Lisbonne

Mathematical Aspects of the Theory of Financial Markets with Transaction Costs

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No arbitrage criteria for frictionless markets (discrete-time)

- Finite Ω : Harrison–Pliska theorem (1981).
- Arbitrary Ω : Dalang–Morton–Willinger theorem (1990).
- Further contributions : Schachermayer, Kabanov–Kramkov, Rogers, Jacod–Shiryaev, Kabanov–Stricker...
- Incomplete information : Kabanov–Stricker (2006)
- Infinite time horizon : Schachermayer (1994)

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Classical theory

1. Classical model
   - Harrison-Pliska theorem
   - Dalang–Morton–Willinger theorem : FTAP
2. Ramifications
   - Restricted information
   - Infinite horizon
3. Hedging theorems

Outline

1. Classical model
   - Harrison-Pliska theorem
   - Dalang–Morton–Willinger theorem : FTAP
2. Ramifications
   - Restricted information
   - Infinite horizon
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Model

- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\dots,T}$ ("history").
- A price process $S = (S_t^1, ..., S_t^d)$, d-dimensional, adapted : S_t is $\mathcal{F}_t$-measurable.
- $S_t^1 = 1$ for all t : the first traded asset is the *numéraire*, say, "bank account". Thus, $\Delta S_t^1 = S_t^1 - S_{t-1}^1 = 0$.
- The value process of a self-financing portfolio with zero initial capital : $V = H \cdot S$ where

$$H \cdot S_t = \sum_{u \le t} H_u \Delta S_u = \sum_{u \le t} \left[H_u^1 \Delta S_u^1 + \sum_{i \ge 2} H_u^i \Delta S_u^i \right]$$

(notation due to P.-A. Meyer). The process $H = (H_t)$ (a strategy) is predictable : H_t is $\mathcal{F}_{t-1}$-measurable, H_t^i, $i \ge 2$, are holdings in stocks. Attention with the interpretation of H_t^1 !

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NA property

- A strategy H is an *arbitrage opportunity* if $H \cdot S_T \geq 0$ and $P(H \cdot S_T > 0) > 0$.
- The model has the *no-arbitrage* property if such H do not exist.
- Equivalently, the NA-property means that

$$R_T \cap L^0_+ = \{0\}$$

where $R_T := \{H \cdot S_T : \ H \text{ is predictable}\}$ is the set of "results" and L^0_+ is the set of non-negative random variables.
- Let $A_T := R_T - L^0_+$ be the set of "results with free disposal" (A_T can be interpreted also as the set of *hedgeable claims*). It is easily seen that the NA-property holds if and only if $A_T \cap L^0_+ = \{0\}$.

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Harrison–Pliska theorem

Formulation

Theorem (Harrison–Pliska (1981))

Suppose that Ω is finite. Then the NA property holds if and only if there is a probability measure $\tilde{P} \sim P$ such that S is a $\tilde{P}$ martingale.

Theorem (Dalang–Morton–Willinger (1990), short version)

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Harrison–Pliska theorem : proof

Theorem (Harrison–Pliska)

Ω is finite. Then $A_T \cap L^0_+ = \{0\} \Leftrightarrow \exists\, \tilde P \sim P$ such that $S \in \mathcal{M}(\tilde P)$.

▷ *Proof* :

- If $S \in \mathcal{M}(\tilde P)$, then $\tilde E H \cdot S_T = 0$. If $H \cdot S_T \geq 0$, then $H \cdot S_T = 0$ $\tilde P$-a.s., hence, P-a.s. That is $R_T \cap L^0_+ = \{0\}$.
- Let $\Omega = \{\omega_1, \dots, \omega_N\}$, $P(\{\omega_i\}) > 0$. The space L^0 with $\langle \xi, \eta \rangle = E\xi\eta$ is Euclidean, A_T is a polyhedral cone, hence, closed. If $A_T \cap L^0_+ = \{0\}$, we can separate A_T and $I_{\{\omega_i\}}$ by a hyperplane, i.e. there is η_i such that

$$\sup_{\xi \in A_T} E\eta_i\xi < E\eta_i I_{\{\omega_i\}}.$$

Since $-L^0_+ \subseteq A_T$, it follows that $\eta_i \geq 0$, $\sup \ldots = 0$, and $\eta_i(\omega_i) > 0$. Thus, $\eta := \sum \eta_i > 0$ and $\eta / E\eta$ is the density $d\tilde P/dP$ of a measure such that $\tilde E\xi \leq 0$ for all $\xi \in R_T$.

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Harrison–Pliska theorem and convex analysis

Facts from convex analysis

- K is a *cone* if it is convex and $\lambda K = K$ for all $\lambda > 0$.
- A cone K defines the partial ordering : $x \geq_K y$ if $x - y \in K$.
- A closed cone K is called *proper* if $K^0 := K \cap (-K) = \{0\}$.
- $\mathrm{cone}\, C$ is the set of all conic combinations of elements of C.
- Let K be a cone in $\mathbb{R}^n$. Its *dual positive cone* $K^* := \{z \in \mathbb{R}^n : \ zx \geq 0 \ \forall x \in K\}$ is closed.
- $\mathrm{int}\, K$ is the interior of K.
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Polyhedral cones

- A cone K is *polyhedral* if it is the intersection of a finite number of half-spaces $\{x : p_i x \geq 0\}$, $p_i \in \mathbb{R}^n$, $i = 1, ..., N$.

Theorem (Farkas–Minkowski–Weyl)

A cone is polyhedral if and only if it is finitely generated.

- Intuitively obvious, but not easy to prove. Useful !
- If K_1, K_2 are polyhedral cones, then $K_1 + K_2$ is also polyhedral.

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Polyhedral cones

- A cone K is *polyhedral* if it is the intersection of a finite number of half-spaces $\{x : \ p_i x \geq 0\}$, $p_i \in \mathbb{R}^n$, $i = 1, ..., N$.

Theorem (Farkas–Minkowski–Weyl)

A cone is polyhedral if and only if it is finitely generated.

- Intuitively obvious, but not easy to prove. Useful !
- If K_1, K_2 are polyhedral cones, then $K_1 + K_2$ is also polyhedral.

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Stiemke lemma

Lemma (Stiemke, modern version)

Let K and R be closed cones in $\mathbb{R}^n$ and K be proper. Then

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

▷ *Proof* :

($\Leftarrow$) The existence of w such that $wx \leq 0$ for all $x \in R$ and $wy > 0$ for all y in $K \setminus \{0\}$ implies that $R \cap (K \setminus \{0\}) = \emptyset$.

($\Rightarrow$) Let C be a convex compact set such that $0 \notin C$ and $K = \operatorname{cone} C$. By the separation theorem (one set is closed and another is compact) there is a non-zero $z \in \mathbb{R}^n$ such that

$$\sup_{x \in R} zx < \inf_{y \in C} zy.$$

Since R is a cone, the sup $\ldots = 0$, hence $z \in -R^*$ and, also, $zy > 0$ for all $y \in C$, so for all $z \in K$, $z \neq 0$, and $z \in \operatorname{int} K$.

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Stiemke lemma implies the HP-theorem

Lemma (Stiemke, modern version (repeated))

Let K and R be closed cones in $\mathbb{R}^n$ and K be proper. Then

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

Take $R = R_T$ and $K = L^0_+$. Then $K^* = L^0_+$. An element η of $(-R^*) \cap \operatorname{int} K^*$ is a strictly positive random variable and $\eta/E\eta$ is a density of "separating" probability measure : $\tilde{E}\xi \le 0$ for all $\xi \in R_T$, hence, $\tilde{E}\xi = 0$ for all $\xi \in R_T$. The novelty in the HP-theorem is just the remark that a separating measure is a martingale one.

Lemma (Stiemke, 1915)

Let $K = \mathbb{R}^n_+$ and $R = \{y \in \mathbb{R}^n : \ y = Bx, \ x \in \mathbb{R}^d\}$ where B is a linear mapping. Then :
either there is $x \in \mathbb{R}^d$ such that $Bx \ge_K 0$ and $Bx \neq 0$ or there is $y \in \mathbb{R}^n$ with strictly positive components such that $B^*y = 0$.

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NA criteria for arbitrary Ω

Theorem (Dalang–Morton–Willinger, 1990, extended version)

The following conditions are equivalent :

(a) $A_T \cap L^0_+ = \{0\}$ *(NA condition)*;

(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$ *(closure in* L^0*)*;

(c) $\bar{A}_T \cap L^0_+ = \{0\}$;

(d) *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}$;

(e) *there is a bounded process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}$;

(f) *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}_{loc}$;

(g) $\{\eta \Delta S_t : \ \eta \in L^0(\mathcal{F}_{t-1})\} \cap L^0_+ = \{0\}$ *for all* $t \le T$ *(NA for 1-step models).*

$S \in \mathcal{M}(\tilde{P})$ *if and only if* $\rho S \in \mathcal{M}(P)$ *where* $\rho_t = E(\rho_T|\mathcal{F}_t)$.

(d′) *there is* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}(\tilde{P})$;

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Auxiliary results

Two simple lemmas

Lemma (Engelbert, von Weizsäcker)

Let $\eta^n \in L^0(\mathbb{R}^d)$ be such that $\underline{\eta} := \liminf |\eta^n| < \infty$. Then there is a strictly increasing sequence of integer-valued random variables (τ_k) such that the sequence of η^{τ_k} converges a.s.

Idea of the proof : in the scalar case we take
$\tau_k := \inf\{n > \tau_{k-1} : \ |\eta^n - \liminf \eta^n| \le k^{-1}\}$, $\tau_0 = 0$.

Lemma (Grigoriev, 2004)

Let $\mathcal{G} = \{\Gamma_\alpha\}$ be a family of measurable sets such any measurable non-null set Γ has the non-null intersection with an element of $\mathcal{G}$. Then there is an at most countable subfamily of sets $\{\Gamma_{\alpha_i}\}$ which union is of full measure.

We may assume wlg that $\mathcal{G}$ is stable under countable unions. Then an element with maximal probability exists and is of full measure.

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Kreps–Yan theorem

Theorem (Kreps, Yan, 1980)

Let $\mathcal{C}$ be a closed convex cone in L^1 such that $-L^1_+ \subseteq \mathcal{C}$ and $\mathcal{C} \cap L^1_+ = \{0\}$. Then there is $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that $\tilde{E}\xi \le 0$ for all $\xi \in \mathcal{C}$.

Proof. By the Hahn–Banach theorem any non-zero $\alpha \in L^1_+$ can be separated from $\mathcal{C}$: there is $\eta_\alpha \in L^\infty$, $||\eta_\alpha||_\infty = 1$, such that

$$\sup_{\xi \in \mathcal{C}} E\eta_\alpha \xi < E\eta_\alpha \alpha.$$

Then $\eta_\alpha \ge 0$, $\sup \ldots = 0$, and $E\eta_\alpha\alpha > 0$. The latter inequality ensures that the family of sets $\Gamma_\alpha := \{\eta_\alpha > 0\}$ satisfies the assumption of the lemma ($E\eta_{I_\Gamma} I_\Gamma > 0$ if $I_\Gamma \ne 0$). Thus, for a certain sequence of indices $\eta := \sum 2^{-i}\eta_{\alpha_i} > 0$ a.s. and we take $\tilde{P} := \eta P$.

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DMW-theorem : proofs of "non-trivial" implications

(c) $\bar{A}_T \cap L^0_+ = \{0\}$;
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DMW-theorem : proof of the "difficult" implication

$A_T \cap L^0_+ = \{0\} \ \Rightarrow \ A_T = \bar{A}_T$ (closure in L^0).

We consider only the case $T = 1$.
Let $H_1^n \Delta S_1 - r^n \to \zeta$ where $H_1^n \in L^0(\mathbb{R}^b, \mathcal{F}_0)$, $r^n \in L^0_+$.
The claim is : $\zeta = H_1 \Delta S_1 - r$ where $H_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0)$, $r \in L^0_+$.
We represent (H_1^n) as the infinite matrix

$$\mathbf{H}_1 := \begin{bmatrix} H_1^{11} & H_1^{21} & \dots & \dots & H_1^{n1} & \dots \\ H_1^{12} & H_1^{22} & \dots & \dots & H_1^{n2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ H_1^{1d} & H_1^{2d} & \dots & \dots & H_1^{nd} & \dots \end{bmatrix}.$$

Suppose that the claim holds when $\mathbf{H}_1$ has, for each ω m zero lines. We show that it holds also when $\mathbf{H}_1$ has $m-1$ zero lines.
Let $\Omega_i \in \mathcal{F}_0$ form a finite partition of Ω. An important (but obvious) observation : we may argue on each Ω_i separately.

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$A_T \cap L^0_+ = \{0\} \;\Rightarrow\; A_T = \bar{A}_T$ (closure in L^0).

We consider only the case $T = 1$.
Let $H^n_1 \Delta S_1 - r^n \to \zeta$ where $H^n_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0)$, $r^n \in L^0_+$.
The claim is : $\zeta = H_1 \Delta S_1 - r$ where $H_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0)$, $r \in L^0_+$.
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$$\mathbf{H}_1 := \begin{bmatrix} H_1^{11} & H_1^{21} & \dots & \dots & H_1^{n1} & \dots \\ H_1^{12} & H_1^{22} & \dots & \dots & H_1^{n2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ H_1^{1d} & H_1^{2d} & \dots & \dots & H_1^{nd} & \dots \end{bmatrix}.$$

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Let $\underline{H}_1 := \liminf |H_1^n|$. On $\Omega_1 := \{\underline{H}_1 < \infty\}$ by the lemma on subsequences, we find a strictly increasing sequence of $\mathcal{F}_0$-measurable r.v. τ_k such that $H_1^{\tau_k}$ converges to some H_1 ; automatically, r^{τ_k} converges to some $r \geq 0$ and we conclude.

On $\Omega_2 := \{\underline{H}_1 = \infty\}$ we put $G_1^n := H_1^n/|H_1^n|$ and $h_1^n := r_1^n/|H_1^n|$. Then $G_1^n \Delta S_1 - h_1^n \to 0$ a.s. By the lemma we find $\mathcal{F}_0$-measurable τ_k such that $G_1^{\tau_k}(\omega)$ converges to some $\tilde{G}_1$. It follows that $\tilde{G}_1 \Delta S_1 = \tilde{h}_1 \geq 0$. Because of the NA-property, $\tilde{G}_1 \Delta S_1 = 0$. As $\tilde{G}_1(\omega) \neq 0$, there exists a partition of Ω_2 into d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $\tilde{G}_1^i \neq 0$ on Ω_2^i.
Define $\bar{H}_1^n := H_1^n - \beta^n \tilde{G}_1$ where $\beta^n := H_1^{ni}/\tilde{G}_1^i$ on Ω_2^i. Then $\bar{H}_1^n \Delta S_1 = H_1^n \Delta S_1$ on Ω_2. The matrix $\bar{\mathbf{H}}_1$ has, for each $\omega \in \Omega_2$, at least m zero lines : our operations did not affect the zero lines of $\mathbf{H}_1$ and a new one has appeared, namely, the ith one on Ω_2^i. We conclude by the induction hypothesis.

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2 Ramifications
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3 Hedging theorems

NA-criteria under restricted information

We are given a filtration $\mathbf{G} = (\mathcal{G}_t)_{t \le T}$ with $\mathcal{G}_t \subseteq \mathcal{F}_t$. The strategies are predictable with respect to $\mathbf{G}$, i.e. $H_{t-1} \in L^0(\mathcal{G}_t)$, a situation when the portfolios are revised on the basis of restricted information, e.g., due to a delay. We define the sets R_T, A_T and give a definition of the arbitrage which, in these symbols, looks exactly as (a) before and we can list the corresponding necessary and sufficient conditions. Notation : $X_t^o := E(X_t|\mathcal{G}_t)$.

Theorem (Kabanov–Stricker, 2006)

The following properties are equivalent :
(a) $A_T \cap L^0_+ = \{0\}$ *(NA condition)* ;
(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$;
(c) $\bar{A}_T \cap L^0_+ = \{0\}$;
(d) *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *with* $(\rho S)^o \in \mathcal{M}(\mathbf{G})$;
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No-Free-Lunch criteria for infinite horizon (Schachermayer)

- $R_\infty := \cup_{T\in\mathbb{N}} R_T$, $A_\infty := R_\infty - L^0_+$.
- *NA-property* : $R_\infty \cap L^0_+ = \{0\}$ (or $A_\infty \cap L^0_+ = \{0\}$).
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NFL holds if and only if there is $P' \sim P$ such that $S \in \mathcal{M}_{loc}(P')$.

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Any L^1-neighborhood of a separating measure contains a measure P' under which S is a local martingale.

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Hedging of European options

- Let $\xi \in L^0(\mathcal{F}_T)$. Define the set of *hedging endowments*

$$\Gamma := \Gamma(\xi) := \{x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S_T \geq \xi\},$$

i.e., Γ is the set of capitals starting from which we can super-replicate the pay-off of *European option* with maturity T by the terminal value of a self-financing portfolio.

- Let $\mathcal{Q}^a$, $\mathcal{Q}^e$ denote the sets of absolute continuous and equivalent martingale measures and let $\mathcal{Z}^a$, $\mathcal{Z}^e$ denote the corresponding sets of density processes.

Theorem

Suppose that NA holds, i.e. $\mathcal{Q}^e \neq \emptyset$. Suppose that $\xi \geq 0$ and $E_Q\xi < \infty$ for every $Q \in \mathcal{Q}^e$. Then $\Gamma = D$ where

$$D := [\bar{x}, \infty[= \{x : \; x \geq E\rho_T\xi \text{ for all } \rho \in \mathcal{Z}^e\}.$$

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Optional decomposition

Theorem (Kramkov, 1996, Föllmer–Kabanov, 1998)

Suppose that $\mathcal{Q}^e \neq \emptyset$. Let $X \geq 0$ be a process which is a supermartingale with respect $Q \in \mathcal{Q}^e$. Then there are a strategy H and an increasing process A such that $X = X_0 + H \cdot S - A$.

Proposition (El Karoui)

Suppose that $Q^e \neq \emptyset$. Let $\xi \in L^0_+$ be such that $\sup_{Q \in \mathcal{Q}^e} E_Q \xi < \infty$. Then the process $X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^e} E_Q(\xi | \mathcal{F}_t)$ is a supermartingale with respect to every $Q \in \mathcal{Q}^e$.

Proof of the hedging theorem. The inclusion $\Gamma \subseteq [\bar{x}, \infty[$ is obvious : if $x + H \cdot S_T \geq \xi$ then $x \geq E_Q \xi$ for every $Q \in \mathcal{Q}^e$. To show the opposite one we suppose that $\sup_{Q \in \mathcal{Q}} E_Q \xi < \infty$ (otherwise both sets are empty). Applying the ODT we get that $X = \bar{x} + H \cdot S - A$. Since $\bar{x} + H \cdot S_T \geq X_T = \xi$, the result follows.

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- For the American-type option the exercise date τ is a stopping time ($\leq T$) and the pay-off is Y_τ, the value at τ of an adapted process Y. The description of the pay-off process $Y = (Y_t)$ is a clause of the contract (as well as the final maturity date T).
- Define the set of initial capitals starting from which we can run a self-financing portfolio which values dominate the pay-off :

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Hedging of American options : the proof

It is based on the optional decomposition theorem applied to the following result where $\mathcal{T}_t$ denotes the set of stopping times $\tau \geq t$.

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