# Small Transaction Costs, Absence of Arbitrage and Consistent Price Systems

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Dedicated to Walter Schachermayer on occasion of his 60th anniversary.

**Abstract** The aim of this note is to establish criterion of absence of arbitrage opportunities under small transaction costs for a family of multi-asset models of financial market.

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# 1 Introduction

The arbitrage theory for financial markets with proportional transaction costs is one of the most advanced and interesting domains of mathematical finance. It success is due to a geometric viewpoint which provides an appropriate language to attack problems. The approach based on convex geometry not only makes arguments much more transparent comparatively with traditional, "parametric", modeling but also allows to put problems in a more general mathematical framework. To the date, for the discrete-time setting there is a plethora of criteria for various types of arbitrage, see Chapter 3 of the book [4]. In a surprising contrast, for continuous-time models only a few results on the no-arbitrage criteria are available. In the recent paper [2] Guasoni, Rásonyi, and

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Schachermayer established an interesting result in this direction. They formulated the question on sufficient and necessary conditions for the absence of arbitrage not for a single model but for a whole family of them. Namely, they considered two-asset models with a fixed continuous price process and constant transaction costs tending to zero. In a rather spectacular way, the resulting no-arbitrage criterion happens to be very simple: the  $NA^w$ -property holds for each model if and only if each model admits a consistent price system. The advantage of such a formulation is clear: topological properties, common in this theory, are not involved. It looks very similar to the no-arbitrage criterion for the model with finite  $\Omega$ , see Th. 3.1.1 in the book [4] and Th. 3.2 in the original paper [6].

Apparently, this result merits to be put in the mainstream of the theory of financial markets with transaction costs. In the present note we extend, using the now "standard" geometric approach, the main theorem of [2] to the case of multi-asset models. The paper [2] serves us as the roadmap.

## 2 Main Result

Let  $\varepsilon \in [0, 1]$  and let  $K^{\varepsilon*} := \mathbf{R}_+(\mathbf{1} + U_{\varepsilon})$ , where  $U_{\varepsilon} := \{x \in \mathbf{R}^d : \max_i |x^i| \le \varepsilon\}$ . That is,  $K^{\varepsilon*}$  is the closed convex cone in  $\mathbf{R}^d$  generated by the max-norm ball of radius  $\varepsilon$  with center at  $\mathbf{1} := (1, \ldots, 1)$ . We denote by  $K^{\varepsilon}$  the (positive) dual cone of  $K^{\varepsilon*}$ .

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$  be a stochastic basis and let  $S = (S_t)_{t \leq T}$  be a **continuous** semimartingale with strictly positive components. We consider the linear controlled stochastic equation

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i, \quad V_0^i = 0, \qquad i \le d,$$

where  $Y^i$  is the stochastic logarithm of  $S^i$ , i.e.  $dY_t^i = dS_t^i/S_t^i$ ,  $Y_0^i = 1$ , and the strategy B is a predictable càdlàg process of bounded variation with  $\dot{B} \in -K^{\varepsilon}$ . The notation  $\dot{B}$  stands for (a measurable version of) the Radon–Nikodym derivative of B with respect to ||B||, the total variation process of B.

A strategy *B* is  $\varepsilon$ -admissible if for the process  $V = V^B$  there is a constant  $\kappa$  such that  $V_t + \kappa S_t \in K^{\varepsilon}$  for all  $t \leq T$ . The set of processes *V* corresponding to  $\varepsilon$ -admissible strategies is denoted by  $A_0^{T\varepsilon}$  while the notation  $A_0^{T\varepsilon}(T)$  is reserved for the set of random variables  $V_T$ ,  $V \in A_0^{T\varepsilon}$ .

Using the random operator

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d)$$

define the random cone  $\widehat{K}_t^{\varepsilon} = \phi_t K^{\varepsilon}$  with the dual  $\widehat{K}_t^{\varepsilon*} = \phi_t^{-1} K^{\varepsilon*}$ . Put  $\widehat{V}_t = \phi_t V_t$ . We denote by  $\mathcal{M}_0^T(\widehat{K}^{\varepsilon*} \setminus \{0\})$  the set of martingales Z such that  $Z_t \in \widehat{K}_t^{\varepsilon*} \setminus \{0\}$  for all  $t \leq T$ .

Theorem 2.1 We have:

$$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\} \quad \forall \varepsilon \in ]0, 1] \quad \Leftrightarrow \quad \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset \quad \forall \varepsilon \in ]0, 1].$$

#### Comments on financial applications.

It is easily seen that for the case d = 2 our model is exactly the same as that of [2] and our theorem is Th. 1.1 therein. The only difference is that we use the "old-fashion"

definition of the value processes. The reader is invited to verify that one can use the more sophisticated one as defined in [4] (following the original paper [1]) and get the same result. In the financial interpretation the cones  $K^{\varepsilon}$  and  $\hat{K}^{\varepsilon}$  are the solvency regions in the terms of the numéraire and physical units, respectively, V and  $\hat{V}$  are value processes, elements of  $\mathcal{M}_0^T(\hat{K}^{\varepsilon*} \setminus \{0\})$  are  $\varepsilon$ -consistent price systems, etc. The condition " $\mathcal{A}_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$  for all  $\varepsilon$ " can be referred to as the universal  $N\mathcal{A}^w$ -property.

In the case d > 2 the considered cones  $K^{\varepsilon}$  and  $K^{\varepsilon*}$  do not correspond to a financial model (though sometimes the traditional terminology is still in use). What is important, our result can be applied to a wide class of financially meaningful models, even with varying transaction costs. To see this, let us consider the family of models of currency markets with the solvency cones given by the matrices of transaction costs coefficients  $\Lambda^{\varepsilon} = (\lambda_{ij}^{\varepsilon})$  as follows:

$$K(\Lambda^{\varepsilon}) = \operatorname{cone} \left\{ (1 + \lambda_{ij}^{\varepsilon})e_i - e_j, \ e_i, \ 1 \le i, j \le d \right\}$$

Suppose that for every  $\varepsilon \in ]0,1]$  there is  $\varepsilon' \in ]0,1]$  such that  $K(\Lambda^{\varepsilon}) \subseteq K^{\varepsilon'}$  and, vice versa, for any  $\delta \in ]0,1]$  there is  $\delta' \in ]0,1]$  such that  $K^{\delta} \subseteq K(\Lambda^{\delta'})$ . It is obvious that under this hypothesis Theorem 2.1 ensures that for the currency markets the  $NA^w(\Lambda^{\varepsilon})$ -property holds for every  $\varepsilon \in ]0,1]$  if and only if an  $\varepsilon$ -consistent price system does exist for every  $\varepsilon \in ]0,1]$ . The hypothesis is fulfilled if  $\Lambda^{\varepsilon} \to 0$  and the duals  $K^*(\Lambda^{\varepsilon})$  have interiors containing 1, e.g., in the case where all  $\lambda_{ij}^{\varepsilon} = \varepsilon$ . Adding some extra arguments one can easily get the following corollary of the main theorem for the family of models with the efficient friction condition.

**Proposition 2.2** Suppose that  $\Lambda^{\varepsilon} \to 0$  and  $\operatorname{int} K^*(\Lambda^{\varepsilon}) \neq \emptyset$  for all  $\varepsilon \in [0, 1]$ . Then

$$NA^{w}(\Lambda^{\varepsilon}) \quad \forall \varepsilon \in ]0,1] \quad \Leftrightarrow \quad \mathcal{M}_{0}^{T}(\widehat{K}^{*}(\Lambda^{\varepsilon}) \setminus \{0\}) \neq \emptyset \quad \forall \varepsilon \in ]0,1].$$

Proof. ( $\Rightarrow$ ) Let  $\delta \in [0,1]$  and  $w \in K^*(\Lambda^{\delta})$ . Then  $w^i/w^j \leq 1 + \lambda_{ij}^{\delta} \to 1$  as  $\delta \to 0$ . It follows that  $K^*(\Lambda^{\delta'}) \subseteq K^{\delta*}$  for some  $\delta' \in [0,1]$ . For the primary cones the inclusion is opposite. Thus, the assumed no-arbitrage property implies the no-arbitrage property in the formulation of Theorem 2.1. Take now  $\varepsilon \in [0,1]$  and a vector  $v \in \operatorname{int} K^*(\Lambda^{\varepsilon}) \cap U_1$ . We define the operator

$$\psi_v : (x^1, ..., x^d) \mapsto (v^1 x^1, ..., v^d x^d).$$

Choose  $\delta \in [0, 1]$  such that  $\psi_v(\mathbf{1} + U_\delta) \subset K^*(\Lambda^{\varepsilon})$ . By virtue of Theorem 2.1 there is  $Z \in \mathcal{M}_0^T(\widehat{K}^{\delta*} \setminus \{0\})$ . The process  $\psi_v Z$  is a martingale. Since  $\psi_v Z = \phi \psi_v \phi^{-1} Z$ , it is an element of  $\mathcal{M}_0^T(\widehat{K}^*(\Lambda^{\varepsilon}) \setminus \{0\})$ .

For the proof of the reverse implication see the beginning of Section 5.  $\Box$ 

## The strategy of the proof of Theorem 2.1.

To prove the nontrivial implication  $(\Rightarrow)$  we exploit the fact that the universal  $NA^w$ -property holds for any imbedded discrete-time model. Using the criterion for  $NA^r$ -property we deduce from here in Section 3 the existence of a "universal chain", that is there exists a sequence of stopping times  $\tau_n$  increasing stationary to T and such that  $\mathcal{M}_0^{\tau_n}(\hat{K}^{\varepsilon*} \setminus \{0\}) \neq \emptyset$  for all  $\varepsilon \in ]0, 1]$  and  $n \geq 1$ . In an analogy with [2], we relate with this "universal chain" functions  $F^i(\varepsilon)$ ,  $i \leq d$ , and check that there is, for each i, an alternative: either  $F^i = 0$ , or  $F^i(0+) = 1$ . This is the most involved part of the proof isolated in Section 4. If all  $F^i = 0$ , the sets  $\mathcal{M}_0^{\tau_n}(\hat{K}^{\varepsilon*} \setminus \{0\})$  are non-empty

and we conclude. If there is a coordinate *i* for which  $F^i(0+) = 1$ , there exists a strict arbitrage opportunity, see Section 5. In Section 6 we discuss the properties of richness of the set of consistent price systems.

## 3 Universal Discrete-Time $NA^w$ -property

We say that the continuous-time model has universal discrete-time  $NA^w$ -property if for any  $\varepsilon > 0$ ,  $N \ge 2$ , and an increasing sequence of stopping times  $\sigma_1, \ldots, \sigma_N$  with values in [0, T] and such that  $\sigma_n < \sigma_{n+1}$  on the set  $\{\sigma_n < T\}$ , we have that

$$L^{0}(\mathbf{R}^{d}_{+},\mathcal{F}_{T})\cap\sum_{n=1}^{N}L^{0}(-\phi_{\sigma_{n}}K^{\varepsilon},\mathcal{F}_{\sigma_{n}})=\{0\}.$$

**Proposition 3.1** Suppose that the model has the universal discrete-time  $NA^w$ -property. Then there is a strictly increasing sequence of stopping times  $\tau_n$  with  $P(\tau_n < T) \to 0$ as  $n \to \infty$  such that for any N and  $\varepsilon \in [0, 1]$  the set  $\mathcal{M}_0^{\tau_N}(\widehat{K}^{\varepsilon*} \setminus \{0\})$  is non-empty.

Proof. Define recursively the increasing sequence of stopping times:

$$\sigma_0 = 0, \quad \sigma_n = \sigma_n^{\varepsilon} := \inf\{t \ge \sigma_{n-1} : \max_{i \le d} |\ln S_t^i - \ln S_{\sigma_{n-1}}^i| \ge \ln(1 + \varepsilon/8)\}, \qquad n \ge 1.$$

This sequence has the following property which we formulate as a lemma.

**Lemma 3.2** For any integer  $N \geq 1$  there exists  $Z \in \mathcal{M}_0^{\sigma_N}(\widehat{K}^{\varepsilon_*} \setminus \{0\})$ .

Proof. To avoid a new notation we suppose without loss of generality that  $S = S^{\sigma_N}$ . Let  $X_n := S_{\sigma_n}$ . By our assumption and in virtue of the criterion for the  $NA^r$ -property there is a discrete-time martingale  $(M_n)_{n \leq N}$  with  $M_n \in L^{\infty}(\phi_{\sigma_n}^{-1}K^{\varepsilon/4*} \setminus \{0\})$ , see Th. 3.2.1 in [4] or Th. 3 in [5]. Put  $Z_t := E(M_N | \mathcal{F}_t)$  and  $\tilde{Z}_t := \phi_t Z_t$ . Let us check that  $Z \in \mathcal{M}_0^{\sigma_N}(\hat{K}^{\varepsilon*} \setminus \{0\})$ . On the set  $\{t \in [\sigma_{n-1}, \sigma_n]\}$ 

$$\tilde{Z}_t = E(\phi_t \phi_{\sigma_n}^{-1} \tilde{Z}_{\sigma_n} | \mathcal{F}_t).$$

Note that

$$(1 + \varepsilon/8)^{-2} \le \frac{S_{\sigma_n}^i}{S_t^i} = \frac{S_{\sigma_{n-1}}^i}{S_t^i} \frac{S_{\sigma_n}^i}{S_{\sigma_{n-1}}^i} \le (1 + \varepsilon/8)^2.$$

Therefore,

$$(1+\varepsilon/8)^{-2}E(\tilde{Z}^i_{\sigma_n}|\mathcal{F}_t) \le \tilde{Z}^i_t \le (1+\varepsilon/8)^2 E(\tilde{Z}^i_{\sigma_n}|\mathcal{F}_t).$$

But  $E(\tilde{Z}_{\sigma_n}|\mathcal{F}_t) = E(\phi_{\sigma_n}M_n|\mathcal{F}_t) \in \operatorname{cone}(\mathbf{1} + U_{\varepsilon/4}) \setminus \{0\}$ , i.e. the components of  $E(\tilde{Z}_{\sigma_n}|\mathcal{F}_t)$  take values in the interval with the extremities  $\lambda(1 \pm \varepsilon/4)$  where  $\lambda > 0$  depends on n and  $\omega$ . Thus,

$$1 - \varepsilon \le (1 + \varepsilon/8)^{-2} (1 - \varepsilon/4) \le \tilde{Z}_t^i / \lambda \le (1 + \varepsilon/8)^2 (1 + \varepsilon/4) \le 1 + \varepsilon$$

This implies the assertion of the lemma.  $\Box$ 

To finish the proof of the proposition, we proceed exactly as at the end of proof of Th. 1.4 in [2]. Namely, we take a sequence of  $\varepsilon_k \downarrow 0$ . For each  $n \ge 1$  we find an integer  $N_{n,k}$  such that

$$P(\sigma_{N_{n,k}}^{\varepsilon_k} < T) < 2^{-(n+k)}$$

Without loss of generality we assume that for each k the sequence  $(N_{n,k})_{n\geq 1}$  is increasing. The increasing sequence of stopping times  $\tau_n := \min_{k \ge 1} \sigma_{N_{n,k}}^{\varepsilon_k}$  converges to T stationary:  $P(\tau_n < T) \leq 2^{-n}$ . Applying the lemma with  $\varepsilon_k$  we obtain that for the process S stopped at  $\sigma_{N_n k}^{\varepsilon_k}$  there exists an  $\varepsilon_k$ -consistent price system. The latter, being stopped at  $\tau_n$ , is an  $\varepsilon_k$ -consistent price system for  $S^{\tau_n}$ .  $\Box$ 

We call the sequence  $(\tau_n)$  which existence was established above *universal chain*.

### 4 Properties of Universal Chains

We explore properties of a universal chain assuming that  $P(\tau_n < T) > 0$  for all n.

Let us introduce the set  $\mathcal{T}_T$  of stopping times  $\sigma$  such that  $P(\sigma < T) > 0$  and, for some n, the inequality  $\sigma \leq \tau_n$  holds on  $\{\sigma < T\}$ . This set is non empty: by the adopted hypothesis it contains all  $\tau_n$ .

Let  $\sigma \in \mathcal{T}_T$  and let *n* be such that  $\sigma \leq \tau_n$  holds on  $\{\sigma < T\}$ .

We denote by  $\mathcal{M}^{i}(\sigma, \varepsilon, n)$  the set of processes Z such that:

1) Z = 0 on  $\{\sigma = T\};$ 

2) Z is a martingale on  $[\sigma, \tau_n]$ , i.e.  $E(Z_{\tau_n} | \mathcal{F}_{\vartheta}) = Z_{\vartheta}$  for any stopping time  $\vartheta$  such that  $\sigma \leq \vartheta \leq \tau_n$  on  $\{\sigma < T\}$ ;

3)  $Z_t(\omega) \in \operatorname{int} \widehat{K}_t^{\varepsilon*}(\omega)$  when  $\sigma(\omega) < T$  and  $t \in [\sigma(\omega), \tau_n(\omega)];$ 4)  $EZ_{\sigma}^{i}I_{\{\sigma < T\}} = 1.$ 

Note that the process  $Z = \tilde{Z}I_{\{\sigma < T\}}/E\tilde{Z}^i_{\sigma}I_{\{\sigma < T\}}$  belongs to  $\mathcal{M}^i(\sigma, \varepsilon, n)$  provided that  $\tilde{Z} \in \mathcal{M}_0^{\tau_n}(\operatorname{int} \hat{K}^{\varepsilon*}).$ Let  $F^i(\varepsilon) := \sup_{\sigma \in \mathcal{T}_T} F^i(\sigma, \varepsilon)$  where

$$F^{i}(\sigma,\varepsilon) := \overline{\lim_{n} \inf_{Z \in \mathcal{M}^{i}(\sigma,\varepsilon,n)} EZ^{i}_{\tau_{n}}I_{\{\tau_{n} < T\}}}.$$

We also put

$$f^{i}(\sigma,\varepsilon,n) := \operatorname{ess}\inf_{Z \in \mathcal{M}^{i}(\sigma,\varepsilon,n)} E((Z^{i}_{\tau_{n}}/Z^{i}_{\sigma})I_{\{\tau_{n} < T\}} | \mathcal{F}_{\sigma}).$$

**Lemma 4.1** For any  $Z \in \mathcal{M}^i(\sigma, \varepsilon, n)$  there is a process  $\overline{Z} \in \mathcal{M}^i(\sigma, \varepsilon, n+1)$  such that  $\bar{Z}^{\tau_n} = Z^{\tau_n}.$ 

Proof. To explain the idea we suppose first that  $Z \in \mathcal{M}^i(\sigma, \varepsilon', n)$  for some  $\varepsilon' < \varepsilon$ . Take  $\delta > 0$  and a process  $\tilde{Z} \in \mathcal{M}^i(\sigma, \delta, n+1)$ . Define the process  $\bar{Z}$  with components

$$\bar{Z}^{j} := Z^{j} I_{[0,\tau_{n}[} + \frac{Z^{j}_{\tau_{n}}}{\tilde{Z}^{j}_{\tau_{n}}} \tilde{Z}^{j} I_{[\tau_{n},T]}.$$

Note that

$$\phi_t Z_t = \lambda_t (1 + u_t^1, \dots, 1 + u_t^d), \quad t \in [\sigma, \tau_n],$$
  
$$\phi_t \tilde{Z}_t = \tilde{\lambda}_t (1 + \tilde{u}_t^1, \dots, 1 + \tilde{u}_t^d), \quad t \in [\tau_n, \tau_{n+1}],$$

where  $\max_{j} |u^{j}| \leq \varepsilon'$ ,  $\max_{j} |\tilde{u}^{j}| \leq \delta$  and  $\lambda_{t}, \tilde{\lambda}_{t} > 0$ . It follows that  $\bar{Z}$  belongs to  $\mathcal{M}^i(\sigma, \bar{\varepsilon}, n+1)$  with

$$\bar{\varepsilon} = \frac{(1+\varepsilon')(1+\delta)}{1-\delta} - 1.$$

Since  $\bar{\varepsilon} < \varepsilon$  for sufficiently small  $\delta = \delta(\varepsilon')$ , the result follows.

In the general case we consider the partition of the set  $\{\sigma < T\}$  on  $\mathcal{F}_{\tau_n}$ -measurable subsets  $A_k$ , on which the process Z evolves, on the interval  $[\sigma, \tau_n]$ , in the cones  $\widehat{K}^{\varepsilon_k *}$ , where  $\varepsilon_k := (\varepsilon - 1/k) \lor 0$ . As above, take processes  $\widetilde{Z}^k \in \mathcal{M}^i(\sigma, \delta_k, n+1)$  with  $\delta_k = \delta(\varepsilon_k)$ . Then the process  $\overline{Z}$  with components

$$\bar{Z}^j := Z^j I_{[0,\tau_n[} + \sum_k \frac{Z^j_{\tau_n}}{\tilde{Z}^{kj}_{\tau_n}} \tilde{Z}^{kj} I_{A_k} I_{[\tau_n,T]}$$

meets the requirements.  $\Box$ 

**Lemma 4.2** The sequence  $(f^i(\sigma,\varepsilon,n))_{n\geq 0}$  is decreasing and its limit  $f^i(\sigma,\varepsilon) \leq F^i(\varepsilon)$ .

Proof. By Lemma 4.1 for any  $Z \in \mathcal{M}^i(\sigma, \varepsilon, n)$  there is a process  $\overline{Z} \in \mathcal{M}^i(\sigma, \varepsilon, n+1)$  such that  $\overline{Z}^{\tau_n} = Z^{\tau_n}$ . Using the martingale property of  $\overline{Z}$  we get that

$$E((Z_{\tau_n}^i/Z_{\sigma}^i)I_{\{\tau_n < T\}} | \mathcal{F}_{\sigma}) = E((\bar{Z}_{\tau_n}^i/\bar{Z}_{\sigma}^i)I_{\{\tau_n < T\}} | \mathcal{F}_{\sigma}) \ge E((\bar{Z}_{\tau_{n+1}}^i/\bar{Z}_{\sigma}^i)I_{\{\tau_{n+1} < T\}} | \mathcal{F}_{\sigma}).$$

It follows that  $f^i(\sigma, \varepsilon, n) \ge f^i(\sigma, \varepsilon, n+1)$ .

Suppose that the claimed inequality  $f^i(\sigma, \varepsilon) \leq F^i(\varepsilon)$  fails. Then there exist a nonnull  $\mathcal{F}_{\sigma}$ -measurable set  $A \subseteq \{\sigma < T\}$  and a constant a > 0 such that for all sufficiently large n

$$f^{i}(\sigma,\varepsilon,n)I_{A} \ge (F^{i}(\varepsilon)+a)I_{A}.$$

Define the stopping time  $\sigma_A := \sigma I_A + TI_{A^c}$  and note that for any  $Z \in \mathcal{M}^i(\sigma, \varepsilon, n)$  the process  $ZI_A/EZI_A$  is an element of  $\mathcal{M}^i(\sigma_A, \varepsilon, n)$ . Since  $E(\xi|\mathcal{F}_{\sigma})I_A = E(\xi|\mathcal{F}_{\sigma_A})I_A$ , we have the bound

$$f^{i}(\sigma_{A},\varepsilon,n)I_{A} \ge f^{i}(\sigma,\varepsilon,n)I_{A}.$$

Thus, for any  $Z \in \mathcal{M}^i(\sigma_A, \varepsilon, n)$  and sufficiently large n

$$EZ_{\tau_n}^i I_{\{\tau_n < T\}} = EZ_{\sigma_A}^i E((Z_{\tau_n}^i / Z_{\sigma_A}^i) I_{\{\tau_n < T\}} | \mathcal{F}_{\sigma_A}) \ge F^i(\varepsilon) + a$$

in contradiction with the definition of  $F^i(\varepsilon)$ .  $\Box$ 

**Lemma 4.3** Let  $\sigma \in \mathcal{T}_T$  be such that  $\sigma \leq \tau_{n_0}$  on the set  $\{\sigma < T\}$  and let  $\varepsilon, \delta > 0$ . Then there are  $n \geq n_0$ ,  $\Gamma \in \mathcal{F}_{\sigma}$  with  $P(\Gamma) \leq \delta$ , and  $Z \in \mathcal{M}^i(\sigma, \varepsilon, n)$  such that  $Z^i_{\sigma} = \eta := I_{\{\sigma < T\}}/EI_{\{\sigma < T\}}$  and

$$E(Z_{\tau_n}^i I_{\{\tau_n < T\}} | \mathcal{F}_{\sigma}) \leq \frac{I_{\{\sigma < T\}}}{EI_{\{\sigma < T\}}} [(F^i(\varepsilon) + \delta)I_{\Gamma^c} + I_{\Gamma}]$$

Proof. Recall that the essential infimum  $\xi$  of a family of random variables  $\{\xi^{\alpha}\}$  is the limit of a decreasing sequence of random variables of the form  $\xi^{\alpha_1} \wedge \xi^{\alpha_2} \wedge \ldots \wedge \xi^{\alpha_m}$ ,  $m \to \infty$ . Thus, for any a > 0 the sets  $\{\xi^{\alpha_k} \leq \xi + a\}$  form a covering of  $\Omega$ . Using the standard procedure, one can construct from this covering a measurable partition of  $\Omega$  by sets  $A^k$  such that  $\xi^{\alpha_k} \leq \xi + a$  on  $A^k$ .

Thus, for any fixed  $n \ge n_0$  there are a countable partition of the set  $\{\sigma < T\}$  into  $\mathcal{F}_{\sigma}$ -measurable sets  $A^{n,k}$  and a sequence of  $Z^{n,k} \in \mathcal{M}^i(\sigma,\varepsilon,n)$  such that

$$E((Z_{\tau_n}^{n,k,i}/Z_{\sigma}^{n,k,i})I_{\{\tau_n < T\}} | \mathcal{F}_{\sigma}) \le f^i(\sigma,\varepsilon,n) + 1/n \quad \text{on } A^{n,k}.$$

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Put, for  $t \in [\sigma, \tau_n]$ ,

$$\tilde{Z}_t^n := \eta \sum_{k=1}^\infty \frac{1}{Z_\sigma^{n,k,i}} Z_t^{n,k} I_{A^{n,k}}.$$

Then  $\tilde{Z}^n \in \mathcal{M}^i(\sigma, \varepsilon, n), \ \tilde{Z}^{n,i}_{\sigma} = \eta$ , and

$$E(\tilde{Z}_{\tau_n}^{n,i}I_{\{\tau_n < T\}}|\mathcal{F}_{\sigma}) = \eta E((\tilde{Z}_{\tau_n}^{n,i}/\eta)I_{\{\tau_n < T\}}|\mathcal{F}_{\sigma}) \le \frac{I_{\{\sigma < T\}}}{EI_{\{\sigma < T\}}}[f^i(\sigma,\varepsilon,n) + 1/n].$$

Note that  $f^i(\sigma, \varepsilon, n) + 1/n$  decreases to  $f^i(\sigma, \varepsilon) \leq F^i(\varepsilon)$ . By the Egorov theorem the convergence is uniform outside of a set  $\Gamma$  of arbitrary small probability. The assertion of the lemma follows from here immediately.  $\Box$ 

**Proposition 4.4** For any  $\varepsilon_1, \varepsilon_2$  we have the inequality

$$F^{i}(\varepsilon_{1})F^{i}(\varepsilon_{2}) \ge F^{i}((1+\varepsilon_{1})(1+\varepsilon_{2})/(1-\varepsilon_{2})-1).$$

$$(4.1)$$

Either  $F^i = 0$ , or there is  $c^i \ge 0$  such that  $F^i(\varepsilon) \ge e^{-c^i \varepsilon^{1/3}}$  for all sufficiently small  $\varepsilon$ . Proof. Fix  $\delta > 0$  and a stopping time  $\sigma \le \tau_{n_0}$  on the set  $\{\sigma < T\}$ . According to the above lemma there are  $n \ge n_0$  and  $Z^1 \in \mathcal{M}^i(\sigma, \varepsilon_1, n)$  such that

$$EZ_{\tau_n}^{1i}I_{\{\tau_n < T\}} \le F^i(\varepsilon_1) + \delta$$

Using the same lemma again (but now with  $\tau_n$  playing the role of  $\sigma$ ), we find m > nand  $Z^2 \in \mathcal{M}^i(\tau_n, \varepsilon_2, m)$  with  $Z_{\tau_n}^{2i} = I_{\{\tau_n < T\}}/EI_{\{\tau_n < T\}}$  such that outside of a set  $\Gamma \in \mathcal{F}_{\tau_n}$  with  $P(\Gamma) \leq \delta$  we have the bound

$$E(Z_{\tau_m}^{2i}I_{\{\tau_m < T\}} | \mathcal{F}_{\tau_n}) \le \frac{I_{\{\tau_n < T\}}}{EI_{\{\tau_n < T\}}} [(F^i(\varepsilon_2) + \delta)I_{\Gamma^c} + I_{\Gamma}].$$

Define on  $[\sigma, \tau_m]$  the martingale Z with  $Z_t^j := Z_t^{1j}$  on  $[\sigma, \tau_n]$  and  $Z_t^j := Z_t^{2j} Z_{\tau_n}^{1j} / Z_{\tau_n}^{2j}$  on  $[\tau_n, \tau_m]$ ,  $j = 1, \ldots, d$ . Then

$$\phi_t Z_t^1 = \lambda_t^1 (1 + u_t^{11}, \dots, 1 + u_t^{1d}), \quad t \in [\sigma, \tau_n],$$
  
$$\phi_t Z_t^2 = \lambda_t^2 (1 + u_t^{21}, \dots, 1 + u_t^{2d}), \quad t \in [\tau_n, \tau_m]$$

where  $\max_j |u^{1j}| \leq \varepsilon_1$ ,  $\max_j |u^{2j}| \leq \varepsilon_2$  and  $\lambda_t^1, \lambda_t^2 > 0$ . It follows that

$$Z \in \mathcal{M}^{i}(\sigma, (1+\varepsilon_{1})(1+\varepsilon_{2})/(1-\varepsilon_{2})-1, m).$$

Note also that

$$EZ_{\tau_m}^{i}I_{\{\tau_m < T\}} = P(\tau_n < T)EZ_{\tau_m}^{2i}Z_{\tau_n}^{1i}I_{\{\tau_m < T\}}$$
  
$$\leq P(\tau_n < T)EZ_{\tau_n}^{1i}I_{\{\tau_n < T\}}E(Z_{\tau_m}^{2i}I_{\{\tau_m < T\}}|\mathcal{F}_{\tau_n}).$$

Hence,

$$EZ_{\tau_m}^i I_{\{\tau_m < T\}} \le (F^i(\varepsilon_1) + \delta)(F^i(\varepsilon_2) + \delta) + EZ_{\tau_n}^{1i} I_{\{\tau_n < T\}} I_{\Gamma}$$

The inequality (4.1) follows from here.

Note that for  $\varepsilon_1, \varepsilon_2 \in ]0, 1/4]$ 

$$\frac{(1+\varepsilon_1)(1+\varepsilon_2)}{1-\varepsilon_2} - 1 = \frac{\varepsilon_1 + 2\varepsilon_2 + \varepsilon_1\varepsilon_2}{1-\varepsilon_2} \le 4(\varepsilon_1 + \varepsilon_2).$$

Since F is decreasing, we obtain from (4.1) that  $F^i(\varepsilon_1)F^i(\varepsilon_2) \ge F^i(4(\varepsilon_1 + \varepsilon_2))$  for all  $\varepsilon_1, \varepsilon_2 \in ]0, 1/8]$ . Using Lemma 4.5 below with  $f = \ln F^i$ , we get the needed bound.  $\Box$ 

**Lemma 4.5** Let  $f : [0, x_0] \to \mathbf{R}$  be a decreasing function such that

$$f(x_1) + f(x_2) \ge f(4(x_1 + x_2)), \qquad \forall x_1, x_2 \le x_0.$$
(4.2)

Then there is c > 0 such that  $f(x) \ge -cx^{1/3}$  for  $x \in [0, x_0]$ .

Proof. In the non-trivial case where  $f(x_0) < 0$ , the constant  $\kappa = -\inf_{x \in ]x_0/8, x_0]} f(x)/x$  is strictly greater than zero. Iterating the inequality  $2f(x) \ge f(8x)$  we obtain that  $2^n f(x) \ge f(2^{3n}x)$  for all  $x \in ]0, 2^{-3n}x_0]$  and all integers  $n \ge 0$ . Therefore,

$$\frac{f(x)}{x} \ge 2^{2n} \frac{f(2^{3n}x)}{2^{3n}x} = \frac{1}{4} x_0^{2/3} \left(\frac{2^{3(n+1)}}{x_0}\right)^{2/3} \frac{f(2^{3n}x)}{2^{3n}x}$$

For  $x \in [2^{-3(n+1)}x_0, 2^{-3n}x_0]$ , the right-hand side dominates  $-cx^{-2/3}$  with the constant  $c := \kappa x_0^{2/3}/4$ . Thus, the inequality  $f(x)/x \ge -cx^{-2/3}$  holds on  $[0, x_0]$ .  $\Box$ 

# 5 Proof of Theorem 2.1

( $\Leftarrow$ ) The arguments are standard. For any  $\xi \in \phi_T A_0^{T\varepsilon}(T)$  and  $Z \in \mathcal{M}_0^T(\widehat{K}^{\varepsilon*} \setminus \{0\})$  we have  $EZ_T \xi \leq 0$  and this inequality is impossible for  $\xi \in L^0(\mathbf{R}^d_+, \mathcal{F}_T), \xi \neq 0$ .

(⇒) In view of Proposition 3.1 we need to consider the case where the universal chain is such that  $P(\tau_n < T) > 0$  for every *n* and we can apply the results on functions  $F^i$ . Now the claim follows from the assertions below (cf. Prop. 3.7 and Th. 3.7 in [2]).

**Proposition 5.1** If  $\sum_{i} F^{i}(\varepsilon) = 0$  for all  $\varepsilon \in [0, 1]$ , then  $\mathcal{M}_{0}^{T}(\widehat{K}^{\varepsilon*} \setminus \{0\}) \neq \emptyset$ .

*Proof.* Fix  $\varepsilon \in [0,1]$  and define a sequence of  $\varepsilon_k \downarrow 0$ , such that  $\bar{\varepsilon}_N \uparrow \varepsilon$  where  $\bar{\varepsilon}_1 = \varepsilon_1$ ,

$$\bar{\varepsilon}_N := (1+\varepsilon_1) \prod_{k=2}^N \frac{1+\varepsilon_k}{1-\varepsilon_k} - 1, \qquad N \ge 2$$

We extend arguments of the proof of Proposition 4.4 in the following way. Namely, we construct inductively an increasing sequence of integers  $(n_N)_{N\geq 0}$  with  $n_0 = 0$  and a sequence of  $Z^{(N)} \in \mathcal{M}_0^{\tau_{n_N}}(\hat{K}^{\varepsilon_N*} \setminus \{0\})$  such that for N = kd + r where  $0 \leq r \leq d-1$ 

$$EZ_{\tau_{n_N}}^{(N)\,r+1}I_{\{\tau_{n_N} < T\}} \le 2^{-N}.$$
(5.3)

Since  $F^1(\varepsilon) = 0$ , Lemma 4.3 ensures the existence of  $Z^1 \in \mathcal{M}^1(0, \varepsilon_1, n_1)$  with

$$EZ_{\tau_{n_1}}^{11}I_{\{\tau_{n_1} < T\}} \le 2^{-1}.$$

Put  $Z^{(1)} := Z^1$ . Take now  $\delta_1 > 0$  such that

$$EZ_{\tau_{n_1}}^{(1)2}I_{\{\tau_{n_1} < T\}}I_A \le 2^{-3}$$

for every  $A \in \mathcal{F}_{\tau_{n_1}}$  with  $P(A) \leq \delta_1$ . Using again Lemma 4.3 (now for the second coordinate), we find an integer  $n_2 > n_1$ , a set  $\Gamma_1 \in \mathcal{F}_{\tau_{n_1}}$  with  $P(\Gamma_1) \leq \delta_1 \wedge 2^{-3}$ , and a process  $Z^2 \in \mathcal{M}^2(\tau_{n_1}, \varepsilon_2, n_2)$  such that  $Z^{22}_{\tau_{n_1}} = I_{\{\tau_{n_1} < T\}}/EI_{\{\tau_{n_1} < T\}}$  and

$$E(Z_{\tau_{n_2}}^{22}I_{\{\tau_{n_2} < T\}} | \mathcal{F}_{\tau_{n_1}}) \le \frac{I_{\{\tau_{n_1} < T\}}}{EI_{\{\tau_{n_1} < T\}}} [2^{-3} + I_{\Gamma_1}].$$

Put  $Z_t^{(2)j} = Z_t^{(1)j}$  on  $[0, \tau_{n_1}]$  and  $Z_t^{(2)j} = Z_t^{2j} Z_{\tau_{n_1}}^{(1)j} / Z_{\tau_{n_1}}^{2j}$  on  $]\tau_{n_1}, \tau_{n_2}], j = 1, \dots, d$ . Note that  $Z^{(2)} \in \mathcal{M}_0^{\tau_{n_2}}(\phi^{-1} \operatorname{cone} \{\mathbf{1} + U_{\bar{\epsilon}_2}\} \setminus \{0\})$  and

$$\begin{split} EZ_{\tau_{n_2}}^{(2)2} I_{\{\tau_{n_2} < T\}} &= P(\tau_{n_1} < T) EZ_{\tau_{n_2}}^{22} Z_{\tau_{n_1}}^{(1)2} I_{\{\tau_{n_2} < T\}} \\ &\leq P(\tau_{n_1} < T) EZ_{\tau_{n_1}}^{(1)2} I_{\{\tau_{n_1} < T\}} E(Z_{\tau_{n_2}}^{22} I_{\{\tau_{n_2} < T\}} | \mathcal{F}_{\tau_{n_1}}) \leq 2^{-2}. \end{split}$$

We continue this procedure passing at each step from the coordinate j to the coordinate j + 1 for  $j \leq d - 1$  and from the coordinate d to the first one.

Since  $P(\tau_n = T) \uparrow 1$ , there is a process Z such that  $Z^{\tau_{n_N}} = Z^{(N)}$  for every N. The components of Z are strictly positive processes on [0, T]. The condition (5.3) ensures that they are martingales. Therefore,  $Z \in \mathcal{M}_0^T(\hat{K}^{\varepsilon*} \setminus \{0\})$ .  $\Box$ 

**Proposition 5.2** Suppose that  $\sum F^i \neq 0$ . Then there is  $\varepsilon \in [0, 1]$  for which the property  $NA^{w\varepsilon}$  (the notation is obvious) does not hold.

Proof. At least one of functions, say,  $F^1$ , is not equal identically to zero. According to Proposition 4.4, we have the bound  $F^1(\varepsilon) > e^{-c\varepsilon^{1/3}}$  for all sufficiently small  $\varepsilon$ . Hence, there is a stopping time  $\sigma$  dominated by certain  $\tau_{n_0}$  on the set { $\sigma < T$ } such that

$$\inf_{Z \in \mathcal{M}^1(\sigma,\varepsilon,n)} EZ^1_{\tau_n} I_{\{\tau_n < T\}} > e^{-c\varepsilon^{1/2}}$$

for all sufficiently large n. Then for every  $Z \in \mathcal{M}^1(\sigma, \varepsilon, n)$  we have that

$$E(Z_{\tau_n}^1 I_{\{\tau_n=T\}} | \mathcal{F}_{\sigma}) \le 1 - e^{-c\varepsilon^{1/3}}.$$

Let us consider the sequence of random variables  $\xi^n \in L^0(\mathbf{R}^d, \mathcal{F}_{\tau_n})$  such that the components  $\xi^{n^2} = \ldots = \xi^{nd} = 0$  and

$$\xi^{n1} = -I_{\{\sigma < T\}} + (1 - e^{-c\varepsilon^{1/3}})^{-1}I_{\{\sigma < T, \tau_n = T\}}$$

Clearly,

$$E(Z_{\tau_n}\xi^n | \mathcal{F}_{\sigma}) \le -I_{\{\sigma < T\}} + (1 - e^{-c\varepsilon^{1/3}})^{-1} E(Z_{\tau_n}^1 I_{\{\tau_n = T\}} | \mathcal{F}_{\sigma}) I_{\{\sigma < T\}} \le 0.$$

We have the inequality  $EZ_{\tau_n}\xi^n \leq 0$ , and, therefore, by the superhedging theorem (see Th. 3.6.3 in [4]),  $\xi^n$  is the terminal value of an admissible process  $\hat{V} = \hat{V}^B$  in the model having  $\sigma$  and  $\tau_n$  as the initial and terminal dates, respectively. Note that on the non-null set  $\{\sigma < T\}$  the limit of  $\xi^{n1}$  is strictly positive. To conclude we use the lemma below which one can get by applying, on each interval  $[0, \tau_n]$ , the Komlós-type result (Lemma 3.6.5 in [4], Lemma 3.5 in [3]) followed by the diagonal procedure.  $\Box$ 

**Lemma 5.3** Suppose that  $\xi^n = \widehat{V}_{\tau_n}^n$  where  $\widehat{V}^n + \mathbf{1} \in \widehat{K}^{\varepsilon}$  and  $\xi^n \to \xi$  a.s. as  $n \to \infty$ . Then there is a portfolio process  $\widehat{V}$  such that  $\widehat{V} + \mathbf{1} \in \widehat{K}^{\varepsilon}$  and  $\xi = \widehat{V}_T$ .

#### 6 Richness of the Set of Consistent Price Systems

The following condition of "richness" of consistent price systems plays an important role in the continuous-time theory of financial markets with transaction costs.

 $\mathbf{B}^{\varepsilon}$  Let  $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ . If  $Z_t \xi \ge 0$  for all  $Z \in \mathcal{M}_0^T(\widehat{K}^{\varepsilon*} \setminus \{0\})$ , then  $\xi \in \widehat{K}_t^{\varepsilon}$  (a.s.).

Simple argument (see, e.g., [4], 3.6.3) shows that  $\mathbf{B}^{\varepsilon}$  is fulfilled for the model with constant transaction costs if S admits an equivalent martingale measure. Its minor changes leads to the next result which seems to be useful interesting in the setting of families of models with vanishing transaction costs:

**Proposition 6.1** Suppose that  $\mathcal{M}_0^T(\widehat{K}^{\varepsilon*} \setminus \{0\}) \neq \emptyset$  for all  $\varepsilon \in ]0, 1]$ . Then the condition  $\mathbf{B}^{\varepsilon}$  holds for all  $\varepsilon \in ]0, 1]$ .

Proof. Take  $w \in \operatorname{int} K^{\varepsilon*}$  with |w| = 1. For all sufficiently small  $\delta > 0$  we have the inclusion  $w + U_{\delta} \subset K^{\varepsilon*}$ . Take  $Z \in \mathcal{M}_0^T(\hat{K}^{\delta*} \setminus \{0\})$  and consider the martingale  $Z^w = (w^1 Z^1, \ldots, w^d Z^d)$ . Note that  $\phi_t Z_t = \rho_t \tilde{Z}_t$  where  $\rho_t > 0$  and  $\tilde{Z}_t \in \mathbf{1} + U_{\delta}$ . Then  $\phi_t Z_t^w = \rho_t \tilde{w}_t$  where  $\tilde{w}_t^i = w^i \tilde{Z}_t^i$ . According to our definition,  $\tilde{w}_t$  takes values in  $w + U_{\delta} \subset K^{\varepsilon*}$ . Therefore,  $Z^w \in \mathcal{M}_0^T(\hat{K}^{\varepsilon*} \setminus \{0\})$  and  $Z^w \xi \ge 0$ . The inequality implies that  $\tilde{w}_t \eta_t \ge 0$  where  $\eta_t(\omega) = \phi_t^{-1}(\omega)\xi(\omega)$ . Letting  $\delta \to 0$ , we obtain that also  $w\eta_t \ge 0$ . The latter inequality holds for all  $w \in K^{\varepsilon*}$ . Hence,  $\phi_t^{-1}\xi \in K^{\varepsilon}$  and  $\xi \in \hat{K}_t^{\varepsilon}$ .  $\Box$ 

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