Outline

- Leland conjectures and Leland–Lott theorem
- K.–Safarian results
- Pergamenshchikov theorem
- Grennan–Swindle scheme
- Gamys–K. theorem
- Denis–K. theorem
- Denis theorems

Black–Scholes : pricing via replication

Classical model, under martingale measure, call option, maturity $T = 1$, price process S is gBM : $dS_t = \sigma S_t dW_t$.
Let $C(t,x)$ be the solution of the Cauchy problem

$$C_t(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) = 0, \qquad C(1,x) = (x-K)^+.$$

That is

$$C(t,x) = C(t,x,\sigma) = x\Phi(d) - K\Phi(d - \sigma\sqrt{1-t}), \qquad t < 1,$$

where Φ is the Gaussian distribution function with the density φ,

$$d = d(t,x) = \frac{1}{\sigma\sqrt{1-t}} \ln\frac{x}{K} + \frac{1}{2}\sigma\sqrt{1-t}.$$

The pay-off is replicated ($V_1 = (S_1 - K)^+$) by the process

$$V_t = C(0,S_0) + \int_0^t C_x(u,S_u)dS_u = C(t,S_t).$$

Leland approximate replication, *J. of Finance*, (1985)

Value process :

$$V_t^n = \widehat{C}(0, S_0) + \int_0^t \sum_{i=1}^n H^n_{t_{i-1}} I_{]t_{i-1},t_i]}(u)\, dS_u - \sum_{t_i<t} k_n S_{t_i} |H^n_{t_i} - H^n_{t_{i-1}}|,$$

where $H^n_{t_i} = \widehat{C}_x(t_i, S_{t_i})$, $t_i = i/n$, the positive parameter $k_n = k_0 n^{-\alpha}$ is the transaction costs coefficient, and $\widehat{C}(t,x)$ is the solution of the Cauchy problem with σ^2 replaced by

$$\widehat{\sigma}^2 = \sigma^2 + \sigma k_0 n^{1/2-\alpha}\sqrt{8/\pi}.$$

That is $\widehat{C}(t,x) = C(t,x,\widehat{\sigma})$: traders can use their old software ! Note that for $\alpha = 1/2$

$$\widehat{\sigma}^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi} = const.$$

Leland : $V_1^n \to V_1 = (S_1 - K)^+$ *in probability for* $\alpha = 0$ (*wrong !*) *and* $\alpha = 1/2$ (*true*).

Results

Theorem (Lott, $\alpha = 1/2$, 1993 ; K.–Safarian, $0 < \alpha < 1/2$, 1997)

$V_1^n \to (S_1 - K)^+$ *in probability for* $\alpha =]0, 1/2]$.

Theorem (K.–Safarian, 1997)

For $\alpha = 0$ *(i.e. for* $k_n = k_0$*)*

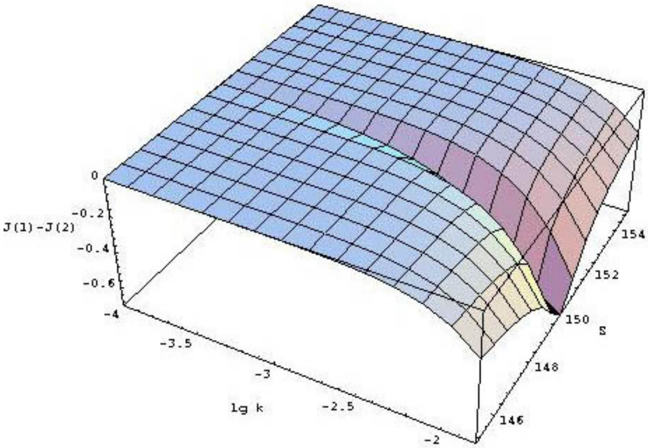
$$V_1^n \to (S_1 - K)^+ + S_1 \wedge K - S_1 F(\ln(S_1/K), k_0) \quad \textit{in probability}$$

where

$$F(y, k_0) := \frac{1}{4} \int_0^\infty \frac{1}{\sqrt{v}} G(y, v, k_0) \exp\left\{ -\frac{v}{2} \left(\frac{y}{v} + \frac{1}{2} \right)^2 \right\} dv,$$

$$G(y, v, k_0) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left| x - \frac{2k_0 y}{\sqrt{2\pi} v} + \frac{k_0}{\sqrt{2\pi}} \right| e^{-x^2/2} dx.$$

Dependence of $J_1 - J_2$ on $k = k_0$ and $S = S_1$ for $K = 150$



Limit theorem

Theorem (Pergamenshchikov, 2003)

Let $k = k_0 > 0$. Then the sequence of random variables

$$\xi_n := n^{1/4}(V_1^n - (S_1 - K)^+ - J_1 + J_2(k_0))$$

where $J_1 := S_1 \wedge K$, $J_2(k_0) := S_1 F(\ln(S_1/K), k_0)$ converges in law to a random variable ξ with a conditionally-Gaussian distribution.

The amplifying factor $n^{1/4}$ was found in the earlier paper[1].

[1] Granditz P., Schachinger W. Leland's approach to option pricing : The evolution of discontinuity. *Mathematical Finance*, **11** (2001), 3.

Grennan–Swindle scheme, Math. Finance, 1996

A generalization of the Leland strategy with $\alpha = 1/2$ (i.e. $k_n = k_0/\sqrt{n}$) to the case of non-uniform revision intervals. Let $f : [0,1] \to [0,1]$ be a function with $f' > 0$, $f(0) = 0$, $f(1) = 1$; $g := f^{-1}$. For fixed n the revision dates are $t_i = g(i/n)$, $1, ..., n$. Example : $g(t) = 1 - (1-t)^\beta$, $\beta \geq 1$. Now

$$\widehat{\sigma}_t^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi} \sqrt{f'(t)}$$

the function $\widehat{C}(t,x)$ is the solution of the Cauchy problem

$$\widehat{C}_t(t,x) + \frac{1}{2}\widehat{\sigma}_t^2 x^2 \widehat{C}_{xx}(t,x) = 0, \qquad \widehat{C}(1,x) = h(x).$$

Put $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t,S_t)$. Then $E(V_1^n - V_1)^2 = A_1(f)n^{-1} + o(n^{-1})$, where the coefficient

$$A_1(f) = \int_0^1 \left[\frac{\sigma^4}{2} \frac{1}{f'(t)} + k_0 \sigma^3 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{f'(t)}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi} \right) \right] \Lambda_t dt.$$

Grennan–Swindle scheme, result

Theorem (Gamys–K., 2007)

Let $h(x) = (x-K)^+$*. Suppose that* $g'', f'' \in C^2([0,1])$ *or* $g(t) = 1-(1-t)^\beta$, $\beta \geq 1$*. Then*

$$E(V_1^n - V_1)^2 = A_1(f)n^{-1} + o(n^{-1}), \quad n \to \infty,$$

$$A_1(f) = \int_0^1 \left[\frac{\sigma^4}{2}\frac{1}{f'(t)} + k_0\sigma^3\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{f'(t)}} + k_0^2\sigma^2\left(1-\frac{2}{\pi}\right)\right]\Lambda_t dt,$$

with $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t,S_t)$*. With the notation* $\rho_t^2 = \int_t^1 \widehat{\sigma}_s^2 ds$,

$$\Lambda_t = \frac{1}{2\pi\rho_t}\frac{K^2}{\sqrt{2\sigma^2 t + \rho_t^2}} \exp\left\{-\frac{\left(\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 t - \frac{1}{2}\rho_t^2\right)^2}{2\sigma^2 t + \rho_t^2}\right\}.$$

Functional limit theorem for Leland–Lott hedging strategy.

Theorem (Denis–K., 2008)

Let $h(x) = (x - K)^+$. Suppose that $g'', f'' \in C^2([0,1])$ or $g(t) = 1 - (1-t)^\beta$, $\beta \geq 1$. Then the distributions of the process $X^n := n^{1/2}(V^n - V)$ in the Skorohod space $\mathcal{D}[0,1]$ converge weakly to the distributions of the diffusion process

$$X_t = \int_0^t F(t, S_t) dW'_t$$

where W' is a Wiener process independent of W and

$$F(t,x) = \left[\frac{\sigma^4}{2}\frac{1}{f'(t)} + k_0 \frac{\sigma^3}{\sqrt{2\pi}\sqrt{f'(t)}} + k_0^2\sigma^2\left(1 - \frac{2}{\pi}\right)\right]^{1/2} \widehat{C}_{xx}(t,x)x^2.$$

Why the proof is difficult ?

We have the representation $V_1^n - V_1 = F_1^n + F_2^n$ where

$$F_1^n = \sigma \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)) S_t dW_t,$$

$$F_2^n = k_0 \sqrt{\frac{2}{\pi}} \sigma \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)} dt - \frac{k_0}{\sqrt{n}} \sum_{i=1}^{n-1} |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})| S_{t_i}.$$

Lemma

Let

$$\tilde{P}_s^{1n} = \frac{1}{2}\sigma^2 \sum_{t_i \le s} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[\Delta t_i - (\Delta W_{t_i})^2\right],$$

$$\tilde{P}_s^{2n} = k_0 \sigma n^{-1/2} \sum_{t_i \le s} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[\sqrt{2/\pi}\sqrt{\Delta t_i} - |\Delta W_{t_i}|\right]$$

Then $nE(\tilde{P}_1^n + \tilde{P}_2^n)^2 \to A_1(f)$ *as* $n \to \infty$.

Comments on the Grennan–Swindle paper

It is not rigorous! The hypothesis that for any $m, n, p \in \mathbb{N}$

$$||\widehat{C}||_{m,n,p} = \sup_{x>0,\ t\in[0,1]} \left[x^m \frac{\partial^{n+p}\widehat{C}(t,x)}{\partial x^n \partial t^p} \right] < \infty$$

is not fulfilled for the call-option with $h(x) = (x-K)^+$ (even for the uniform grid) : explicit formulae show that derivatives of $\widehat{C}(t,x)$ have singularities at the point $(1, K)$. So, the results do not cover practically interesting cases. The obtained formula is used in numerical analysis of the approximate hedging of call-options. Even in such a restricted case the arguments are obscure : the authors do not care about divergence of the integral due to singularities of $1/f'$ not excluded by their assumptions. Neglecting the singularities may lead to an erroneous answer as in Leland's paper.

The Grannan–Swindle paper contains another interesting idea : to minimize the functional $A_1(f)$ with respect to the scale f in a hope to improve the performance of the strategy by an appropriate choice of the revision dates...

Approximate hedging of more general options

(G) : $g'' \in C[0,1[$, such that $g''(t)(1-t)^\lambda$ is bounded for some $\lambda \in [0,1[$.

(H) : $h \in C^2(\mathbf{R}_+)$, $|h''(x)| \le M(1+x^{-\beta})$, $\beta \ge 3/2$.

Theorem (Denis, 2007)

If (G) and (H) hold, then $P\text{-}\lim_n V_1^n = h(S_1) + \varepsilon_\alpha$, where

$$\varepsilon_\alpha = \frac{1}{2}\int_0^\infty \frac{1}{S_1}[\theta_1(x,S_1) - |\theta_1(x,S_1)|]dx, \quad \alpha \in]0,1/2[,$$

$$\varepsilon_{1/2} = \frac{1}{2}\sigma k_0 \sqrt{\frac{8}{\pi}} \int_0^1 \sqrt{f'(t)}\left(\widehat{C}_{xx}(t,S_t) - |\widehat{C}_{xx}(t,S_t)|\right) dt,$$

$$\theta_1(x,S_1) := \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} h'(S_1 e^{\sqrt{x}y + x/2}) y\varphi(y)dy.$$

Theorem (Denis, 2007)

Let $\alpha = 0$. Suppose the h is convex or concave and the assumptions (G) and (H) hold. Then

$$P\text{-}\lim_n V_1^n = h(S_1) + J_1 - J_2(k_0) + \varepsilon_0,$$

$$J_1 = \frac{1}{2} S_1 \int_0^\infty \frac{1}{\sqrt{x}} \theta_1(S_1, x) dx$$

$$\theta_1(S, x) = \frac{1}{\sqrt{x}} \int_{-\infty}^\infty h'(Se^{\sqrt{x}y + x/2}) y \varphi(y) dy,$$

$$J_2(k_0) = \frac{1}{2} S_1 \int_0^\infty j_2(S_1, x) dx$$

$$j_2(S, x) = \theta_1(S, x) \exp\left\{ -\frac{k_0^2 \theta_2^2(S, x)}{\pi \theta_1^2(S, x)} \right\} + k_0 \left[2\Phi\left(k_0 \frac{\sqrt{2}}{\pi} \frac{\theta_2(S, x)}{\theta_1(S, x)} \right) - 1 \right] \theta_2(S, x),$$

$$\theta_2(S, x) = \frac{1}{x} \int_{-\infty}^\infty h'(Se^{\sqrt{x}y + x/2})(-y^2 - \sqrt{x}y + 1) \varphi(y) dy.$$