Mathematical Aspects of the Theory of Financial Markets with Transaction Costs

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No arbitrage criteria for frictionless markets (discrete-time)

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- Infinite time horizon : Schachermayer (1994)
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Classical theory

1. Classical model
   - Harrison-Pliska theorem
   - Dalang–Morton–Willinger theorem: FTAP

2. Ramifications
   - Restricted information
   - Infinite horizon

3. Hedging theorems
Outline

1. Classical model
   - Harrison-Pliska theorem
   - Dalang–Morton–Willinger theorem: FTAP

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   - Restricted information
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Model

- A probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t=0,1,...,T}\) ("history").
- A price process \(S = (S^1_t, ..., S^d_t)\), \(d\)-dimensional, adapted: \(S_t\) is \(\mathcal{F}_t\)-measurable.
- \(S^1_t = 1\) for all \(t\): the first traded asset is the numéraire, say, "bank account". Thus, \(\Delta S^1_t = S^1_t - S^1_{t-1} = 0\).
- The value process of a self-financing portfolio with zero initial capital: \(V = H \cdot S\) where

\[
H \cdot S_t = \sum_{u \leq t} H_u \Delta S_u = \sum_{u \leq t} \left[ H^1_u \Delta S^1_u + \sum_{i \geq 2} H^i_u \Delta S^i_u \right]
\]

(notation due to P.-A. Meyer). The process \(H = (H_t)\) (a strategy) is predictable: \(H_t\) is \(\mathcal{F}_{t-1}\)-measurable, \(H^i_t, i \geq 2\), are holdings in stocks. Attention with the interpretation of \(H^1_t\)!
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A strategy $H$ is an *arbitrage opportunity* if $H \cdot S_T \geq 0$ and $P(H \cdot S_T > 0) > 0$.

The model has the *no-arbitrage* property if such $H$ do not exist.

Equivalently, the NA-property means that

$$ R_T \cap L_0^+ = \{0\} $$

where $R_T := \{H \cdot S_T : H \text{ is predictable}\}$ is the set of “results” and $L_0^+$ is the set of non-negative random variables.

Let $A_T := R_T - L_0^+$ be the set of “results with free disposal” ($A_T$ can be interpreted also as the set of *hedgeable claims*). It is easily seen that the NA-property holds if and only if $A_T \cap L_0^+ = \{0\}$. 
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Harrison–Pliska theorem
Formulation

Theorem (Harrison–Pliska (1981))
Suppose that $\Omega$ is finite. Then the NA property holds if and only if there is a probability measure $\tilde{P} \sim P$ such that $S$ is a $\tilde{P}$ martingale.

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The NA property holds if and only if there is a probability measure $\tilde{P} \sim P$ such that $S$ is a $\tilde{P}$ martingale.

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Harrison–Pliska theorem : proof

**Theorem (Harrison–Pliska)**

Ω is finite. Then \( A_T \cap L_0^+ = \{0\} \iff \exists \tilde{P} \sim P \) such that \( S \in \mathcal{M}(\tilde{P}) \).

**Proof :**

- If \( S \in \mathcal{M}(\tilde{P}) \), then \( \tilde{E}H \cdot S_T = 0 \). If \( H \cdot S_T \geq 0 \), then \( H \cdot S_T = 0 \) \( \tilde{P} \)-a.s., hence, \( P \)-a.s. That is \( R_T \cap L_0^+ = \{0\} \).

- Let \( \Omega = \{\omega_1, ..., \omega_N\} \), \( P(\{\omega_i\}) > 0 \). The space \( L_0^+ \) with \( \langle \xi, \eta \rangle = E\xi\eta \) is Euclidean, \( A_T \) is a polyhedral cone, hence, closed. If \( A_T \cap L_0^+ = \{0\} \), we can separate \( A_T \) and \( I_{\{\omega_i\}} \) by a hyperplane, i.e. there is \( \eta_i \) such that

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\sup_{\xi \in A_T} E\eta_i \xi < E\eta_i I_{\{\omega_i\}}.
\]

Since \( -L_0^+ \subseteq A_T \), it follows that \( \eta_i \geq 0 \), \( \sup \ldots = 0 \), and \( \eta_i(\omega_i) > 0 \). Thus, \( \eta := \sum \eta_i > 0 \) and \( \eta/E\eta \) is the density \( d\tilde{P}/dP \) of a measure such that \( \tilde{E}\xi \leq 0 \) for all \( \xi \in R_T \).
Harrison–Pliska theorem: proof

**Theorem (Harrison–Pliska)**

\( \Omega \) is finite. Then \( A_T \cap L_0^+ = \{0\} \Leftrightarrow \exists \tilde{P} \sim P \text{ such that } S \in \mathcal{M}(\tilde{P}). \)

\[ \text{Proof:} \]

- If \( S \in \mathcal{M}(\tilde{P}) \), then \( \tilde{E}H \cdot S_T = 0 \). If \( H \cdot S_T \geq 0 \), then \( H \cdot S_T = 0 \) \( \tilde{P} \)-a.s., hence, \( P \)-a.s. That is \( R_T \cap L_0^+ = \{0\} \).
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Harrison–Pliska theorem and convex analysis
Facts from convex analysis

- **$K$ is a cone** if it is convex and $\lambda K = K$ for all $\lambda > 0$.
- A cone $K$ defines the partial ordering: $x \geq_K y$ if $x - y \in K$.
- A closed cone $K$ is called **proper** if $K^0 := K \cap (-K) = \{0\}$.
- Cone $C$ is the set of all conic combinations of elements of $C$.
- Let $K$ be a cone in $\mathbb{R}^n$. Its **dual positive cone** $K^* := \{z \in \mathbb{R}^n : zx \geq 0 \ \forall x \in K\}$ is closed.
- $\text{int} \ K$ is the interior of $K$.
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- A **closed cone $K \subseteq \mathbb{R}^n$ is proper** if and only if there is a compact convex set $C$ such that $0 \notin C$ and $K = \text{cone} \ C$.
  - One can take $C = \text{conv} (K \cap \{x \in \mathbb{R}^n : |x| = 1\})$.
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### Facts from convex analysis

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  If \( K \) is proper then \( \text{int} \ K^* = \{w : wx > 0 \ \forall x \in K, \ x \neq 0\} \).
A cone $K$ is *polyhedral* if it is the intersection of a finite number of half-spaces $\{x : p_i x \geq 0\}, p_i \in \mathbb{R}^n, i = 1, \ldots, N$.

**Theorem (Farkas–Minkowski–Weyl)**

A cone is polyhedral if and only if it is finitely generated.

- Intuitively obvious, but not easy to prove. Useful!
- If $K_1, K_2$ are polyhedral cones, then $K_1 + K_2$ is also polyhedral.
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Harrison–Pliska theorem and convex analysis

Stiemke lemma

Lemma (Stiemke, modern version)

Let $K$ and $R$ be closed cones in $\mathbb{R}^n$ and $K$ be proper. Then

$$R \cap K = \{0\} \iff (-R^*) \cap \text{int } K^* \neq \emptyset.$$

▷ Proof:

($\Leftarrow$) The existence of $w$ such that $wx \leq 0$ for all $x \in R$ and $wy > 0$ for all $y$ in $K \setminus \{0\}$ implies that $R \cap (K \setminus \{0\}) = \emptyset$.

($\Rightarrow$) Let $C$ be a convex compact set such that $0 \notin C$ and $K = \text{cone } C$. By the separation theorem (one set is closed and another is compact) there is a non-zero $z \in \mathbb{R}^n$ such that

$$\sup_{x \in R} zx < \inf_{y \in C} zy.$$

Since $R$ is a cone, the sup $\ldots = 0$, hence $z \in -R^*$ and, also, $zy > 0$ for all $y \in C$, so for all $z \in K$, $z \neq 0$, and $z \in \text{int } K$. 
Harrison–Pliska theorem and convex analysis
Stiemke lemma implies the HP-theorem

**Lemma (Stiemke, modern version (repeated))**

Let $K$ and $R$ be closed cones in $\mathbb{R}^n$ and $K$ be proper. Then

$$R \cap K = \{0\} \iff (-R^*) \cap \text{int } K^* \neq \emptyset.$$  

Take $R = R_T$ and $K = L_0^\infty$. Then $K^* = L_0^\infty$. An element $\eta$ of $(-R^*) \cap \text{int } K^*$ is a strictly positive random variable and $\eta/E\eta$ is a density of “separating” probability measure: $\tilde{E}\xi \leq 0$ for all $\xi \in R_T$, hence, $\tilde{E}\xi = 0$ for all $\xi \in R_T$. The novelty in the HP-theorem is just the remark that a separating measure is a martingale one.

**Lemma (Stiemke, 1915)**

Let $K = \mathbb{R}_+^n$ and $R = \{y \in \mathbb{R}^n : y = Bx, x \in \mathbb{R}^d\}$ where $B$ is a linear mapping. Then:

- either there is $x \in \mathbb{R}^d$ such that $Bx \geq_K 0$ and $Bx \neq 0$ or there is $y \in \mathbb{R}^n$ with strictly positive components such that $B^*y = 0$. 

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Financial markets with transaction costs.
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The following conditions are equivalent:

(a) $A_T \cap L^0_+ = \{0\}$ (NA condition);
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(d) there is a process $\rho \in \mathcal{M}$, $\rho > 0$, such that $\rho S \in \mathcal{M}$;
(e) there is a bounded process $\rho \in \mathcal{M}$, $\rho > 0$, such that $\rho S \in \mathcal{M}$;
(f) there is a process $\rho \in \mathcal{M}$, $\rho > 0$, such that $\rho S \in \mathcal{M}_{loc}$;
(g) $\{\eta \Delta S_t : \eta \in L^0(\mathcal{F}_{t-1})\} \cap L^0_+ = \{0\}$ for all $t \leq T$ (NA for 1-step models).

$S \in \mathcal{M}(\tilde{P})$ if and only if $\rho S \in \mathcal{M}(P)$ where $\rho_t = E(\rho_T|\mathcal{F}_t)$.

(d’) there is $\tilde{P} \sim P$ such that $S \in \mathcal{M}(\tilde{P})$;
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NA criteria for arbitrary $\Omega$

**Theorem (Dalang–Morton–Willinger, 1990, extended version)**

The following conditions are equivalent:

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Auxiliary results
Two simple lemmas

**Lemma (Engelbert, von Weizsäcker)**

Let $\eta^n \in L^0(\mathbb{R}^d)$ be such that $\underline{\eta} := \lim \inf |\eta^n| < \infty$. Then there is a strictly increasing sequence of integer-valued random variables $(\tau_k)$ such that the sequence of $\eta^{\tau_k}$ converges a.s.

Idea of the proof: in the scalar case we take $\tau_k := \inf\{n > \tau_{k-1} : |\eta^n - \lim \inf \eta^n| \leq k^{-1}\}$, $\tau_0 = 0$.

**Lemma (Grigoriev, 2004)**

Let $\mathcal{G} = \{\Gamma_\alpha\}$ be a family of measurable sets such any measurable non-null set $\Gamma$ has the non-null intersection with an element of $\mathcal{G}$. Then there is an at most countable subfamily of sets $\{\Gamma_{\alpha_i}\}$ which union is of full measure.

We may assume wlg that $\mathcal{G}$ is stable under countable unions. Then an element with maximal probability exists and is of full measure.
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Auxiliary results
Kreps–Yan theorem

**Theorem (Kreps, Yan, 1980)**

Let $C$ be a closed convex cone in $L^1$ such that $-L^1_+ \subseteq C$ and $C \cap L^1_+ = \{0\}$. Then there is $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that $\tilde{E}\xi \leq 0$ for all $\xi \in C$.

**Proof.** By the Hahn–Banach theorem any non-zero $\alpha \in L^1_+$ can be separated from $C$: there is $\eta_\alpha \in L^\infty$, $\|\eta_\alpha\|_\infty = 1$, such that

$$\sup_{\xi \in C} E\eta_\alpha \xi < E\eta_\alpha \alpha.$$  

Then $\eta_\alpha \geq 0$, $\sup \ldots = 0$, and $E\eta_\alpha \alpha > 0$. The latter inequality ensures that the family of sets $\Gamma_\alpha := \{\eta_\alpha > 0\}$ satisfies the assumption of the lemma ($E\eta_{I_{\Gamma}} l_{I_{\Gamma}} > 0$ if $l_{I_{\Gamma}} \neq 0$). Thus, for a certain sequence of indices $\eta := \sum 2^{-i} \eta_{\alpha_i} > 0$ a.s. and we take $\tilde{P} := \eta P$. 

Yuri Kabanov
Financial markets with transaction costs.
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Yuri Kabanov
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DMW-theorem: proofs of “non-trivial” implications

**DMW-theorem**

(c) \( \bar{A}_T \cap L_0^+ = \{0\} \);
(e') there is \( \tilde{P} \sim P \) with \( d\tilde{P}/dP \in L^\infty \) such that \( S \in \mathcal{M}(\tilde{P}) \).

(c) \( \Rightarrow \) (e') Let \( X := \sum_{t\leq T} |S_t|, Z' := e^{-X}/Ee^{-X}, P' := Z'P, A_1^\uparrow := A_T \cap L^1(P') \). Then \( \bar{A}_1^\uparrow \cap L_0^+ = \{0\} \). By the Kreps-Yan theorem there is bounded \( Z'' \) such that \( E'Z''\xi \leq 0 \) for all \( \xi \in A_1^\uparrow \), in particular, for \( \xi = \pm I_\Gamma(S_{t+1} - S_t) \) where \( \Gamma \in \mathcal{F}_t \). But this means that \( \tilde{P} = Z'Z''P \) is a martingale measure.

(a) \( A_T \cap L_0^+ = \{0\} \);
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(f') \( \Rightarrow \) (a) Let \( \xi \in A_T \cap L_0^+ \), i.e. \( 0 \leq \xi \leq H \cdot S_T \). Since the conditional expectation with respect to the local martingale measure \( \tilde{E}(H_t\Delta S_t|\mathcal{F}_{t-1}) = 0 \), we obtain by consecutive conditioning that \( \tilde{E}H \cdot S_T = 0 \). Thus, \( \xi = 0 \).
DMW-theorem: proofs of “non-trivial” implications

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DMW-theorem : proof of the “difficult” implication

\[ A_T \cap L^0_+ = \{0\} \Rightarrow A_T = \overline{A_T} \text{ (closure in } L^0). \]

We consider only the case \( T = 1 \).

Let \( H^n_1 \Delta S_1 - r^n \rightarrow \zeta \) where \( H^n_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0) \), \( r^n \in L^0_+ \).

The claim is : \( \zeta = H_1 \Delta S_1 - r \) where \( H_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0) \), \( r \in L^0_+ \).

We represent \((H^n_1)\) as the infinite matrix

\[
H_1 := \begin{bmatrix}
H_{11}^{11} & H_{11}^{21} & \ldots & \ldots & H_{11}^{n1} & \ldots \\
H_{12}^{11} & H_{12}^{21} & \ldots & \ldots & H_{12}^{n1} & \ldots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
H_{1d}^{11} & H_{1d}^{21} & \ldots & \ldots & H_{1d}^{n1} & \ldots \\
\end{bmatrix}.
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Suppose that the claim holds when \( H_1 \) has, for each \( \omega m \) zero lines. We show that it holds also when \( H_1 \) has \( m - 1 \) zero lines.

Let \( \Omega_i \in \mathcal{F}_0 \) form a finite partition of \( \Omega \). An important (but obvious) observation : we may argue on each \( \Omega_i \) separately.
DMW-theorem: proof of the “difficult” implication

\( A_T \cap L_0^0 = \{0\} \Rightarrow A_T = \bar{A}_T \) (closure in \( L_0^0 \)).

We consider only the case \( T = 1 \).

Let \( H_1^n \Delta S_1 - r^n \rightarrow \zeta \) where \( H_1^n \in L_0^0(\mathbb{R}^b, \mathcal{F}_0) \), \( r^n \in L_0^+ \).

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H_{11}^{11} & H_{12}^{21} & \cdots & \cdots & H_{11}^{n1} & \cdots \\
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Let \( \Omega_i \in \mathcal{F}_0 \) form a finite partition of \( \Omega \). An important (but obvious) observation : we may argue on each \( \Omega_i \) separately.
DMW-theorem : proof of the “difficult” implication

\[ A_T \cap L^0_+ = \{0\} \Rightarrow A_T = \bar{A}_T \text{ (closure in } L^0). \]

We consider only the case \( T = 1 \).

Let \( H^n_1 \Delta S_1 - r^n \rightarrow \zeta \) where \( H^n_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0) \), \( r^n \in L^0_+ \).

The claim is : \( \zeta = H_1 \Delta S_1 - r \) where \( H_1 \in L^0(\mathbb{R}^b, \mathcal{F}_0) \), \( r \in L^0_+ \).

We represent \((H^n_1)\) as the infinite matrix

\[
H_1 := \begin{bmatrix}
H^{11}_1 & H^{21}_1 & \cdots & \cdots & H^{n1}_1 & \cdots \\
H^{12}_1 & H^{22}_1 & \cdots & \cdots & H^{n2}_1 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
H^{1d}_1 & H^{2d}_1 & \cdots & \cdots & H^{nd}_1 & \cdots \\
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\[ A_T \cap L_0^+ = \{0\} \Rightarrow A_T = \bar{A}_T \text{ (closure in } L^0). \]

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\cdots & \cdots & \cdots & \cdots & \cdots \\
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DMW-theorem : proof of the “difficult” implication

Let $H_1 := \liminf |H_1^n|$. On $\Omega_1 := \{H_1 < \infty\}$ by the lemma on subsequences, we find a strictly increasing sequence of $\mathcal{F}_0$-measurable r.v. $\tau_k$ such that $H_1^{T_k}$ converges to some $H_1$; automatically, $r^{T_k}$ converges to some $r \geq 0$ and we conclude.

On $\Omega_2 := \{H_1 = \infty\}$ we put $G_1^n := H_1^n / |H_1^n|$ and $h_1^n := r_1^n / |H_1^n|$. Then $G_1^n \Delta S_1 - h_1^n \to 0$ a.s. By the lemma we find $\mathcal{F}_0$-measurable $\tau_k$ such that $G_1^{T_k}(\omega)$ converges to some $\tilde{G}_1$. It follows that $\tilde{G}_1 \Delta S_1 = \tilde{h}_1 \geq 0$. Because of the NA-property, $\tilde{G}_1 \Delta S_1 = 0$.

As $\tilde{G}_1(\omega) \neq 0$, there exists a partition of $\Omega_2$ into $d$ disjoint subsets $\Omega^i_2 \in \mathcal{F}_0$ such that $\tilde{G}_1^i \neq 0$ on $\Omega^i_2$.

Define $\bar{H}_1^n := H_1^n - \beta^n \tilde{G}_1$ where $\beta^n := H_1^{ni} / \tilde{G}_1^i$ on $\Omega^i_2$. Then $\bar{H}_1^n \Delta S_1 = H_1^n \Delta S_1$ on $\Omega_2$. The matrix $\bar{H}_1$ has, for each $\omega \in \Omega_2$, at least $m$ zero lines: our operations did not affect the zero lines of $H_1$ and a new one has appeared, namely, the $i$th one on $\Omega^i_2$. We conclude by the induction hypothesis.
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Outline

1. Classical model
   - Harrison-Pliska theorem
   - Dalang–Morton–Willinger theorem: FTAP

2. Ramifications
   - Restricted information
   - Infinite horizon

3. Hedging theorems
NA-criteria under restricted information

We are given a filtration $G = (G_t)_{t\leq T}$ with $G_t \subseteq F_t$. The strategies are predictable with respect to $G$, i.e. $H_{t-1} \in L^0(G_t)$, a situation when the portfolios are revised on the basis of restricted information, e.g., due to a delay. We define the sets $R_T$, $A_T$ and give a definition of the arbitrage which, in these symbols, looks exactly as (a) before and we can list the corresponding necessary and sufficient conditions. Notation : $X^o_t := E(X_t|G_t)$.

**Theorem (Kabanov–Stricker, 2006)**

The following properties are equivalent:

(a) $A_T \cap L^0_+ = \{0\}$ (NA condition);
(b) $A_T \cap L^0_+ = \{0\}$ and $A_T = \tilde{A}_T$;
(c) $\tilde{A}_T \cap L^0_+ = \{0\}$;
(d) there is a process $\rho \in \mathcal{M}$, $\rho > 0$, with $(\rho S)^o \in \mathcal{M}(G)$;
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Yuri Kabanov

Financial markets with transaction costs.
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No-Free-Lunch criteria for infinite horizon (Schachermayer)

- $R_\infty := \bigcup_{T \in \mathbb{N}} R_T$, $A_\infty := R_\infty - L_0^+$.
- **NA-property**: $R_\infty \cap L_+^0 = \{0\}$ (or $A_\infty \cap L_+^0 = \{0\}$).
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**Theorem**

NFL holds if and only if there is $P' \sim P$ such that $S \in \mathcal{M}_{loc}(P')$.

**Theorem**

Any $L_1^\infty$-neighborhood of a separating measure contains a measure $P'$ under which $S$ is a local martingale.

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Hedging of European options

- Let $\xi \in L^0(\mathcal{F}_T)$. Define the set of hedging endowments
  \[ \Gamma := \Gamma(\xi) := \{ x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S_T \geq \xi \}, \]
  i.e., $\Gamma$ is the set of capitals starting from which we can super-replicate the pay-off of European option with maturity $T$ by the terminal value of a self-financing portfolio.

- Let $Q^a$, $Q^e$ denote the sets of absolute continuous and equivalent martingale measures and let $Z^a$, $Z^e$ denote the corresponding sets of density processes.

Theorem

Suppose that NA holds, i.e. $Q^e \neq \emptyset$. Suppose that $\xi \geq 0$ and $E_Q \xi < \infty$ for every $Q \in Q^e$. Then $\Gamma = D$ where

\[ D := [\bar{x}, \infty[ = \{ x : x \geq E\rho_T \xi \text{ for all } \rho \in Z^e \}. \]
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Optional decomposition

Theorem (Kramkov, 1996, Föllmer–Kabanov, 1998)

Suppose that $Q^e \neq \emptyset$. Let $X \geq 0$ be a process which is a supermartingale with respect $Q \in Q^e$. Then there are a strategy $H$ and an increasing process $A$ such that $X = X_0 + H \cdot S - A$.

Proposition (El Karoui)

Suppose that $Q^e \neq \emptyset$. Let $\xi \in L^0_+$ be such that $\sup_{Q \in Q^e} E_Q \xi < \infty$. Then the process $X_t = \text{ess sup}_{Q \in Q^e} E_Q (\xi | \mathcal{F}_t)$ is a supermartingale with respect to every $Q \in Q^e$.

Proof of the hedging theorem. The inclusion $\Gamma \subseteq [\bar{x}, \infty]$ is obvious: if $x + H \cdot S_T \geq \xi$ then $x \geq E_Q \xi$ for every $Q \in Q^e$. To show the opposite one we suppose that $\sup_{Q \in Q} E_Q \xi < \infty$ (otherwise both sets are empty). Applying the ODT we get that $X = \bar{x} + H \cdot S - A$. Since $\bar{x} + H \cdot S_T \geq X_T = \xi$, the result follows.
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Hedging of American options

- For the American-type option the exercise date $\tau$ is a stopping time ($\leq T$) and the pay-off is $Y_\tau$, the value at $\tau$ of an adapted process $Y$. The description of the pay-off process $Y = (Y_t)$ is a clause of the contract (as well as the final maturity date $T$).

- Define the set of initial capitals starting from which we can run a self-financing portfolio which values dominate the pay-off:

$$\Gamma := \Gamma(Y) := \{x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S \geq Y\}.$$  

**Theorem**

Suppose that $Q^e \neq \emptyset$. Let $Y \geq 0$ be an adapted process such that $E_{Q^e} Y_t < \infty$ for every $Q \in Q^e$ and $t \leq T$. Then

$$\Gamma = \{x : x \geq E_{\rho_\tau} Y_\tau \text{ for all } \rho \in Z^e \text{ and all stopping times } \tau \leq T\}.$$
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Theorem

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Hedging of American options: the proof

It is based on the optional decomposition theorem applied to the following result where $\mathcal{I}_t$ denotes the set of stopping times $\tau \geq t$.

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Suppose that $Q^e \neq \emptyset$. Let $\xi \in L_+^0$ be such that $\sup_{Q \in Q^e} E_Q \xi < \infty$. Then the process $X_t = \operatorname{ess sup}_{Q \in Q^e, \tau \in \mathcal{I}_t} E_Q (Y_\tau | \mathcal{F}_t)$ is a supermartingale with respect to every $Q \in Q^e$. 