-\Theta')^{-1}x^+, x) = (x + \Theta'(I-\Theta')^{-1}x^+)^+ \le x^+ + \Theta'(I-\Theta')^{-1}x^+ = (I-\Theta')^{-1}x^+.$$

It follows that the restriction of $f_1(.,x)$ on the order interval $[0,(I-\Theta')^{-1}x^+]$ maps the latter into itself and, by the Knaster–Tarski theorem, this restriction has the minimal $\underline{v}$ and the maximal $\bar{v}$ fixed points.

Similarly, $f_2((I-\Theta')^{-1}x^+,x) \le (I-\Theta')^{-1}x^+$ and the restriction of $f_2(.,x)$ on the order interval $[x,(I-\Theta')^{-1}x^+]$ has the minimal $\underline{w}$ and the maximal $\bar{w}$ fixed points.

Uniqueness. Let $\tilde{v}$ be a fixed point of the mapping $v \mapsto f_1(v,x)$ not necessary laying in $[0,(I-\Theta')^{-1}x^+]$. Suppose that the set of indices $B := \{i \in \mathcal{N}:\ \tilde{v}^i > \underline{v}^i\}$ is non-empty. Note that $\Lambda_B\tilde{v} = \Lambda_B x + \Lambda_B\Theta'\tilde{v}$ and $\Lambda_B\underline{v} \ge \Lambda_B x + \Lambda_B\Theta'\underline{v}$. Hence,

$$\Lambda_B(\underline{v}-\tilde{v}) \ge \Lambda_B\Theta'(\underline{v}-\tilde{v}) \ge \Lambda_B\Theta'\Lambda_B(\underline{v}-\tilde{v}).$$

We get from here that

$$\mathbf{1}'_B\Lambda_B(I-\Theta')\Lambda_B(\underline{v}-\tilde{v}) \ge 0$$

in contradiction with Lemma 4.2. So, $B=\emptyset$ and $\tilde{v} \le \underline{v}$. It follows that the fixed point $\tilde{v}$ also belongs to the interval $[0,I-\Theta')^{-1}x^+]$ and $\tilde{v}=\underline{v}$.

Taking the positive part of both sides (4.2) we conclude that for any solution w of this equation w^+ solves (4.1), i.e. coincides with the unique solution of the latter. But w is uniquely determined by w^+.

Positive homogeneity. If $\lambda \in \mathbf{R}_+$ and $v(x)$ solves (4.1), then $\lambda v(x)$ solves the equation $v = (\lambda x+\Theta' v)$ and, by the uniqueness of solution, $v(\lambda x) = \lambda v(x)$. Similar argument works for (4.2).

Convexity and the Lipschitz property. For any $x \in \mathbf{R}^N$ the recursively defined sequence

$$v_{n+1}(x) = (x+\Theta' v_n(x))^+, \quad n \ge 1, \qquad v_0(x) = x,$$

evolves in the interval $[x,(I-\Theta')^{-1}x^+]$ and any its limit point is the solution of (4.1). Since the latter admits a unique solution, the sequence $v_n(x)$ has a limit $v(x)$. The obvious induction argument shows that the function $x \mapsto v_n(x)$ is convex and so is the function $x \mapsto v(x)$. But a finite convex function is locally Lipschitz. Since v is positive homogeneous, it has the Lipschitz property. The claimed properties for the solution w of (4.2) holds because they hold for w^+ solving (4.1).

Monotonicity. Let $\Delta := w(x+h) - w(x)$ where $h \in \mathbf{R}^N_+$. Put $A := \{i:\ \Delta^i < 0\}$ and define the diagonal matrix $\Lambda := \operatorname{diag} \mathbf{1}_A$. The elementary inequality $a<b$ implies that $a^+ - b^+ \ge a-b$, the inequality $a \ge b$ implies that $a^+ - b^+ \ge 0$. Therefore,

$$\Theta'(w^+(x+h) - w^+(x)) \ge \Theta'\Lambda\Delta$$

and

$$\Lambda\Delta = \Lambda h + \Lambda\Theta'(w^+(x+h) - w^+(x)) \ge \Lambda h + \Lambda\Theta'\Lambda\Delta.$$

Regrouping terms and summing up the components we get that

$$\mathbf{1}'\Lambda(I-\Theta')\Lambda\Delta \ge \mathbf{1}\Lambda h \ge 0.$$

If $A \ne \emptyset$, we arrive to a contradiction since the left-hand side above is

$$\sum_{j\in A}\Delta^j - \sum_{j\in A}\Big(\sum_{i\in A}\theta^{ij}\Big)\Delta^j < 0 \tag{4.3}$$

in virtue of the hypothesis **H**: all sums in parentheses are less than unit and at least one should be strictly less. Thus, $A=\emptyset$, i.e. $w(x)$ is order preserving.

It remains to prove the monotonicity of v. Let $h \in \mathbf{R}^N_+$, $\Delta := v(x+h) - v(x)$, $A := \{i: \ \Delta^i < 0\}$,

$$
\begin{aligned}
B_0 &:= \{i\colon \ x^i + (\Theta' v(x))^i = 0\},\\
B_1 &:= \{i\colon \ x^i + (\Theta' v(x))^i > 0\},\\
B_2 &:= \{i\colon \ x^i + (\Theta' v(x))^i < 0\}.
\end{aligned}
$$

Define the diagonal matrices $\Lambda := \operatorname{diag} \mathbf{1}_A$ and $\Lambda_k := \operatorname{diag} \mathbf{1}_{B_k}$ for $k = 0, 1, 2$.

Note that for $i \in B_2$ we have obviously $v^i(x+h) \geq v^i(x) = 0$. Moreover,

$$
\Delta^i = v^i(x+h) = (x^i + h^i + (\Theta' v(x+h))^i)^+ = 0
$$

when $|h|$ is sufficiently small. For $i \in B_0$ we have $\Delta^i = v^i(x+h) \geq 0$. For $i \in B_1$ we have $x^i + h + (\Theta' v(x+h))^i > 0$ when $|h|$ is sufficiently small and, therefore,

$$
\Lambda_1 \Delta = \Lambda_1 (h + \Theta' \Lambda_1 \Delta + \Theta' \Lambda_0 \Delta) \geq \Lambda_1 (h + \Theta' \Lambda_1 \Delta).
$$

Since $A \subseteq B_1$ we get that $\Lambda \Delta \geq \Lambda h + \Lambda \Theta' \Lambda \Delta$ and, as above, $A = \emptyset$. □

We consider the following system of equations whose set of solutions will be denoted by $\Gamma_1 \subseteq [0, l] \times \mathbf{R}^N_+$:

$$
\begin{aligned}
p &= (e + \Pi' p + \Theta' V)^+ \wedge l, && (4.4)\\
V &= (e + \Pi' p - p + \Theta' V)^+. && (4.5)
\end{aligned}
$$

For $(p, V) \in \Gamma_1$ the components p and V are called, respectively, *clearing vector* and *equity*.

Accordingly to Lemma 4.3 for every p the equation (4.5) admits a unique solution, namely, $V(p) := v(e + \Pi' p - p)$ which is Lipschitz in p. Thus, the equation

$$
p = (e + \Pi' p + \Theta' V(p))^+ \wedge l \tag{4.6}
$$

has a solution in virtue of the Brouwer theorem claiming that a continuous mapping (given by the left-hand side above) of a convex compact set ($[0, l]$ in our case) has a fixed point. So,

$$
\Gamma_1 = \{(p, V(p))\colon \ p \text{ solves } (4.6)\} \neq \emptyset.
$$

We also introduce the system

$$
\begin{aligned}
p &= (e + \Pi' p + \Theta' U)^+ \wedge l, && (4.7)\\
U &= (e + \Pi' p - l + \Theta' U)^+ && (4.8)
\end{aligned}
$$

with the set of solutions $\Gamma_2 \subseteq [0, l] \times \mathbf{R}^N_+$ and the system

$$
\begin{aligned}
p &= (e + \Pi' p + \Theta' W^+)^+ \wedge l, && (4.9)\\
W &= e + \Pi' p - l + \Theta' W^+ && (4.10)
\end{aligned}
$$

with the set of solutions $\Gamma_3 \subseteq [0, l] \times \mathbf{R}^N$.

Introducing the equation

$$
p = (e + \Pi' p + \Theta' U(p))^+ \wedge l \tag{4.11}
$$

and using Lemma 4.3 we can prove that

$$\Gamma_2 = \{(p, U(p)) : \ p \text{ solves } (4.11)\} \neq \emptyset,$$

where $U(p) := v(e + \Pi' p - l)$. Since the latter function is monotone, we can apply to the equation (4.11) the Knaster–Tarski theorem providing additional useful information: there exists the smallest $\underline{p}$ and the largest $\bar{p}$ solutions of (4.11). The monotonicity of $U(p)$ allows to conclude that Γ_2 has the minimal and the maximal elements, namely, $(\underline{p}, U(\underline{p}))$ and $(\bar{p}, U(\bar{p}))$.

In the same way, introducing the equation

$$p = (e + \Pi' p + \Theta' W^+(p))^+ \wedge l \tag{4.12}$$

and defining the function $W(p) = w(e + \Pi' p - l)$, we prove that the set

$$\Gamma_3 = \{(p, W(p)) : \ p \text{ solves } (4.12)\}$$

contains the minimal and the maximal elements $(\underline{p}, W(\underline{p}))$ and $(\bar{p}, W(\bar{p}))$, respectively.

It remains to show that Γ_1 also has the minimal and the maximal elements and establish the relations between all these sets.

Let us introduce the function $\varphi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N \times \mathbf{R}^N_+$ with $\varphi(x, y) := (x, y^+)$.

Lemma 4.4 *$\Gamma_1 = \Gamma_2 = \varphi(\Gamma_3)$.*

Proof. ($\Gamma_1 \subseteq \Gamma_2$) Let $(p, V(p)) \in \Gamma_1$. If $V^i(p) > 0$, then $(e + \Pi' p - p + \Theta' V(p))^i > 0$. Rewriting the last inequality as $p^i < (e + \Pi' p + \Theta' V(p))^i$ we obtain in view of (4.6) that $p^i = l^i$. Thus, for such i we have that

$$V^i(p) = ((e + \Pi' p - l + \Theta' V(p))^i)^+.$$

If $V^i(p) = 0$, then the above equality holds trivially due to (4.6). That is, $V(p)$ solves the equation (4.8) for U. Hence, due to the uniqueness of solution, $V(p) = U(p)$ and $(p, V(p)) \in \Gamma_2$.

($\Gamma_2 \subseteq \Gamma_1$) Let $(p, U(p)) \in \Gamma_2$. If $0 \le p^i < l^i$, then, accordingly to equation (4.11), $p^i = ((e + \Pi' p + \Theta' U(p))^i)^+$, implying via (4.8) that $U^i(p) = 0$ and

$$U^i(p) = ((e + \Pi' p - p + \Theta' U(p))^i)^+.$$

If $p^i = l^i$, this equality follows directly from the definition of $U(p)$. Thus, $U(p)$ solves (4.5) and, therefore, coincides with $V(p)$. But this means that $(p, U(p)) \in \Gamma_2$.

($\varphi(\Gamma_3) \subseteq \Gamma_2$) Let $(p, W(p)) \in \Gamma_3$. By definition, $W(p)$ satisfies the equation (4.10). Taking the positive part of both sides of this equation, we obtain that $W^+(p)$ satisfies the equation (4.8). Hence, $(p, W^+(p)) \in \Gamma_2$.

($\Gamma_2 \subseteq \varphi(\Gamma_3)$) Let $(p, U(p)) \in \Gamma_2$. Note that

$$W(p) := e + \Pi' p - l + \Theta' U(p) \le (e + \Pi' p - l + \Theta' U(p))^+ = U(p)$$

and $W^i(p) = U^i(p)$ if $W^i(p) \ge 0$. Thus, $W^+(p) = U(p)$. It follows that $W(p)$ solves (4.10) and $(p, W(p)) \in \Gamma_3$. $\square$

In the sequel we use the abbreviations: $\underline{V} := V(\underline{p})$, $\bar{V} := V(\bar{p})$,

$$\Delta_p := \bar{p} - \underline{p} \geq 0, \qquad \Delta_V := \bar{V} - \underline{V} \geq 0,$$

$A_p := \{i\colon\ \Delta_p > 0\}$, $A_V := \{i\colon\ \Delta_V > 0\}$. We define the diagonal matrices

$$\Lambda_p := \operatorname{diag} \mathbf{1}_{A_p}, \qquad \Lambda_V := \operatorname{diag} \mathbf{1}_{A_V}, \qquad \Lambda := \operatorname{diag} \mathbf{1}_{A_p \cup A_V}.$$

Lemma 4.5 *The following identities hold:*

$$\Lambda(I - \Theta')\Lambda_V = 0, \qquad \Lambda(\Pi' - I)\Lambda_p = 0. \tag{4.13}$$

Proof. Note that $\underline{V} \geq e + \Pi\underline{p} - \underline{p} + \Theta'\underline{V}$ and for each $i \in A_p \cup A_V$ necessarily $\bar{p}^i > 0$ and $\bar{V}^i = e^i + (\Pi'\bar{p})^i - \bar{p}^i + (\Theta'\bar{p})^i$. Thus,

$$\Lambda\Delta_V \leq \Lambda(\Pi' - I)\Delta_p + \Lambda\Theta'\Delta_V.$$

Taking into account that $\Delta_V = \Lambda\Delta_V$ and $\Delta_p = \Lambda\Delta_p$, we get from here that

$$\mathbf{1}'\Lambda(I - \Theta')\Lambda\Delta_V \leq \mathbf{1}'\Lambda(\Pi' - I)\Lambda\Delta_p.$$

Inspecting the explicit expressions (similar to that in (4.3)) we conclude that the right-hand side above is less or equal to zero while the left-hand side is greater or equal to zero. So,

$$\mathbf{1}'\Lambda(I - \Theta')\Lambda\Delta_V = 0, \qquad \mathbf{1}'\Lambda(\Pi' - I)\Lambda\Delta_p = 0. \tag{4.14}$$

Since $\Delta_V^i > 0$ on A_V and $\Delta_p^i > 0$ on A_p, these equalities are equivalent to (4.13). □

Theorem 4.6 *Suppose that for any subset of indices $A \neq \emptyset$ there is $j \in A$ such that*

$$\sum_{i \in A} \theta^{ij} < 1, \qquad \sum_{i \in A} \pi^{ij} < 1.$$

Then $(\underline{p}, V(\underline{p})) = (\bar{p}, V(\bar{p}))$.

Proof. The identities (4.14) (equivalent to (4.13)) can be written as

$$\sum_{j \in A_V} \Delta_V^j - \sum_{j \in A_V} \Big(\sum_{i \in A_V \cup A_p} \theta^{ij} \Big) \Delta_V^j = 0,$$
$$\sum_{j \in A_p} \Delta_p^j - \sum_{j \in A_p} \Big(\sum_{i \in A_V \cup A_p} \pi^{ij} \Big) \Delta_p^j = 0.$$

Applying the assumption with $A = A_p \cup A_V$ we get the result. □

Theorem 4.7 *Suppose that for any subset of indices A such that for all $i \in A$*

$$\sum_{j \in A} \theta^{ij} = 1 \quad \text{or} \quad \sum_{j \in A} \pi^{ij} = 1$$

it holds that

$$\sum_{i \in A} e^i > \sum_{i \in A} \Big(1 - \sum_{j \in A} \pi^{ij} \Big) l^i.$$

Then the clearing vector is unique. In particular, in the case of the Eisenberg–Noe model where $\Theta = 0$, if for any subset of indices A such that $\sum_{j \in A} \pi^{ij} = 1$ for all $i \in A$ also holds the inequality $\sum_{i \in A} e^i > 0$, then the clearing vector is unique.

Proof. We start from the equality

$$\Lambda\bar{V} = \Lambda(e + (\Pi' - I)\bar{p} + \Theta'\bar{V}).$$

Regrouping terms and multiplying from the left by $\mathbf{1}'$ we obtain the identity

$$\mathbf{1}'\Lambda(I - \Theta')\Lambda\bar{V} + \mathbf{1}'\Lambda(I - \Pi')\Lambda\bar{p} = \mathbf{1}'\Lambda e + \mathbf{1}'\Lambda\Pi'(I - \Lambda)\bar{p} + \mathbf{1}'\Lambda\Theta'(I - \Lambda)\bar{V}.$$

Note that $\bar{V}^i = 0$ for $i \in A_p \setminus A_V$. Therefore, $(\Lambda - \Lambda_V)\bar{V} = 0$. Combining with (4.13) we conclude that the first term in the left-hand side of the above identity is zero.

If $i \in A_V \setminus A_p$, that is, $\bar{p}^i = \underline{p}^i$, and $\bar{V}^i > \underline{V}^i \geq 0$, then in virtue of definitions $\bar{V}^i = e^i + (\Pi'\bar{p})^i - \bar{p}^i + (\Theta' V(\bar{p}))^i > 0$, implying, via (4.6), that $\bar{p}^i = l^i$. Thus,

$$(\Lambda - \Lambda_p)\bar{p} = (\Lambda - \Lambda_p)l. \tag{4.15}$$

Using the second relation in (4.13) we obtain that the second term in the left-hand side of the identity is equal to $\mathbf{1}'\Lambda(I - \Pi')\Lambda l$. So,

$$\mathbf{1}'\Lambda(I - \Pi')\Lambda l = \mathbf{1}'\Lambda e + \mathbf{1}'\Pi'(I - \Lambda)\bar{p} + \mathbf{1}'\Theta'(I - \Lambda)\bar{V} \geq \mathbf{1}'\Lambda e.$$

That is,

$$\sum_{i \in A_p \cup A_V} \Big(1 - \sum_{j \in A_p \cup A_V} \pi^{ij}\Big) l^i \geq \sum_{i \in A_p \cup A_V} e^i.$$

Applying the assumption with $A = A_p \cup A_V$ we get the result. □

Remark. In the early paper by Suzuki the model was analyzed using a different approach. In the above notations the equations (2), (3) of [16], with the positive part (2), can be written as

$$\begin{aligned} p &= (e + \Pi' p + \Theta' V)^+ \wedge l =: g_1(p, V), \\ V &= (e + \Pi' p - l + \Theta' V)^+ =: g_2(p, V), \end{aligned}$$

where the second equation is the equation for W^+. If $\lambda := |\Pi'|_1 \vee |\Theta'|_1 < 1$, then the mapping $(p, V) \mapsto g(p, V)$ is a contraction in $(\mathbf{R}_+^{2N}, |.|_1)$. Indeed, the elementary inequality

$$|a^+ \wedge c - b^+ \wedge c| + |(a - c)^+ - (b - c)^+| \leq |a - b|, \qquad a, b \in \mathbf{R},\ c \in \mathbf{R}_+,$$

implies that

$$\begin{aligned} |g(p, V) - g(\tilde{p}, V)|_1 &\leq |\Pi'(p - \tilde{p})|_1 + |\Theta'(V - \tilde{V})|_1 \leq |\Pi'|_1 |p - \tilde{p}|_1 + |\Theta'|_1 |V - \tilde{V}|_1 \\ &\leq \lambda(|p - \tilde{p}|_1 + |V - \tilde{V}|_1). \end{aligned}$$

With this the existence and uniqueness of equilibrium is obvious.

5 The Elsinger model: debts of different seniorities

We consider a version of the Elsinger model where the interbank debts may be senior and junior. In this model the system of N banks is described by the vector of cash reserves and by M matrices $L_1 = (l_1^{ij})$, ..., $L_M = (l_M^{ij})$ representing the hierarchy of liabilities with decreasing seniority. That is, the element l_1^{ij} represents the debt of the bank i to the bank j of the highest seniority etc., $\sum_j l_S^{ij}$ is the total of debts of the bank i of the seniority S.

The relative liabilities are defined by the matrix Π_S with

$$\pi_S^{ij} = \frac{l_S^{ij}}{l_S^i} = \frac{l_S^{ij}}{\sum_j l_S^{ij}}$$

(by convention, $0/0 := 1$).

The clearing procedure requires the complete reimbursement of the debts starting from the highest priority and, for each seniority, the distribution is proportional to the volume of obtained credits of this seniority. For the bank i we denote by p_S^i the value distributed to cover the debts of the seniority S. So, the clearing is described by the set of vectors p_S, $S = 1, \dots, M$, which can be considered as a single "long" vector from $(\mathbf{R}^N)^M$ satisfying the system of equations

$$p_1^i = \Big(e^i + \sum_S \sum_j \pi_S^{ji} p_S^j\Big) \wedge l_1^i,$$

$$p_S^i = \Big(e^i + \sum_S \sum_j \pi_S^{ji} p_S^j - \sum_{r<S} l_r^i\Big)^+ \wedge l_S^i, \qquad 1 < S \le M.$$

In a vector form they can be written as follows:

$$p_S = \Big(e + \sum_S \Pi'_S p_S - \sum_{r<S} l_r\Big)^+ \wedge l_S, \qquad S = 1, ..., M. \tag{5.1}$$

It is clear that, for the partial ordering in $(\mathbf{R}^N)^M$ induced by the cone $(\mathbf{R}_+^N)^M$, the function

$$(p_1, ..., p_M) \mapsto \left(\Big(e + \sum_S \Pi'_S p_S^*\Big)^+ \wedge l_1, ..., \Big(e + \sum_S \Pi'_S p_S^* - \sum_{r<M} l_r\Big)^+ \wedge l_M\right)$$

is a monotone mapping of the order interval $[0, l_1] \times ... \times [0, l_M] \subset (\mathbf{R}^N)^M$ into itself. Thus, according to the Knaster–Tarski theorem the set of fixed points of this mapping, i.e. the solutions of the equation (5.1), is non-empty and has the maximal and the minimal elements.

In the case of liabilities of different seniority after clearing by the vector $p \in (\mathbf{R}^N)^M$ the equity vector $c \in \mathbf{R}^N$ has the form:

$$c = \Big(e + \sum_S \Pi'_S p_S - \sum_S l_S\Big)^+.$$

Lemma 5.1 *The equity vector does not depend on the clearing vector.*

Proof. Note that

$$\Big(e+\sum_S \Pi'_S p_S\Big) \wedge \sum_S l_S = \sum_S p_S.$$

Therefore,

$$\Big(e+\sum_S \Pi'_S p_S - \sum_S l_S\Big)^+ = e+\sum_S \Pi'_S p_S - \sum_S p_S.$$

With this identity the reasoning is analogous to that with a single seniority class. □

In an attempt to use language of graphs in the uniqueness theorem in the spirit of Eisenberg–Noe the authors used in [6] a specific graph structure induced by the matrices Π_S.

For a given clearing vector p we define the *default index* d^i of the node i as the smallest r such that

$$\bar{p}^i_r = e^i + \sum_S \sum_j \pi^{ji}_S \bar{p}^j_S - \sum_{r'<r} l^i_{r'}.$$

In another words, d^i is the lowest seniority for which the bank equity after clearing is equal to zero. Define the matrix $\Delta=\Delta(p)$ by putting $\Delta^{ij}=1$ if $\pi^{ij}_{d(i)}>0$, and $\Delta^{ij}=0$ otherwise. We use the notation $i \rightsquigarrow j$ if $\Delta^{ij}=1$ and denote by $O(i)$ *the* Δ*-orbit of* i, that is the set of all $j\neq i$ for which there is a directed path $i\rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \ldots \rightsquigarrow j$.

Arguments similar to those in the proof of Theorem 2.3 lead to the following result:

Theorem 5.2 *Suppose that for the clearing vector* $\bar{p}$ *any non-empty* Δ*-orbit* $O(i)$ *is such that* $\mathbf{1}'_{O(i)}e>0$*. Then the clearing vector is unique.*

6 The Fischer model: clearing with derivatives

In the paper [9] Fisher generalized the Suzuki–Elsinger model to cover systems where banks besides of straight debts may have liabilities in terms of derivatives having different seniorities.

Mathematically, this means that matrices L_S may depend on the clearing vectors. The clearing equations for the situation with cross-holdings can be represented as follows:

$$p_S = \Big(e+\Theta' V + \sum_{r\le M} \Pi'_r p_r - \sum_{r<S} l_r(p)\Big)^+ \wedge l_S(p), \qquad S=1,\ldots,M, \tag{6.1}$$

$$V = \Big(e+\Theta' V + \sum_{r\le M} \Pi'_r p_r - \sum_S p_S\Big)^+. \tag{6.2}$$

Economically, Fisher's model is quite different from those previously discussed because now the matrices Π_S are disconnected from $L_S(p)$ and become input parameters of the model.

Theorem 6.1 *Suppose that the functions* $p\mapsto l_S(p)$ *are bounded and continuous,* $|\Theta|<1$*. Then the system (6.1), (6.2) has a solution.*

Proof. By virtue of Lemma 4.3 the equation (6.2) has a solution $V(p)$ for any p and this solution is continuous in p. Plugging $V(p)$ into (6.1) we obtain in the right-hand side a continuous function which maps into itself the compact convex set $[0,l_1^*]\times\dots\times[0,l_M^*]$ where $l_S^*=\sup_p l_S(p)$. The application of the Brouwer theorem leads to the claim. □

In particular, the above theorem ensures the existence of clearing vector in the model with credit default swaps (CDS) where L_1 is the matrix of the straight debts having the highest priority and

$$l_S^{ij}:=\lambda_S^{ij}(l_S-p_S)^+, \qquad S\ge 2,$$

where $\lambda_S^{ij}\ge 0$ are arbitrary constants.

Lemma 6.2 *The system (6.1), (6.2) is equivalent to the system*

$$p_S=\Big(e+\Theta'V+\sum_{r\le M}\Pi'_r p_r-\sum_{r<S} l_r(p)\Big)^+\wedge l_S(p), \qquad S=1,...,M, \tag{6.3}$$

$$V=\Big(e+\Theta'V+\sum_{r\le M}\Pi'_r p_r-\sum_{S} l_S(p)\Big)^+. \tag{6.4}$$

Proof. If we fix V and take p that satisfies the relations (6.1), then the right-hand sides of (6.2) and (6.4) coincide. □

The paper [9] contains results on the existence and uniqueness of solution of (6.3), (6.4) without the assumption on boundedness of l_S but with a more stringent condition on coefficients, namely, on the matrices Π_S.

Theorem 6.3 *Suppose that $e\ge 0$, the functions $p\mapsto l_S(p)$ are continuous, and $|\Theta|_\infty<1$, $|\Pi_S|_\infty<1$ for all S. Then the system (6.3), (6.4) has a solution.*

Proof. Put $\Pi_{M+1}:=\Theta$, $p_{M+1}:=V$, and $l_{M+1}:=\infty=(\infty,\dots,\infty)$. Slightly abusing notations we retain the symbol p for the $N(M+1)$-dimensional vector $(p_1,...,p_M,p_{M+1})$ and write the system (6.3), (6.4) in a more compact form $p=\Phi(p)$ where

$$\Phi_S(p):=\Big(e+\sum_{r\le M+1}\Pi'_r p_r-\sum_{r<S} l_r(p)\Big)^+\wedge l_S(p), \qquad S=1,...,M+1.$$

For $y\in\mathbf{R}$, $a_1,\dots,a_M\in\mathbf{R}_+$ we have the identity

$$\sum_{S=1}^{M}\Big(y-\sum_{r<S}a_r\Big)^+\wedge a_S+\Big(y-\sum_{r\le M}a_r\Big)^+=y \tag{6.5}$$

easily verified by induction based on the observation that $w^+\wedge a=w^+-(w-a)^+$ when $w\in\mathbf{R}$ and $a\in\mathbf{R}_+$.

Using it, we infer that for any $p\in\mathbf{R}_+^{N(M+1)}$

$$\sum_{S\le M+1}\Phi_S(p)=e+\sum_{S\le M+1}\Pi'_S p_S$$

and

$$\sum_{S\le M+1}|\Phi_S(p)|_1=|e|_1+\sum_{S\le M+1}|\Pi'_S p_S|_1\le |e|_1+\theta\sum_{S\le M+1}|p_S|_1$$

where $\theta := \max_{S\le M+1} |\Pi'_S|_1$. In particular, if

$$\sum_{r\le M+1} |p_r|_1 \le \frac{1}{1-\theta}|e|_1, \tag{6.6}$$

then also

$$\sum_{S\le M+1} |\Phi_S(p)|_1 \le \frac{1}{1-\theta}|e|_1.$$

So, the continuous function $p \mapsto \Phi(p)$ maps the convex compact set of $p \in \mathbf{R}_+^{N(M+1)}$ satisfying (6.6) into itself and, by the Brouwer theorem, has a fixed point, i.e. the equation $p = \Phi(p)$ has a solution.

Note that if $p = \Phi(p)$, then the so-called *accounting equation* is fulfilled

$$\sum_{S\le M+1} p_S = e + \sum_{S\le M+1} \Pi'_S p_S$$

and, therefore,

$$\sum_{S\le M+1} |p_S|_1 = |e|_1 + \sum_{S\le M+1} |\Pi'_S p_S|_1 \le |e|_1 + \sum_{S\le M+1} |\Pi'_S|_1 |p_S|_1$$

implying that

$$(1-\theta)\sum_{S\le M+1} |p_S|_1 \le \sum_{S\le M+1} (1-|\Pi'_S|_1)|p_S|_1 \le |e|_1.$$

Thus, any solution of the equation $p = \Phi(p)$ satisfies (6.6).

Remark. It is easily seen that the claim of the theorem holds also in the case where the matrices Π'_S depend on p continuously and $\theta := \sup_p |\Pi'_S(p)|_1 < 1$ for $S = 1, \dots, M+1$.

The uniqueness result of [9] is based on the following elementary statement:

Lemma 6.4 *Let $a_r, b_r \in \mathbf{R}$ be such that $b_r \ge a_r \ge 0$, $r \ge 1$. Let $A_0 := 0$, $B_0 := 0$, $A_r := \sum_{j\le r} a_j$, $B_r := \sum_{j\le r} b_j$ for $r \ge 1$. If $w, z \in \mathbf{R}$ are such that*

$$z - w \ge B_M - A_M, \tag{6.7}$$

then

$$z - w = \sum_{r\le M} \big|(z-B_{r-1})^+ \wedge b_r - (w-A_{r-1})^+ \wedge a_r\big| + \big|(z-B_M)^+ - (w-A_M)^+\big|.$$

Proof. Since $b_M - a_M \ge 0$ the inequality (6.7) implies that $z - w \ge B_{M-1} - A_{M-1}$. As for (6.5) we can use induction arguments but based this time on the identity

$$|v^+ \wedge b - u^+ \wedge a| = |v^+ - u^+| - |(v-b)^+ - (v-a)^+|$$

which holds when $b \ge a \ge 0$ and $v - u \ge b - a$. □

Theorem 6.5 *In addition to the assumptions of preceding theorem suppose that*

$$l_r^i(p) = \psi_r^i\Big(\sum_{r\le M+1} (\Pi_r' p_r)^i\Big)$$

where $\psi_r^i : \mathbf{R}_+ \to \mathbf{R}_+$ are increasing functions such that for any $u, v \in \mathbf{R}_+$, $v \ge u$, we have the bound

$$v - u \ge \sum_{r\le M} \big(\psi_r^i(v) - \psi_r^i(u)\big), \quad i = 1, \dots, N.$$

Then the system (6.3), (6.4) has a unique solution.

Proof. We check that Φ is a contraction mapping in the space $\mathbf{R}_+^{N(M+1)}$ in the metric induced by the l^1-norm. Let p and $\tilde p$ be two vectors. Define

$$x^i := \sum_{r\le M+1} (\Pi_r' p_r)^i, \qquad y^i := e^i + x^i, \qquad \Sigma_r^i := \sum_{j\le r} \psi_r^j(x^i),$$

and $\tilde x^i$, $\tilde y^i$, $\tilde\Sigma_r^i$ similarly; put also $\psi_{M+1}^i(x^i) = \psi_{M+1}^i(\tilde x^i) := \infty$. With these definitions

$$|\Phi(p) - \Phi(\tilde p)|_1 = \sum_{i\le N}\sum_{r\le M+1} \big|(y^i - \Sigma_{r-1}^i)^+ \wedge \psi_r^i(x^i) - (\tilde y^i - \tilde\Sigma_{r-1}^i)^+ \wedge \psi_r^i(\tilde x^i)\big|.$$

The hypothesis of the theorem allows us to apply Lemma 6.4, choosing a correspondence with its notations in dependence of the sign of the difference $x^i - \tilde x^i$, and conclude that the interior sum is equal to $|y^i - \tilde y^i| = |x^i - \tilde x^i|$. Thus,

$$\begin{aligned}|\Phi(p) - \Phi(\tilde p)|_1 &= \sum_{i\le N}\left|\sum_{r\le M+1} (\Pi_r'(p_r - \tilde p_r))^i\right| \le \sum_{r\le M+1} |\Pi_r'(p_r - \tilde p_r)|_1 \\ &\le \sum_{r\le M+1} |\Pi_r'|_1 |p_r - \tilde p_r|_1 \le \theta |p - \tilde p|_1,\end{aligned}$$

where $\theta := \max_{S\le M+1} |\Pi_S'|_1 < 1$. □

7 Models with illiquid assets and a price impact

7.1 Selling with equal proportions

In the paper [2], developing ideas of [3], it is considered a setting where banks own also a single illiquid asset. Its selling implies replacing the nominal price in the balance sheets by the market price decreasing with respect to the supply volume. More interesting are situations where banks own several illiquid assets selling of which leads to price impacts depending on selling strategies. We present here the simplest generalization considered in [6] supposing that each bank is obliged to sell the illiquid assets in equal proportions as in [4].

Let us consider single seniority clearing problem where the bank i owns not only the cash e^i but also K illiquid assets, in quantities $y^{i1}, \dots, y^{iK}$ represented in the model by y_i, the ith row of the matrix $Y = (y^{im})$, $i \le N$, $m \le K$. The nominal prices per unit of illiquid assets are strictly positive numbers $Q^1, ..., Q^K$. The clearing requires their partial or total sale influencing the market price. If the bank sells $u^{im} \in [0, y^{im}]$ units of the m-th assets for the price q_m, its total increase in cash is

$$(Uq)^i = \sum_{m=1}^{K} u^{im} q^m.$$

The price formation is modeled by the inverse demand function $F_0 : \mathbf{R}^K \to \mathbf{R}^K$ assumed to be continuous and monotone decreasing (that is, $F_0(z) \le F_0(x)$ when $z \ge x$ in the component-wise sense) and such that $F_0(0) = Q$ and $F_0^m(Y'\mathbf{1}) > 0$ for $m = 1, \dots K$. So, in the absence of supply the market prices are the nominal prices and they are strictly positive even when all illiquid assets are sold.

The clearing rules: each bank pays debts in accordance to the matrix of relative liabilities and sell illiquid assets if it has insufficient amount of cash. The result of clearing should be: all debts of the bank are covered or its resources are completely exhausted.

Suppose that the bank i must sell all assets in the same proportion

$$\alpha^i(q) = \frac{\left(l^i - e^i - \sum_j \pi^{ji} p^j\right)^+}{\sum_k y^{ik} q^k} \wedge 1, \quad i \in \mathcal{N}. \tag{7.1}$$

For a fixed market price the bank does not sell illiquid assets if its cash reserve and the collected debts cover the liabilities. In the another extreme case where

$$l^i - e^i - \sum_j \pi^{ji} p^j \ge \sum_k y^{ik} q^k = (Yq)^i$$

all illiquid assets have to be sold and the bank defaults. In the intermediate case the bank sells a share $\alpha^i \in]0, 1[$ of the mth asset, adding to its cash account the amount

$$\frac{l^i - e^i - \sum_j \pi^{ji} p^j}{\sum_k y^{ik} q^k} y^{im} q_m.$$

The total increase in cash allows to cover the liabilities.

Under such a rule the ith bank sells u^{im} units of the mth asset where

$$u^{im} := u^{im}(p, q) := \frac{y^{im}\left(l^i - e^i - \sum_j \pi^{ji} p^j\right)^+}{\sum_k y^{ik} q^k} \wedge y^{im}.$$

The total supply of the illiquid assets is the vector $U'(p, q)\mathbf{1}$ where $U(p, q)$ is the matrix with entries given by the above formula.

An equilibrium vector $(p^*, q^*) \in [0, l] \times [F_0(Y'\mathbf{1}), Q]$ is a solution of the system of $N + K$ equations written in the matrix form as

$$p = (e + U(p, q)q + \Pi' p) \wedge l, \tag{7.2}$$

$$q = F_0(U'(p, q)\mathbf{1}). \tag{7.3}$$

It is not difficult to verify that $(e + U(p, q)q + \Pi' p) \wedge l = (e + Yq + \Pi' p) \wedge l$ and the equation (7.2) can be written in the form

$$p = (e + Yq + \Pi' p) \wedge l, \tag{7.4}$$

see [2] and [6].

The existence of the equilibrium is easy. Indeed, we check that

$$U'(p,q)\mathbf{1} \geq U'(\tilde{p},\tilde{q})\mathbf{1}, \qquad U(p,q)q + \Pi' p \leq U(\tilde{p},\tilde{q})\tilde{q} + \Pi'\tilde{p}$$

when $(\tilde{p},\tilde{q}) \geq (p,q)$. Denoting $F(p,q)$ the right-hand side of the first equation we obtain that $(p,q) \mapsto (F(p,q), F_0(U'(p,q))\mathbf{1})$ is a monotone mapping of the order interval $[0,l] \times [F_0(Y'\mathbf{1}), Q]$ into itself. Accordingly to Knaster–Tarski theorem the set of its fixed points is nonempty and contains the minimal and maximal elements $(\underline{p}^*, \underline{q}^*)$ and $(\bar{p}^*, \bar{q}^*)$.

For a fixed q the function $p \to F(p,q)$ is monotone. Thus, by the Knaster–Tarski theorem the set of solutions of the equation (7.2) is nonempty and contains, in particular, the maximal element $\bar{p}(q)$.

For any fixed $q \in [F_0(Y'\mathbf{1}), Q]$ the largest solution $\bar{p} = \bar{p}(q)$ of (7.2) is

$$\bar{p} = \sup\{p \in [0,l] : \; p \leq (e + U(p,q)q + \Pi' p) \wedge l\}.$$

Thus, $q \mapsto \bar{p}(q)$ is an increasing (and continuous) function on $[F_0(Y'\mathbf{1}), Q]$. It follows that the supply function

$$q \mapsto \zeta(q) := U'(\bar{p}(q), q)\mathbf{1}$$

is decreasing and, therefore, the $q \mapsto F_0(\zeta(q))$ is an increasing (and continuous) mapping of the interval $[F_0(Y'\mathbf{1}), Q]$ into itself. Hence, it has the minimal and maximal fixed points we shall denote q_1 and q_2.

Theorem 7.1 *Let the function $x \mapsto x'F_0(x)$ be strictly increasing on $[F_0(Y'\mathbf{1}), Q]$. Then there is q^* such that the set of solutions of the system (7.2), (7.3) is contained in the interval with the extremities $(\underline{p}(q^*), q^*)$ and $(\bar{p}(q^*), q^*)$. In particular, if for each q the solution of (7.2) is a unique, then the solution of the system is also unique.*

For the proof and discussion of this result see [6].

7.2 Selling with maximization of goal functionals

In the above model the banks are constrained to sell their illiquid assets in equal proportions and the strategy matrix $U = U(p,q)$ has a specific form. In a natural extension of this model the row $u_i = (u^{i1}, \dots, u^{iK})$ representing the strategy of the bank i is composed by arbitrary functions $u^{im} = u^{im}(p,q)$ on $[0,l] \times [F_0(Y'\mathbf{1}), Q]$ with values in $[0, y^{im}]$ such that $Uq = (l - e - \Pi' p)^+$, i.e. each bank sells the illiquid assets either to cover, without excess, the shortfall or to exhaust all resources.

For a fixed strategy U the equilibrium vector $(p^*, q^*) \in [0,l] \times [F_0(Y'\mathbf{1}), Q]$ defined as the solution of the system (7.4), (7.3) always exists if the function $(p,q) \mapsto U(p,q)$ is continuous (by the Brouwer theorem). More interesting is the game-theoretic setting in which strategies are also elements of equilibrium.

For a given strategy matrix U and i define the maximization problem

$$\Phi^i(v, U) \to \max,$$

over the set

$$\Gamma^i(p,q) := \{v \in \mathbf{R}^K : \; v'q = (y_i q) \wedge (l^i - e^i - (\Pi' p)^i)^+, \; v^m \in [0, y^{im}], \; m = 1, \dots, K\}.$$

It is assumed that each function Φ^i is continuous and does not depend on the row u_i of the matrix U. Economically interesting is the case where Φ^i depends only on the vector $U'\mathbf{1}-u_i'$, i.e. the decision of the bank i is based on the knowledge of the total supply of each illiquid asset by all other banks.

Let us denote by $G^i(p,q,U)$ the set of solutions of the above problem and by $V^i(p,q,U)$ its optimal value.

Define the set-valued mapping Ψ by putting

$$\Psi(p,q,U) := \{(e+Yq+\Pi' p)\wedge l\} \times \{F_0\big(U'\mathbf{1}\big)\} \times \prod_{i\le N} G^i(p,q,U).$$

Theorem 7.2 *Suppose that for each U the functions $v \mapsto \Phi^i(v,U)$ are quasiconcave on $[0,Y'\mathbf{1}]$. Then there exists the triple $(p_*,q_*,U_*) \in \Psi(p_*,q_*,U_*)$.*

Proof. The function $v\mapsto \Phi^i(v,U)$ is continuous and $\Gamma^i(p,q)\neq\emptyset$ is compact. Thus, the set $G^i(p,q,U)$ is nonempty. Moreover, it is closed and convex, being an intersection of all nonempty convex closed subsets $\{v\in\Gamma^i(p,q)\colon \Phi^i(v,U)\ge a\}$ over all $a<V^i(p,q,U)$. Hence, Ψ has nonempty compact convex values and the existence of the equilibrium (p_*,q_*,U_*) follows from the Kakutani fixpoint theorem. The only hypothesis of the latter remaining to verify is that the graph of Ψ is closed. It is sufficient to check that the graph of each set-valued mapping G^i is closed.

So, let $(p_n,q_n,U_n,v_n)\to(\tilde p,\tilde q,\tilde U,\tilde v)$ where $v_n\in G^i(p_n,q_n,U_n)$. We need to prove that $\Phi^i(\tilde v,\tilde U)=V^i(\tilde p,\tilde q,\tilde U)$. Take $\varepsilon>0$. Since a function defined on a compact is uniformly continuous, we have, for all sufficiently large n, that for any v with the components $v^m\in[0,y^{im}]$

$$\Phi^i(v,U_n)-\varepsilon \le \Phi^i(v,\tilde U) \le \Phi^i(v,U_n)+\varepsilon. \tag{7.5}$$

In particular, for $v=v_n$ we obtain that

$$V^i(p_n,q_n,U_n)-\varepsilon \le \Phi^i(v_n,\tilde U) \le V^i(p_n,q_n,U_n)+\varepsilon. \tag{7.6}$$

Taking in (7.5) supremum over v in $\Gamma^i(p,q)$ we get that

$$V^i(p_n,q_n,U_n)-\varepsilon \le \sup_{v\in\Gamma^i(p_n,q_n)} \Phi^i(v,\tilde U) \le V^i(p_n,q_n,U_n)+\varepsilon.$$

Since the market prices q^i are bounded away from zero, $\Gamma^i(p_n,q_n)\to\Gamma^i(\tilde p,\tilde q)$ in the Haussdorf metric. It follows that for sufficiently large n

$$V^i(p_n,q_n,U_n)-2\varepsilon \le \sup_{v\in\Gamma^i(\tilde p,\tilde q)} \Phi^i(v,\tilde U) \le V^i(p_n,q_n,U_n)+2\varepsilon. \tag{7.7}$$

Taking in (7.6) and (7.7) liminf in n, we get that $|\Phi^i(\tilde v,\tilde U)-V^i(\tilde p,\tilde q,\tilde U)|\le 3\varepsilon$ implying the required property. □

Remark. In [8] the goal functionals are $\Phi^i(v,U):=y_iF_0(U'\mathbf{1}-u_i'+v)$. This means that the bank i maximizes the total value of its available illiquid assets calculated in the prices q using the clearing vector p and knowing the total sell of each asset by other banks (note that functional does not depend on u_i). To our opinion, a more natural choice of the goal functional could be $\Phi^i(v,U):=(y_i-v')F_0(U'\mathbf{1}-u_i'+v)$.

8 Appendix. Knaster–Tarski fixpoint theorem

Let X be a set with a partial ordering $\geq$ and let A be its nonempty subset. By definition, $\sup A$ is an element $\bar{x}$ such that $\bar{x} \geq x$ for all $x \in A$ and if y is such that $y \geq x$ for all $x \in A$ then $y \geq \bar{x}$. The definition of $\inf A$ follows the same pattern but with the dual ordering $\leq$. A partially ordered set X is *complete lattice* if for any its nonempty subset A there exist $\inf A$ and $\sup A$.

Theorem 8.1 *Let X be a complete lattice and let $f : X \to X$ be an order-preserving mapping, $L := \{x : \ f(x) \leq x\}$, $U := \{x : \ f(x) \geq x\}$. The set $L \cap U$ of fixed points of f is non-empty and has the smallest and the largest fixed points which are, respectively, $\underline{x} := \inf L$ and $\bar{x} := \sup U$.*

Proof. Note that $L \neq \emptyset$ since it contains the element $\sup X$. Take arbitrary $x \in L$. Then $\underline{x} \leq x$ implying that $f(\underline{x}) \leq f(x) \leq x$. Thus, $f(\underline{x}) \leq \underline{x}$ as $\underline{x}$ is $\inf L$. So, $\underline{x} \in L$. Since $f(L) \subseteq L$, also $f(\underline{x}) \in L$, hence, $\underline{x} \leq f(\underline{x})$, i.e. $\underline{x} = f(\underline{x})$. All fixed points belong to L and, therefore, $\underline{x}$ is the smallest one.

The proof of the statement for the largest fixed point is analogous. $\square$

Corollary 8.2 *Let f_i, $i = 1, 2$, be two order-preserving mappings of a complete lattice $(X, \geq)$ into itself such that $f_2 \geq f_1$. Let $\underline{x}_i := \inf L_i$ and $\bar{x}_i := \sup U_i$ be their smallest and largest fixed points. Then $\underline{x}_2 \geq \underline{x}_1$ and $\bar{x}_2 \geq \bar{x}_1$.*

The claim is obvious because $L_1 = \{x : \ f_1(x) \leq x\} \supseteq \{x : \ f_2(x) \leq x\} = L_2$ and $U_1 = \{x : \ f_1(x) \geq x\} \subseteq \{x : \ f_2(x) \geq x\} = U_2$, see [11].

These general results are applied in this paper to the order intervals $[a, b] \subset \mathbf{R}^d$ with the component-wise ordering, i.e. induced by the cone $\mathbf{R}^d_+$.

Acknowledgements The research is funded by the grant of RSF n° 15-11-30042. The authors expressed their thanks to T. Suzuki for fruitful discussions. The hospitality of Laboratories of quantitative finance of Higher School of Economics and Tokyo Metropolitan University is highly appreciated.

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