Lisbonne

Mathematical Aspects of the Theory of Financial Markets with Transaction Costs

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Contributions

Contributions to no-arbitrage criteria

No-arbitrage criteria

- Finite Ω : Kabanov–Stricker (2001).
- Arbitrary Ω : Kabanov–Rásonyi–Stricker (2002), Grigoriev (2005).
- Robust NA : Schachermayer (2004), Kabanov–Rásonyi–Stricker (2003).
- Incomplete information : Bouchard (2007), De Vallière–Kabanov–Stricker (2007).
- Model with bid-ask spread : Jouini–Kallal (1995).

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Contributions to hedging theorems

Hedging theorems for discrete- and continuous time models

- Two-asset continuous time model : Cvitanić–Karatzas (1996).
- Multi-asset models : Kabanov (1999), Kabanov–Last (2002), Delbaen–Kabanov–Valkeila (2002), Campi–Schachermayer (2006).
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Outline

Basic model

Values versus physical units

- There are d assets which we prefer to interpret as currencies. Their quotes are given in units of a certain *numéraire* which may not be a traded security. At time t the quotes are expressed by the vector of prices $S_t = (S_t^1, \dots, S_t^d)$; its components are strictly positive. We assume that $S_0 = \mathbf{1} = (1, ..., 1)$.
- The agent's positions can be described either by the vector of "physical" quantities $\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d)$ or by the vector $V = (V_t^1, \dots, V_t^d)$ of values invested in each asset ; they are related as follows :

$$\widehat{V}_t^i = V_t^i / S_t^i, \quad i \le d.$$

- Formally, $\widehat{V}_t = \phi_t V_t$, where

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d).$$

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Basic model
Dynamics

The portfolio evolution can be described by the initial condition $V_{-0} = v$ (the endowments of the agent when entering the market) and the increments at dates $t \geq 0$:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i,$$

$$B_t^i := \sum_{j=1}^{d} L_t^{ji} - \sum_{j=1}^{d} (1 + \lambda_t^{ij}) L_t^{ij},$$

where $L_t^{ji} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ represents the accumulated net amount transferred from the position j to the position i at the date t ; (ΔL_t^{ij}), interpreted as an "order" matrix, is a control ; (λ_t^{ij}) is the matrix of transaction costs coefficients : $\lambda_t^{ij} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, $\lambda^{ii} = 0$.

Basic model

Dynamics – mathematically, nothing new !

- The portfolio dynamics can be described in more conventional way by a controlled linear difference equation :

$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta B_t^i, \qquad i = 1, ..., d,$$

where Y^i, a "stochastic logarithm" of S^i, is given as follows :

$$\Delta Y_t^i = \frac{\Delta S_t^i}{S_{t-1}^i}, \quad Y_0^i = 1.$$

- We can take ΔB_t as the control. Any $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$ defines $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$ where

$$M_t := \left\{ x \in \mathbf{R}^d : \exists a \in \mathbf{M}_+^d \text{ such that } x^i = \sum_j [(1+\lambda_t^{ij})a^{ij} - a^{ji}] \right\}.$$

A measurable selection arguments show that any increment $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$ is generated by a certain (in general, not unique) order $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$.

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Dynamics in physical units and the Cauchy formula

- The portfolio dynamics in physical units is surprisingly simple and, financially, obvious :

$$\Delta \widehat{V}_t^i = \frac{\Delta B_t^i}{S_t^i}, \qquad i = 1, ..., d.$$

- We can write this as :

$$\Delta \widehat{V}_t = \widehat{\Delta B}_t, \qquad -\widehat{\Delta B}_t \in \widehat{M}_t := \phi_t M_t.$$

- It follows that

$$V_t^i = S_t^i \widehat{V}_t^i = S_t^i \left(v^i + \sum_{s=0}^{t} \frac{\Delta B_s^i}{S_s^i} \right).$$

This is just the Cauchy formula for the solution of the non-homogeneous linear difference equation.

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Solvency cones

- The cone $K_t := M_t + \mathbb{R}^d$ is the solvency region : $x \in K_t$ if and only if one can find a matrix $a \in \mathbf{M}^d_+$ such

$$x^i + \sum_j [a^{ji} - (1 + \lambda_t^{ij})a^{ij}] \geq 0, \; i \leq d.$$

In other words, K_t is the set of portfolios (denominated in units of the numéraire) which can be converted at time t, paying the transactions costs, to portfolios without short positions (i.e. without debts in any asset).

- $\widehat{K}_t = \widehat{M}_t + \mathbb{R}^d_+$ is the solvency cone when the accounting of assets (e.g., currencies) is done in terms of physical units.
- Note that M_t is a polyhedral cone, namely, $M_t = \Psi(\mathbf{M}^d_+)$ where $\Psi : \mathbf{M}^d \to \mathbb{R}^d$ is a linear mapping with

$$[\Psi((a^{ij}))]^i := \sum_j [(1 + \lambda_t^{ij})a^{ij} - a^{ji}].$$

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Solvency cones and their duals

- Generators of $\mathbf{M}_+^d$ are the matrices with all zero entries except a single one equal to unit. Thus,

$$M_t = \mathrm{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ 1 \le i,j \le d\}.$$

Its dual positive cone $M_t^* := \{w:\ wx \ge 0\ \forall\, x \in M_t\}$ is

$$M_t^* = \{w:\ (1+\lambda_t^{ij})w^i - w^j \ge 0,\ 1 \le i,j \le d\}.$$

- The cone K_t is also polyhedral :

$$K_t = \mathrm{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\},$$

and its positive dual is

$$K_t^* = M_t^* \cap \mathbf{R}_+^d = \{w \in \mathbf{R}_+^d:\ (1+\lambda_t^{ij})w^i - w^j \ge 0,\ 1 \le i,j \le d\}.$$

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- Since $\widehat{K}_t = \phi_t K_t$, we have

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- Note that if there is a non-zero transaction costs coefficient λ_t^{ij}, then all vectors e_i belong to $M_t = K_t$.
- The solvency cone K_t can be generated by many matrices Λ_t. Sometimes it is convenient to consider the matrix such that

$$1+\lambda_t^{ij} \le (1+\lambda_t^{ik})(1+\lambda_t^{kj}), \qquad \forall\, i,j,k.$$

The financial interpretation is obvious : an "intelligent" investor will first try all possible chains of transfers from the ith position to the position j and act accordingly to a cheapest one, i.e. as the above property is fulfilled.

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Interpretation of K_0

The linear space $K_t^0 := K_t \cap (-K_t)$ is composed by the positions which can be converted to zero without paying transaction costs and vice versa.

Indeed, let $x \in K_t \cap (-K_t)$. According to definition,

$$
\begin{aligned}
x^i &= \sum_j [(1+\lambda_t^{ij})a^{ij} - a^{ji}] + h^i, \\
-x^i &= \sum_j [(1+\lambda_t^{ij})\tilde{a}^{ij} - \tilde{a}^{ji}] + \tilde{h}^i.
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Summing up, we get that

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\sum_{i=1}^d \sum_{j=1}^d \lambda_t^{ij}(a^{ij} + \tilde{a}^{ij}) + \sum_{i=1}^d (h^i + \tilde{h}^i) = 0.
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Modelling in physical units domain

- The model is given by the adapted matrix-valued bid-ask process $\Pi = (\pi^{ij})$ where $\pi^{ij} > 0$ represents a number of units of the ith asset needed to get in exchange one unit of the jth asset (of course, $\pi^{ii} = 1$). In the literature it is usually assumed that $\pi^{ij} \le \pi^{ik}\pi^{kj}$ (i.e. the investor is "intelligent").
- The solvency region, i.e. the set of $y \in \mathbf{R}^d$ for which one can find $c \in \mathbf{M}^d_+$ such that

$$y^i \ge \sum_j [\pi^{ij}_t(\omega)c^{ij} - c^{ji}], \quad i \le d,$$

is cone $\{\pi^{ij}e_i - e_j,\ e_i,\ 1 \le i, j \le d\}$, i.e. coincides with $\widehat{K}_t$.
- Is this model more general? No. Take any $S_t \in L^0(\widehat{K}^*_t \setminus \{0\}, \mathcal{F}_t)$ and put $\lambda^{ij}_t := \pi^{ij}_t S^i_t / S^j_t - 1$. Then $S^i_t > 0$ and $\lambda^{ij}_t \ge 0$ because $S_t e_i > 0$, $S_t(\pi^{ij}_t e_i - e_j) \ge 0$...

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Model of stock market

- All transactions pass through the money : so the orders are either "buy a stock", or "sell a stock", i.e. they are the vectors $(\Delta L_t^2, ..., \Delta L_t^d)$ and $(\Delta M_t^2, ..., \Delta M_t^d)$.
- The corresponding d-asset dynamics is given by the system

$$\begin{aligned}\Delta V_t^1 &= \sum_{j\geq 2}(1-\mu_t^j)\Delta M_t^j - \sum_{j\geq 2}(1+\lambda_t^j)\Delta L_t^j,\\ \Delta V_t^i &= V_{t-1}^i \Delta Y_t^i + \Delta L_t^i - \Delta M_t^i, \quad i = 2, ..., d.\end{aligned}$$

- $M_t = \operatorname{cone}\{-(1+\lambda_t^j)e_1 + e_j,\ (1-\mu_t^j)e_1 - e_j,\ j = 2, ..., d\}$,

$$K_t = \Big\{x \in \mathbb{R}^d :\ x^1 + \sum_{j\geq 2}^{d}[(1-\mu_t^j)x^j I_{\{x^j>0\}} - (1+\lambda_t^j)x^j I_{\{x^j<0\}}] \geq 0\Big\}.$$

- The model can be imbedded into the model of currency market by choosing sufficiently large transaction costs coefficients for the direct exchange of stocks.

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- The model can be imbedded into the model of currency market by choosing sufficiently large transaction costs coefficients for the direct exchange of stocks.

Model of stock market

- All transactions pass through the money : so the orders are either "buy a stock", or "sell a stock", i.e. they are the vectors $(\Delta L_t^2, ..., \Delta L_t^d)$ and $(\Delta M_t^2, ..., \Delta M_t^d)$.
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Model with a price spread

- This is a model of stock market, i.e. transactions are only buying or selling shares according to two price processes $\bar{S}$ and $\underline{S}$ where $\bar{S}^j \geq \underline{S}^j > 0$, $j = 2, ..., d$. It can be given in terms of a single price (quote) process and transaction cost coefficients. E.g., one can put $S_t := (\bar{S}_t + \underline{S}_t)$ and define $\lambda_t^j := \bar{S}_t^j / S_t^j - 1$, $\mu_t^j := 1 - \underline{S}_t^j / S_t^j$. The absence of arbitrage opportunities means that $R_T \cap L_+^0 = \{0\}$ where the "results" here are terminal values of the money component of the portfolio processes (in our terminology this will correspond to the NA^w-property).
- Historically, the first criterion of absence of arbitrage was obtained for such a model. The Jouini–Kallal theorem claims (under some conditions) that there is no-arbitrage if and only if there exist a probability measure $\tilde{P} \sim P$ and an $\mathbb{R}^{d-1}$-valued $\tilde{P}$-martingale $\tilde{S}$ such that $\underline{S}_t^j \leq \tilde{S}_t^i \leq \bar{S}_t^i$, $i = 2, ..., d$. If $\underline{S} = \bar{S}$, the assertion coincides with the DMW theorem.

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Transactions charging the bank account

- The dynamics is given as follows :

$$\begin{aligned}\Delta V_t^1 &= \sum_{j\geq 2}(\Delta L_t^{j1} - \Delta L_t^{1j}) - \sum_{i,j} \gamma_t^{ij} \Delta L_t^{ij},\\ \Delta V_t^i &= \widehat{V}_{t-1}^i \Delta S_t^i + \sum_j \Delta L_t^{ji} - \sum_j \Delta L_t^{ij}, \quad i = 2, ..., d,\end{aligned}$$

  where $\gamma_t^{ij} \in [0, 1[$, $\gamma^{ii} = 0$.

- For this model, linear and with polyhedral cone constraints on the controls, the solvency cone is again a polyhedral one :

$$K_t = \mathrm{cone}\,\{\gamma^{ij} e_1 + e_i,\ (1+\gamma^{1i})e_1 - e_i,\ (-1+\gamma^{j1})e_1 + e_j,\ e_i,\ i,j \leq d\}.$$

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Outline

Principal problems

Problem 1. What are analogs of no-arbitrage criteria ?

Problem 2. What are analogs of hedging theorem ?

No-arbitrage problem : definitions

- We consider the basic model in the case where Ω is finite and use the Stiemke theorem to get an idea.
- Let R_T be the set of all V_T which are the terminal variables of the processes

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i, \quad V_{-1}^i = 0,$$

$$A_T := R_T - L^0(K_T, \mathcal{F}_T) = R_T - L^0(\mathbb{R}_+, \mathcal{F}_T).$$

- We denote $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ the set of martingales $Z = (Z_t)_{t \le T}$ such that $Z_t \in L^0(\widehat{K}_t^* \setminus \{0\})$ for all t. Elements of $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ are called consistent price systems.
- We define the *strict arbitrage opportunity* as a strategy B such that the terminal value V_T of the portfolio process $V = V^B$ with $V_{-1} = 0$ belongs to $L^0(\mathbb{R}_+^d)$ but is not equal to zero.

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No-arbitrage problem : NA^w for finite Ω

- We say that a model has the *weak no-arbitrage property* (in symbols : NA^w) if it does not admit strict arbitrage opportunities, i.e. $R_T \cap L^0(\mathbb{R}^d_+) = \{0\}$ or, equivalently, $\widehat{R}_T \cap L^0(\mathbb{R}^d_+) = \{0\}$ where $\widehat{R}_T = \phi_T R_T$ is the set of attainable results in physical units.
- Other ("obviously") equivalent conditions :
- $A_T \cap L^0(\mathbb{R}^d_+) = \{0\}$.
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Theorem (Kabanov–Stricker, 1999)

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Proof of the NA^w-criterion for finite Ω

We apply (in the finite-dimensional space $L^0(\mathbb{R}^d, \mathcal{F}_T)$) :

Lemma (Stiemke, modern version)

Let K and R be closed cones in $\mathbb{R}^n$ and K be proper. Then

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

- Take $R = \widehat{R}_T$ and $K = L^0(\mathbb{R}^d_+, \mathcal{F}_T)$. These sets are polyhedral cones. By the lemma $\widehat{R}_T \cap L^0(\mathbb{R}^d_+) = \{0\}$ if and only if there exists η in the interior of $L^0(\mathbb{R}^d_+, \mathcal{F}_T)$ which belongs to $-\widehat{R}^*_T$. This means that the components of η are strictly positive and $E\xi\eta \leq 0$ for all $\xi \in \widehat{R}_T$.
- It remains to note that the martingale $Z_t = E(\eta|\mathcal{F}_t)$ belongs to $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$. For $\zeta \in L^0(\widehat{K}_t, \mathcal{F}_t) \subseteq -\widehat{R}_T + L^0(\mathbb{R}^d_+)$ we have that $EZ_t\zeta = E\eta\zeta \geq 0$. This means that $Z_t \in L^0(\widehat{K}^*_t, \mathcal{F}_t)$.

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Relation with the Harrison–Pliska theorem

- Suppose that $\Lambda = 0$ and the first asset is the numéraire, i.e. $\Delta S^1_t = 0$. Let $\bar{V}_t = \sum_{i \le d} V^i_t$. It follows that

$$\Delta \bar{V}_t = \sum_{i=1}^{d} \widehat{V}^i_{t-1} \Delta S^i_t = H_t \Delta S_t,$$

where $H_t \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. There is a linear relations for the components $\widehat{V}^i_{t-1}$ but it is of no importance : $\Delta S^1_t = 0$. The set of $\bar{V}_T$ is exactly R_T of the model of frictionless market and the classical NA-condition $R_T \cap L^0_+ = \{0\}$ is equivalent to the NA^w-condition.

- If $\Lambda = 0$, then the cone $\widehat{K}^*_t = \mathbb{R}_+ S_t$. The property $Z_t \in L^0(\widehat{K}^*_t, \mathcal{F}_t)$ means that $Z_t = \rho_t S_t$ for some $\rho_t \ge 0$. Thus, $Z \in \mathcal{M}^T_0(\widehat{K}^* \setminus \{0\})$ if and only if there is a martingale $\rho > 0$ such that ρS is a martingale; we may assume that $E\rho_t = 1$.

Relation with the Harrison–Pliska theorem

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No-arbitrage problem : NA^s_T for finite Ω

- A strategy B is a *weak arbitrage opportunity* at time $t \leq T$ if $V^B_t \in K_t$ but $P(V^B_t \notin K^0_t) > 0$ where $K^0_t := K_t \cap (-K_t)$. The absence of such strategies at time t is referred to as the *strict no arbitrage* property NA^s_t :

$$R_t \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K^0_t, \mathcal{F}_t),$$

or, equivalently, in the realm of physical values :

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(b) $A_T \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(K^0_T, \mathcal{F}_T)$;
(c) there is $Z^{(T)} \in \mathcal{M}^T_0(\widehat{K}^* \setminus \{0\})$ with $Z^{(T)}_T \in L^1(\mathrm{ri}\, \widehat{K}^*_T, \mathcal{F}_T)$.

No-arbitrage problem : NA^s for finite Ω

- The proof is based on a generalization of the Stiemke lemma.
- Note that NA^s_T does not imply NA^s_t for $t < T$. In other words, a weak arbitrage opportunities may disappear next day.
- We use the notation NA^s when NA^s_t holds for every $t \le T$ and formulate the following corollary :

Theorem (Kabanov–Stricker, 1999)

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No-arbitrage problem in an abstract setting

- By the experience with models of frictionless markets one may guess that the above no-arbitrage criteria hold true also for arbitrary Ω.

  But not !

  Mathematically, the problem of no-arbitrage for market with transaction costs is very intriguing.

- As we observed, the portfolio dynamics is given by a controlled linear difference equation with conic constrains on the controls. So, it is quite natural to treat the no-arbitrage criteria in the general framework of such equations. The Cauchy formula provides an explicit representation for the solution, corresponding, in financial context, to the dynamics given in the physical units domain. These considerations leads a fairly simple abstract setting.

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No-arbitrage problem in an abstract setting

- We are given a sequence of set-valued mappings $G = (G_t)$ called *$\mathcal{C}$-valued process* specified by a countable sequence of adapted $\mathbb{R}^d$-valued processes $X^n = (X^n_t)$ such that for every t and ω only a finite but non-zero number of $X^n_t(\omega)$ is different from zero and $G_t(\omega) := \operatorname{cone}\{X^n_t(\omega),\ n \in \mathbb{N}\}$, i.e. $G_t(\omega)$ is polyhedral. [Think that there is only a finite number of generators.]
- Let G and $\tilde{G}$ be closed cones. We say that G is *dominated* by $\tilde{G}$ if $G \setminus G^0 \subseteq \operatorname{ri} \tilde{G}$ where $G^0 := G \cap (-G)$. We extend this notion to $\mathcal{C}$-valued processes. It can be formulated in terms of the dual cones : $G \setminus G^0 \subseteq \operatorname{ri} \tilde{G} \ \Leftrightarrow\ \tilde{G}^* \setminus \tilde{G}^{*0} \subseteq \operatorname{ri} G^*$.
If G has an interior (as in the case of financial models where $G_t = \widehat{K}_t \supseteq \mathbb{R}^d$),

$$G \setminus G^0 \subseteq \operatorname{int} \tilde{G} \quad \Leftrightarrow \quad \tilde{G}^* \setminus \{0\} \subseteq \operatorname{ri} G^*.$$

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- Let G be a $\mathcal{C}$-valued process, $A_0^t(G) := A_t(G) := -\sum_{s=0}^t L^0(G_s, \mathcal{F}_s)$.
- We say that G satisfies :
  – *weak no-arbitrage property* NA^w if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(\partial G_t, \mathcal{F}_t) \qquad \forall\, t \le T;$$

  – *strict no-arbitrage property* NA^s if

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  – *robust no-arbitrage property* NA^r if G is dominated by $\tilde{G}$ satisfying NA^w.
- It is an easy exercise to check that if G dominates the constant process $\mathbb{R}_+^d$ then NA^w holds if and only if $A_T(G) \cap L^0(\mathbb{R}_+^d) = \{0\}$.

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No-arbitrage problem in an abstract setting

Theorem (Schachermayer, 2004, KRS, 2003)

Assume that G dominates $\mathbb{R}^d_+$. *Then*

$$NA^r \quad \Leftrightarrow \quad \mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset.$$

Theorem (Penner, 2003)

Assume that $L^0(G^0_s, \mathcal{F}_{s-1}) \subseteq L^0(G^0_{s-1}, \mathcal{F}_{s-1})\ \forall\, s \leq T$. *Then*

$$NA^s \quad \Leftrightarrow \quad \mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset.$$

The hypothesis of the 2nd theorem holds trivially when $G^0 = \{0\}$ (the efficient friction condition in financial context). More interesting, it is fulfilled for the the market model for which the subspace $K^0_t = K_t \cap (-K_t)$ is constant over time (e.g., the transaction costs are constant) and NA^s holds. In such a case NA^r and NA^s coincide.

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No-arbitrage problem in an abstract setting

Grigoriev theotem

Theorem

Let $d = 2$. Then the following conditions are equivalent :
(A) $A_0^T \cap L^0(\mathbf{R}_+^d) = \{0\}$;
(C) $\bar{A}_0^T \cap L^0(\mathbf{R}_+^d) = \{0\}$;
(D) $\mathcal{M}_0^T(G^* \setminus \{0\}) \neq \emptyset$.

Example A two-asset one-period model satisfying NA^w for which A_0^1 is not closed. Let $\Omega = \mathbb{N}$, $\mathcal{F} = 2^\Omega$, $P(k) = 2^{-k}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$. Take $G_0 = \operatorname{cone}\{2e_2 - e_1, e_1 - e_2\}$ and $G_1 = \operatorname{cone}\{2e_1 - e_2, e_2 - e_1\}$. The vector $e_1 + e_2$ belongs to both G_0^* and G_1^* and, hence, the constant process $Z = e_1 + e_2$ is an element of $\mathcal{M}_0^1(G^* \setminus \{0\})$. The random variable ξ with $\xi(k) = k(e_2 - e_1)$ does not belong to the set A_0^1 but lays in the closure of the latter.

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No-arbitrage problem in an abstract setting

Example : NA^w holds but $\mathcal{M}_0^1(G^* \setminus \{0\}) = \emptyset$

A three-dimensional one-period model. Take $G_0^* = \mathbb{R}_+\eta$, $G_1^* = \operatorname{cone}\{\eta_1, \eta_2\}$ where $\eta = (3,1,1)$ and $\eta_1 = (4,1,1)$ are deterministic vectors in $\mathbb{R}^3_+$ while η_2 is a random one with $\eta_2(k) = (2,1,1+1/k)$.
Clearly, $\mathcal{M}_0^1(G^* \setminus \{0\}) = \emptyset$: one cannot find random variables $\alpha, \beta \geq 0$ to meet the conditions $E\alpha = E\beta = 1/2$ and $E\beta\gamma = 0$, where $\gamma(k) = 1/k$, needed to ensure that $EZ_1 = Z_0$.
Let $\xi_0 \in -G_0$ and $\xi_1 \in -L^0(G_1, \mathcal{F})$ be such that $\xi = \xi_0 + \xi_1$ takes values in $\mathbb{R}^3_+$. The latter condition implies that $\eta_1\xi \geq 0$. Since $\eta_1\xi_1 \leq 0$, we have $\eta_1\xi_0 \geq 0$. Also $\eta_2(k)\xi_0 \geq 0$ whatever is k. But

$$\eta_1\xi_0 + \lim_k \eta_2(k)\xi_0 = 2\eta\xi_0 \leq 0$$

and, therefore, both terms in the lhs are zero. So, $\eta_1\xi_0 = 0$. As a result, $\eta_1\xi = \eta_1\xi_2 \leq 0$. With ξ taking values in $\mathbb{R}^3_+$ this is possible only when $\xi = 0$ and NA^w holds.

No-arbitrage problem in an abstract setting

One more example

- Thus, a straightforward generalization of the Grigoriev theorem for an arbitrary $\mathcal{C}$-valued process fails to be true already in dimension three. However, the above counterexample does not exclude that it holds in a narrower class of financial models.
- There is a rather complicated example of four-asset two-period model satisfying NA^s for which $\mathcal{M}_0^2(G^* \setminus \{0\}) = \emptyset$.

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Outline

Hedging theorem for European options

Finite Ω

- The formal description of the convex set of *hedging endowments* (in values or in physical units since we use a convention that at all $S_0^i = 1$) is as follows :

$$\Gamma := \{v \in \mathbb{R}^d : \ \exists\ B \in \mathcal{B} \text{ such that } \ v + V_T^B \succeq_T C\}$$

- It is easy to see that $\Gamma = \{v \in \mathbb{R}^d : \ \widehat{C} \in v + \widehat{A}_0^T\}$.
- We introduce also the closed convex set

$$D := \left\{ v \in \mathbb{R}^d : \ \sup_Z E(Z_T \widehat{C} - Z_0 v) \le 0 \right\}$$

where Z runs the set $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ assumed to be non-empty.

Theorem (K.–Stricker, 2001)

Let Ω be finite and $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\}) \neq \emptyset$. Then $\Gamma = D$.

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Hedging theorem for European options

Finite Ω : proof

- Let $\xi = \sum_{t=0}^{T} \xi_t$ with $\xi_t \in -L^0(\widehat{K}_t, \mathcal{F}_t)$. Then
$$EZ_T\widehat{C} \le EZ_T\Big(v + \sum_{t\le T} \xi_t\Big) = Z_0 v + \sum_{t\le T} EZ_t\xi_t \le Z_0 v.$$
if $Z \in \mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ and the "easy" inclusion $\Gamma \subseteq D$ holds.
- Take now $v \notin \Gamma$. To show that $v \notin D$ it is sufficient to find $Z \in \mathcal{M}_0^T(\widehat{K}^*)$ with $Z_0 v < EZ_T\widehat{C}$. Since $\widehat{C} \notin v + \widehat{A}_0^T$, it can be separated :
$$\sup_{\xi \in v + \widehat{A}_0^T} E\eta\xi < E\eta\widehat{C}$$
for some d-dimensional random variable η. Define a martingale $Z_t := E(\eta|\mathcal{F}_t)$. It follows that $EZ_t\xi_t \ge 0$ for all $\xi_t \in L^0(\widehat{K}_t, \mathcal{F}_t)$ implying that $Z \in \mathcal{M}_0^T(\widehat{K}^*)$. Taking $\xi = v$ and using the martingale property, we get the desired inequality $EZ_0 v < E\eta\widehat{C}$.

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Abstract setting, arbitrary Ω

- We fix a d-dimensional random variable ζ (which correspond in financial context to $\widehat{C}$, the contingent claim expressed in physical units). Define the set

$$\Gamma = \{v \in \mathbb{R}^d : \ \zeta \in v + A_0^T\}.$$

- Let $\mathcal{Z}$ be the set of martingales from $\mathcal{M}_0^T(\mathrm{ri}\, G^*)$ such that $E(Z_T\zeta)^- < \infty$. Put

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Theorem (K.–Rasonyi–Stricker, 2002)

Suppose that $\mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset$. Then $\Gamma = D$.

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Hedging theorem for European options

Abstract setting, arbitrary Ω

- We fix a d-dimensional random variable ζ (which correspond in financial context to $\widehat{C}$, the contingent claim expressed in physical units). Define the set

$$\Gamma = \{v \in \mathbb{R}^d : \ \zeta \in v + A_0^T\}.$$

- Let $\mathcal{Z}$ be the set of martingales from $\mathcal{M}_0^T(\mathrm{ri}\, G^*)$ such that $E(Z_T\zeta)^- < \infty$. Put

$$D := \left\{ v \in \mathbb{R}^d : \ \sup_{Z \in \mathcal{Z}} E(Z_T\zeta - Z_0 v) \le 0 \right\}.$$

Theorem (K.–Rasonyi–Stricker, 2002)

Suppose that $\mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset$. Then $\Gamma = D$.

Hedging theorem for American options
Finite Ω

- Abstract setting : the model is given by $\mathcal{C}$-valued process $G = (G_t)$, $t \geq T$, dominating $\mathbb{R}^d_+$.
- The pay-off process $Y = (Y_t)$ is now $\mathbb{R}^d$-valued.
- we denote by $\mathcal{X}^0$ the set of $X = (X_t)$ with $X_{-1} = 0$ and $\Delta X_t \in -L^0(G_t, \mathcal{F}_t)$ for $t = 0, 1, ..., T$ and put

$$\Gamma := \{v \in \mathbb{R}^d : \ \exists X \in \mathcal{X}^0 \text{ such that } \ v + X_t - Y_t \in G_t \ \forall\, t\}.$$

- We introduce the set $A_0^T(.)$ of hedgeable American claims consisting of all processes Y which can be dominated by a portfolio process with zero initial capital.
- By analogy with the results available for frictionless market and the hedging theorems for European-type options under transaction costs one may guess that

$$\Gamma = \{v \in \mathbb{R}^d : \ E(Z_\tau Y_\tau - Z_0 v) \leq 0 \ \forall Z \in \mathcal{M}(G^*), \ \tau \in \mathcal{T}\}.$$

Surprisingly, it is not true.

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Hedging theorem for American options

Finite Ω : a theorem

- To formulate the correct result we introduce the notation

$$\bar{Z}_t := \sum_{r=t}^{T} E(Z_r|\mathcal{F}_t).$$

- Define the set of adapted bounded processes

$$\mathcal{Z}(G^*, P) := \{Z : \ Z_t, \bar{Z}_t \in L^\infty(G_t^*, \mathcal{F}_t), \ t = 0, 1, ..., T\}.$$

- Clearly, all bounded martingales from $\mathcal{M}(G^*, P)$ belongs to $\mathcal{Z}(G^*, P)$.

Theorem (Bouchard–Temam, 2005)

Suppose that Ω is finite. Then

$$\Gamma = \left\{ v \in \mathbb{R}^d : \ E \sum_t Z_t Y_- \bar{Z}_0 v \le 0 \ \ \forall Z \in \mathcal{Z}(G^*, P) \right\}.$$

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