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Consumption-Investment Problem with Transaction Costs for Lévy-driven Price Processes

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Abstract We consider an optimal control problem of linear stochastic integro-differential equation with conic constraints on the phase variable and the control of singular-regular type. Our setting includes consumption-investment problems for models of financial markets in the presence of proportional transaction costs where that the prices are geometric Lévy processes and the investor is allowed to take short positions. We prove that the Bellman function of the problem is a viscosity solution of the HJB equation. A uniqueness theorem for the solution of the latter is established. Special attention is paid to the Dynamic Programming Principle.

Keywords Consumption-investment problem · Lévy process · Transaction costs · Bellman function · Dynamic programming · HJB equation.

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1 Introduction

In this paper we study the classical consumption-investment model with infinite horizon in the presence of transaction costs. Our aim is to extend the results of [18] to the case where the price processes are geometric Lévy process. Namely, we show that the Bellman function is a viscosity solution of the corresponding Hamilton–Jacobi–Bellman equation. We also prove a uniqueness theorem for the latter.

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Mathematically, the consumption-investment problem with transaction costs is a regular-singular control problem for a linear stochastic equation in a cone. Its specificity is that the Bellman function is not smooth and, therefore, one cannot use the verification theorem (at least, in its traditional form) because the Itô formula cannot be applied. Nevertheless, one can show that the Bellman function is a solution of the HJB equation in viscosity sense. Though the general line of arguments is common, one needs to re-examine each step of the proof. In particular, for the considered jump-diffusion model, the HJB equation contains an integro-differential operator and the test functions involved in the definition of the viscosity solution must be “globally” defined. It seems that already in 1986 H.M. Soner noticed that the control problems with jump parts can be considered in the framework of the theory of viscosity solutions, [26], [27].

There is a growing literature on extension of the concept of viscosity solutions to equations with integro-differential operators, see, e.g., [23], [2], [22], [9], [8], [3], [4]. There are several variants of the definition of viscosity solution. Our choice is intended to serve the model with a positive utility function. The definition can be viewed as a simplified version of that adopted in [15].

A rather detailed study of consumption-investment problems under transaction costs when the the prices follows exponential Lévy processes and the investor is constrained to keep long positions in all assets, money included, was undertaken in papers by Benth et *al.* [10] and [11]. Our geometric approach seems to be more general than that of the mentioned papers where the authors consider a ”parametric” version of the stock market with transactions always passing through money (i.e. either “buy stock” or “sell stock”). A more important difference is that in our setting the investor may take short positions as was always assumed in the classical papers [21], [13], [25]. If short positions are admitted, the ruin may happen due to a jump of the price process. That is why the classical setting we consider here leads to a different HJB equation of a more complicated structure. Following the ideas from the paper [18] we derive the Dynamic Programming Principle splitted into two separate assertions. Though it is the principal tool which allows to check that the Bellman function is a viscosity solution of the HJB equation, it is rarely discussed in the literature (and even taken for granted, see, e.g., in [1], [25], [10]).

The main results of the paper is Theorem 10.1 claiming that if the Bellman function is continuous up to the boundary then it is a viscosity solution of the HJB equation and the uniqueness theorem for the Dirichlet problem arising in the model, Th. 11.2. We formulate the latter in terms of the Lyapunov function, an object that is defined in terms of the truncated operator, in which the utility function is not involved. Its introduction allows us to disconnect the uniqueness of a solution and the existence of a classical supersolution.

The structure of the problem is the following. In Sections 2 and 3 we introduce the model dynamics and describe the goal functional providing comments on the concavity of the Bellman function W. In Section 4 we show that the Bellman function, if finite, then it is continuous in the interior of the solvency cone. In Section 5 we give a formal description of the HJB equation. Sections 6 and 7 contain a short account of basic facts on viscosity solutions for integro-differential operators. In Section 8 we explain the role of classical supersolutions to the HJB equations. Section 9 is devoted to the Dynamic Programming Principle. In Section 10 we use it to show that the Bellman function is the solution of our HJB equation. Section 11 contains a uniqueness theorem formulated in terms of a Lyapunov function. In Section 12 we provide examples of Lyapunov functions and classical supersolutions.

2 The Model

Our setting is more general than that of the standard model of financial market under constant proportional transaction costs. In particular, the cone K is not supposed to be polyhedral. We assume that the asset prices are geometric Lévy processes. Our framework appeals to a theory of viscosity solutions for non-local integro-differential operators.

Let $Y = (Y_t)$ be an $\mathbf{R}^d$-valued semimartingale on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the trivial initial σ-algebra. Let K and $\mathcal{C}$ be **proper** closed cones in $\mathbf{R}^d$ such that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. Define the set $\mathcal{A}$ of controls $\pi = (B, C)$ as the set of **predictable** càdlàg processes of bounded variation such that, up to an evanescent set,

$$\dot{B} \in -K, \qquad \dot{C} \in \mathcal{C}. \tag{2.1}$$

Here $\dot{B}$ denotes a (measurable version) of the Radon–Nikodym derivative of B with respect to the total variation process $||B||$. The notation $\dot{C}$ has a similar sense. Though models with arbitrary C is of interest, we restrict ourselves in the present paper by considering consumption processes admitting intensity. To this and we define $\mathcal{A}_a$ as the set of controls π with absolutely continuous components C such that the increment $\Delta C_0 := C_0 - C^i_{0-} = 0$ where $C_{0-} = 0$. For the elements of $\mathcal{A}_a$ we have $c := dC/dt \in \mathcal{C}$.

The controlled process $V = V^{x,\pi}$ is the solution of the linear system

$$dV^i_t = V^i_{t-} dY^i_t + dB^i_t - dC^i_t, \qquad V^i_{0-} = x^i, \quad i = 1, ..., d. \tag{2.2}$$

In general, $\Delta V_0 = \Delta B_0$ is not is not equal to zero: the investor may revise the portfolio when entering the market.

The solution of (2.2) can be expressed explicitly using the Doléans-Dade exponentials

$$\mathcal{E}_t(Y^i) = e^{Y^i_t - (1/2)\langle Y^{ic}\rangle_t} \prod_{s \le t} (1 + \Delta Y^i_s) e^{-\Delta Y^i_s}. \tag{2.3}$$

Namely,

$$V^i_t = \mathcal{E}_t(Y^i) x^i + \mathcal{E}_t(Y^i) \int_{[0,t]} \mathcal{E}^{-1}_{s-}(Y^i)(dB^i_s - dC^i_s), \quad i = 1, ..., d. \tag{2.4}$$

We introduce the stopping time

$$\theta = \theta^{x,\pi} := \inf\{t : \ V^{x,\pi}_t \notin \operatorname{int} K\}.$$

For $x \in \operatorname{int} K$ we consider the subsets $\mathcal{A}^x$ and $\mathcal{A}^x_a$ of "admissible" controls for which $\pi = I_{[0,\theta^{x,\pi}]}\pi$ and $\{V_- + \Delta B \in \operatorname{int} K\} = \{V_- \in \operatorname{int} K\}$. This means that for an admissible control the process $V^{x,\pi}$ stops at the moment when it leaves the interior of the solvency cone and there is no more consumption. Moreover, the process V does not leave the interior of K due to a jump of B: the investor is reasonable enough not to ruin himself by making too expensive portfolio revision.

The important hypothesis that the cone K is proper, i.e. $K \cap (-K) = \{0\}$, or equivalently, $\operatorname{int} K^* \neq \emptyset$, corresponds to the model of financial market with *efficient friction*. In a financial context K (usually containing $\mathbf{R}^d_+$) is interpreted as the solvency region and $C = (C_t)$ as the consumption process; the process $B = (B_t)$ describes accumulated fund transfers. In the "standard" model with proportional transaction costs (sometimes referred to as the model of currency market)

$$K = \operatorname{cone}\{(1+\lambda^{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\}$$

where $\lambda^{ij} \ge 0$ are transaction costs coefficients, see Section 3.1 in the book [19] for details and other examples.

The process Y represents the relative price movements. If S^i is the price process of the ith asset, then $dS^i_t = S^i_{t-}dY^i_t$ and $S^i_t = S^i_0\mathcal{E}_t(Y^i)$. Without loss of generality we assume that $S^i_0 = 1$ for all i. In this case Y^i is the so-called stochastic logarithm of S^i. The formula (2.4) can be re-written as follows:

$$V^i_t = S^i_t x^i + S^i_t \int_{[0,t]} \frac{1}{S^i_{s-}}(dB^i_s - dC^i_s), \quad i = 1, ..., d. \tag{2.5}$$

We shall work assuming that

$$dY_t = \mu t + \Xi dw_t + \int z(p(dz,dt) - q(dz,dt)) \tag{2.6}$$

where $\mu \in \mathbf{R}^d$, w is a m-dimensional standard Wiener process and $p(dz,dt)$ is a Poisson random measure with the compensator $q(dz,dt) = \Pi(dz)dt$ such that $\Pi(dz)$ is a measure concentrated on $]-1,\infty[^d$. Note that the latter property of the Lévy measure corresponds to the financially meaningful case where $S^i > 0$. For the $m \times d$-dimensional matrix Ξ we put $A = \Xi\Xi^*$. We assume that

$$\int (|z|^2 \wedge |z|)\Pi(dz) < \infty. \tag{2.7}$$

It is important to note that the jumps of Y and B cannot occur simultaneously. More precisely, the process $|\Delta B||\Delta Y|$ is indistinguishable of zero. Indeed, for any $\varepsilon > 0$ we have, using the predictability of the process $\Delta B = B - B_-$, that

$$\begin{aligned} E\sum_{s \ge 0} |\Delta B_s||\Delta Y_s| I_{\{|\Delta Y_s| > \varepsilon\}} &= E\int_0^\infty \int |\Delta B_s| I_{\{|z|>\varepsilon\}} |z| p(dz,ds) \\ &= E\int_0^\infty \int |\Delta B_s||z| I_{\{|z|>\varepsilon\}} \Pi(dz)ds = 0 \end{aligned}$$

because for each ω the set $\{s : \Delta B_s(\omega) \ne 0\}$ is at most countable and its Lebesgue measure is equal to zero. Thus, the process $|\Delta B||\Delta Y| I_{\{|\Delta Y| > \varepsilon\}}$ is indistinguishable of zero and so is the process $|\Delta B||\Delta Y|$.

It follows that $\Delta B_\theta = 0$. Since the predictable process $I_{\{V_- \in \partial K\}} I_{[0,\theta]}$ has at most countable number of jumps, the same reasoning as above leads to the conclusion that $I_{\{V_- \in \partial K\}}|\Delta Y| I_{[0,\theta]}$ is indistinguishable of zero. This means that θ is the first moment when either V or V_- leaves $\operatorname{int} K$. This property will be used in the proof that W is lower semicontinuous on $\operatorname{int} K$.

In our proof of the dynamic programming principle (needed to derive the HJB equation) we shall assume that the stochastic basis is a canonical one, that is the space of càdlàg functions under which the coordinate process is the Lévy process.

We denote by $C_\rho(K)$ the subspace of the space of continuous functions f on K such that $\sup_{x\in K}|f(x)|(1+|x|^\rho)<\infty$. The notation $f\in C^2(x)$ means that f is smooth in some neighborhood of x.

Let $f\in C_1(K)\cap C^2(\operatorname{int}K)$. Using the abbreviation

$$I(z,x):=I_{\{z:\ x+\operatorname{diag}xz\in\operatorname{int}K\}}=I_{\operatorname{int}K}(x+\operatorname{diag}xz)$$

we introduce the function

$$\mathcal{I}(f,x):=\int(f(x+\operatorname{diag}xz)-f(x)-\operatorname{diag}xzf'(x))I(z,x)\Pi(dz),\quad x\in\operatorname{int}K.$$

It is well-defined and continuous in x. Indeed, let $\varepsilon>0$ be such that the ball $\mathcal{O}_\varepsilon(x)\subset K$ and $\delta:=\varepsilon/(2|x|)$. Then, using the Taylor formula, we have the bound

$$|f(x+\operatorname{diag}xz)-f(x)-\operatorname{diag}xzf'(x))|\le\kappa_1|z|^2I_{\mathcal{O}_{\varepsilon/2}(x)}(z)+\kappa_2|z|I_{K\setminus\mathcal{O}_{\varepsilon/2}(x)}(z)$$

which right-hand side is integrable with respect to Π.

3 Goal Functionals and Concavity of the Bellman Function

Let $U:\mathcal{C}\to\mathbf{R}_+$ be a concave function such that $U(0)=0$ and $U(x)/|x|\to0$ as $|x|\to\infty$. With every $\pi=(B,C)\in\mathcal{A}_a^x$ we associate the "utility process"

$$J_t^\pi:=\int_0^t e^{-\beta s}U(c_s)\,ds\,,\quad t\ge0\,,$$

where $\beta>0$. We consider the infinite horizon maximization problem with the *goal functional* EJ_∞^π and define its *Bellman function* W by

$$W(x):=\sup_{\pi\in\mathcal{A}_a^x}EJ_\infty^\pi\,,\quad x\in\operatorname{int}K\,. \tag{3.8}$$

Since $\mathcal{A}_a^{x_1}\subseteq\mathcal{A}_a^{x_2}$ when $x_2-x_1\in K$, the function W is **increasing** with respect to the partial ordering $\ge_K$ generated by the cone K.

If π_i, $i=1,2$, are admissible strategies for the initial points x_i, then the strategy $\lambda\pi_1+(1-\lambda)\pi_2$ is an admissible strategy for the initial point $\lambda x_1+(1-\lambda)x_2$, $\lambda\in[0,1]$, laying on the interval connecting x_1 and x_2. In the case where the relative price process Y is continuous, the corresponding ruin time for the process

$$V^{\lambda x_1+(1-\lambda)x_2,\lambda\pi_1+(1-\lambda)\pi_2}=\lambda V^{x_1,\pi_1}+(1-\lambda)V^{x_2,\pi_2} \tag{3.9}$$

dominates the maximum of the ruin times for processes V^{x_i,π_i}. The concavity of u implies that

$$J_t^{\lambda\pi_1+(1-\lambda)\pi_2}\ge\lambda J_t^{\pi_1}+(1-\lambda)J_t^{\pi_2}. \tag{3.10}$$

and, hence, the function W is concave on $\operatorname{int}K$.

Unfortunately, in our main case of interest, where Y has jumps, the ruin times cannot be related in such a simple way. One can easily imagine a situation where $\theta^{x_1,\pi_1}=\theta^{\lambda x_1+(1-\lambda)x_2,\lambda\pi_1+(1-\lambda)\pi_2}<\infty$ while $\theta^{x_2,\pi_2}=\infty$ and the relations (3.9) and (3.10) do not hold. Therefore, we cannot guarantee, by the above argument, that the Bellman function is concave. Of course, these considerations show only that the concavity of W cannot be obtained in a straightforward way as claimed in some publications.

It is not excluded. Moreover, the concavity is rather plausible because one may guess that for the optimal strategies there are no short positions in the risky assets and the ruin by jumps is impossible.

The concavity of the Bellman function W is not a property just interesting *per se*. The classical definition of viscosity solution, as was given by the famous "User's guide" [12], requires the continuity. On the other hand, a concave function is continuous in the interior of its domain (and even locally Lipschitz), see, e.g., [5]. Of course, the model must contains a provision which ensures that W is finite. But the latter property in the case of continuous price processes implies that W is continuous on $\operatorname{int} K$. In the case of processes with jumps one needs to analyze the continuity of W using other arguments.

In the next section we show that the finiteness of W still guarantees its continuity in the interior of K. We do this using the following assertion.

Lemma 3.1 *Suppose that W is a finite function. Let $x \in \operatorname{int} K$. Then the function $\lambda \mapsto W(\lambda x)$ is right-continuous on $\mathbf{R}_+$.*

Proof. Let $\lambda > 0$. Then $\lambda\pi \in \mathcal{A}_a^{\lambda x}$ if and only if $\pi \in \mathcal{A}_a^x$. For a concave function U with $U(0) = 0$ we have, for any $\varepsilon > 0$ the inequality $U(c) \geq (1+\varepsilon)^{-1}U((1+\varepsilon)c)$. Hence, for an arbitrary strategy $\pi \in \mathcal{A}_a^x$ we have that

$$\begin{aligned} J_\infty^{(1+\varepsilon)\pi} - J_\infty^\pi &= E\int_0^\infty e^{-\beta t}\Big(U((1+\varepsilon)c_t) - U(c_t)\Big)dt \\ &\leq \varepsilon E\int_0^\infty e^{-\beta t}U(c_t))dt \leq \varepsilon W(x). \end{aligned}$$

It follows that $W((1+\varepsilon)x) \leq (1+\varepsilon)W(x)$. Since $W(x) \leq W((1+\varepsilon)x)$, we infer from here that $\lambda \mapsto W(\lambda x)$ is right-continuous at the point $\lambda = 1$. Replacing x by λx we obtain the claim. □

If U is a homogeneous function of order γ with $\gamma \in]0,1[$, i.e. $U(\lambda x) = \lambda^\gamma U(x)$ for all $\lambda > 0$, $x \in K$, then $W(\lambda x) = \lambda^\gamma W(x)$. Thus, the function $\lambda \mapsto W(\lambda x)$ is concave and, therefore, continuous if finite.

Remark 1. In financial models usually $\mathcal{C} = \mathbf{R}_+ e_1$ and $\sigma^0 = 0$, i.e. the only first (non-risky) asset is consumed. Correspondingly, $U(c) = u(e_1 c) = u(c^1)$ where u is a utility function of a scalar argument. Our presentation is oriented to the power utility function $u_\gamma(x) = x^\gamma/\gamma$ with $\gamma \in]0,1[$. The case of $\gamma \leq 0$, where, by convention, $u_0(x) = \ln x$, is of interest but it is not covered by the present study.

Remark 2. We consider here a model with mixed "regular-singular" controls. In fact, the assumption that the consumption process has an intensity $c = (c_t)$ and the agent's utility depends on this intensity is not very satisfactory from the economical point of view. One can consider models with an intertemporal substitution and the consumption by "gulps", i.e. dealing with "singular" controls of the class $\mathcal{A}^x$ and the goal functionals like

$$J_t^\pi := \int_0^t e^{-\beta s}U(\bar{C}_s)ds\,,$$

where

$$\bar{C}_s = \int_0^s K(s,r)dC_r$$

with a suitable kernel $K(s,r)$ (the exponential kernel $e^{-\gamma(s-r)}$ is the common choice).

4 Continuity of the Bellman Function

Proposition 4.1 *Suppose that $W(x) < \infty$ for all $x \in \operatorname{int} K$. Then W is continuous on* $\operatorname{int} K$.

Proof. First, we show that the function W is upper semicontinuous on $\operatorname{int} K$. Suppose that this is not the case and there is a sequence x_n converging to some $x_0 \in \operatorname{int} K$ such that $\limsup_n W(x_n) > W(x_0)$. Without loss of generality we way assume that the sequence $W(x_n)$ converges. The points $\tilde{x}_k = (1+1/k)x_0$, $k \geq 1$, belong to the ray $\mathbf{R}_+ x_0$ and converges to x_0. We find a subsequence x_{n_k} such that $\tilde{x}_k \geq_K x_{n_k}$ for all $k \geq 1$. Indeed, since

$$(1+1/k)x_0 \in x_0 + \operatorname{int} K,$$

there exists $\varepsilon_k > 0$ such that

$$(1+1/k)x_0 + \mathcal{O}_{\varepsilon_k}(0) \in x_0 + \operatorname{int} K.$$

It follows that

$$(1+1/k)x_0 + (x_n - x_0) + \mathcal{O}_{\varepsilon_k}(0) \in x_n + \operatorname{int} K$$

and, therefore, $(1+1/k)x_0 \in x_n + \operatorname{int} K$ for all n such that $|x_n - x_0| < \varepsilon_k$. Any strictly increasing sequence of indices n_k with $|x_{n_k} - x_0| < \varepsilon_k$ gives us in a subsequence of points x_{n_k} having the needed property. The function W is increasing with respect to the partial ordering $\geq_K$. Thus,

$$\lim_k W(\tilde{x}_k) \geq \lim_k W(x_{n_k}) > W(x_0).$$

On the other hand, the function $\lambda \mapsto W(\lambda x_0)$ is right-continuous at $\lambda = 1$ and, hence, $\lim_k W(\tilde{x}_k) = W(x_0)$. This contradiction shows that W is upper semicontinuous on $\operatorname{int} K$.

Let us show now that $\liminf_n W(x_n) \geq W(x_0)$ as $x_n \to x_0$, i.e. W is lower semicontinuous on $\operatorname{int} K$.

Fix $\varepsilon > 0$. Due to the finiteness of the Bellman function there are a strategy π and $T \in \mathbf{R}_+$ such that for $\theta = \theta^{x_0,\pi}$ we have the bound

$$E \int_0^{T\wedge\theta} e^{-\beta s} U(c_s)ds \geq W(x_0) - \varepsilon.$$

It remains to show that

$$\liminf_n \theta_n \wedge T \geq \theta \wedge T \qquad \text{a.s.,} \tag{4.11}$$

where we use the abbreviation $\theta_n := \theta^{x_n,\pi}$. Indeed, with this bound we get, using the Fatou lemma, that

$$\begin{aligned} \liminf_n W(x_n) &\geq \liminf_n E \int_0^{\theta_n\wedge T} e^{-\beta s} U(c_s)ds \geq E \liminf_n \int_0^{\theta\wedge T} e^{-\beta s} U(c_s)ds \\ &\geq E \int_0^{\theta\wedge T} e^{-\beta s} U(c_s)ds \geq W(x_0) - \varepsilon \end{aligned}$$

and the claim follows since ε is arbitrarily small.

To prove (4.11), we observe that on $[0, \theta_n \wedge \theta \wedge T]$ we have the representation

$$V_t^{x_n,\pi} - V_t^{x_0,\pi} = \operatorname{diag}(x_n - x_0) S_t$$

implying that

$$\sup_{t \le \theta_n \wedge \theta \wedge T} |V_t^{x_n,\pi} - V_t^{x_0,\pi}| \le S_T^* |x_n - x_0|,$$

where $S_T^* := \sup_{t \le T} |S_t|$. Fix arbitrary, "small", $\delta > 0$. For almost all ω the distance $\rho(\omega)$ of a trajectory $V_t^{x_0,\pi}(\omega)$ on the interval $[0, \theta \wedge T - \delta]$ is strictly positive. The above bound shows that for sufficiently large n the $V_t^{x_n,\pi}(\omega)$ does not deviate from $V_t^{x_0,\pi}(\omega)$ more than on $\rho(\omega)/2$ on the interval $[0, \theta_n(\omega) \wedge \theta(\omega) \wedge T]$. It follows that $\theta_n(\omega) \ge \theta(\omega) \wedge T - \delta$. Thus,

$$\liminf_n \theta_n \wedge T \ge \theta \wedge T - \delta \qquad \text{a.s.}$$

and (4.11) holds. □

5 The Hamilton–Jacobi–Bellman Equation

Let $G := (-K) \cap \partial \mathcal{O}_1(0)$ where $\mathcal{O}_r(y) := \{x \in \mathbf{R}^d : \ |x - y| < r\}$. The set G is compact and $-K = \operatorname{cone} G$. We denote by Σ_G the *support function* of G, given by the relation $\Sigma_G(p) = \sup_{x \in G} px$. The convex function $U^*(.)$ is the Fenchel dual of the convex function $-U(-.)$ whose domain is $-\mathcal{C}$, i.e.

$$U^*(p) = \sup_{x \in \mathcal{C}} (U(x) - px).$$

We introduce a function of five variables by putting

$$F(X, p, \mathcal{I}(f,x), W, x) := \max\{F_0(X, p, \mathcal{I}(f,x), W, x) + U^*(p), \Sigma_G(p)\},$$

where X belongs to $\mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$, $f \in C_1(K) \cap C^2(x)$ and the function F_0 is given by

$$F_0(X, p, \mathcal{I}(f,x), W, x) := \frac{1}{2} \operatorname{tr} A(x) X + \mu(x) p + \mathcal{I}(f,x) - \beta W$$

where $A(x)$ is the matrix with $A^{ij}(x) := a^{ij} x^i x^j$, $\mu^i(x) := \mu^i x^i$, $1 \le i,j \le d$.

In a more detailed form we have that

$$F_0(X, p, \mathcal{I}(f,x), W, x) = \frac{1}{2} \sum_{i,j=1}^d a^{ij} x^i x^j X^{ij} + \sum_{i=1}^d \mu^i x^i p^i + \mathcal{I}(f,x) - \beta W.$$

Note that F_0 is increasing in the argument f in the same sense as $\mathcal{I}$.

If ϕ is a smooth function, we put

$$\mathcal{L}\phi(x) := F(\phi''(x), \phi'(x), \mathcal{I}(\phi, x), \phi(x), x).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.

We show, under mild hypotheses, that W is the unique viscosity solution of the Dirichlet problem for the HJB equation

$$F(W''(x), W'(x), \mathcal{I}(W,x), W(x), x) = 0, \qquad x \in \operatorname{int} K, \tag{5.12}$$

$$W(x) = 0, \qquad x \in \partial K, \tag{5.13}$$

with the boundary condition understood in the usual classical sense.

6 Viscosity Solutions for Integro-Differential Operators

Since, in general, W may have no derivatives at some points $x \in \operatorname{int} K$ (and this is, indeed, the case for the model considered here), the notation (5.12) needs to be interpreted. The idea of viscosity solutions is to substitute W in F by suitable test functions. Formal definitions (adapted to the case we are interested in) are as follows.

A function $v \in C(K)$ is called *viscosity supersolution* of (5.12) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \geq f$ the inequality $\mathcal{L}f(x) \leq 0$ holds.

A function $v \in C(K)$ is called *viscosity subsolution* of (5.12) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \leq f$ the inequality $\mathcal{L}f(x) \geq 0$ holds.

A function $v \in C(K)$ is a *viscosity solution* of (5.12) if v is simultaneously a viscosity super- and subsolution.

At last, a function $v \in C_1(K) \cap C^2(\operatorname{int} K)$ is called *classical supersolution* of (5.12) if $\mathcal{L}v \leq 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\operatorname{int} K$.

For the sake of simplicity and having in mind the specific case we shall work on, we incorporated in the definitions the requirement that the viscosity super- and subsolutions are continuous on K including the boundary. For other cases this might be too restrictive and more general and flexible formulations can be used.

Lemma 6.1 *Suppose that the function v is a viscosity solution of (5.12). If v is twice differentiable at $x_0 \in \operatorname{int} K$, then it satisfies (5.12) at this point in the classical sense.*

Proof. One needs to be more precise with definitions since it is not assumed that v' is defined at every point of a neighborhood of x_0. "Twice differentiable" means here that the Taylor formula at x_0 holds:

$$v(x) = P_2(x - x_0) + (x - x_0)^2 h(|x - x_0|)$$

where

$$P_2(x - x_0) := v(x_0) + \langle v'(x_0), x - x_0 \rangle + \frac{1}{2} \langle v''(x_0)(x - x_0), x - x_0 \rangle$$

and $h(r) \to 0$ as $r \downarrow 0$.

We introduce the notation $\Gamma_r := \{z \in \mathbf{R}^d : \ |\operatorname{diag} x_0 z| \leq r\}$, $r \geq 0$.

Let $\varepsilon > 0$. We choose a number $\delta_0 \in]0,1[$ such that $|h(s)| \leq \varepsilon$ for $s \leq \delta_0$ and define $\delta := \delta_0/(1 + |x_0|)$. Take $\Delta \in]\delta, 1[$ sufficiently close to δ to insure that $x_0 + \mathcal{O}_\Delta(0) \subset K$, $\Pi(\mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0)) \leq \varepsilon$, and $\Pi(\Gamma_\Delta \setminus \Gamma_\delta) \leq \varepsilon$.

We define the function $f_\varepsilon \in C_1(K) \cap C^2(x_0)$ by the formula

$$f_\varepsilon(x) = \begin{cases} P_2(x - x_0) + \varepsilon (x - x_0)^2, & x \in x_0 + \mathcal{O}_\delta(0), \\ g(x) \vee v(x), & x \in x_0 + \mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0), \\ v(x), & x \in x_0 + \mathcal{O}^c_\Delta(0), \end{cases}$$

where

$$g(x) := P_2\left(\delta \frac{x - x_0}{|x - x_0|}\right) + \varepsilon\delta + \frac{\delta - |x - x_0|}{\Delta - |x - x_0|}.$$

Clearly, $f_\varepsilon(x_0) = v(x_0)$ and $f_\varepsilon \geq v$. Since v is a viscosity subsolution, $\mathcal{L}f_\varepsilon(x_0) \geq 0$. Note that,

$$|\mathcal{L}f_\varepsilon(x_0) - \mathcal{L}v(x_0)| \leq \varepsilon \sum_{i=1}^n a^{ii}(x_0^i)^2 + |\mathcal{I}(f_\varepsilon - v, x_0)|.$$

It is not difficult to show that $|\mathcal{I}(f_\varepsilon - v, x_0)|$ is also proportional to ε. Indeed,

$$|\mathcal{I}((f_\varepsilon - v)I_{\mathcal{O}_\delta(0)}, x_0)| \le \varepsilon |x_0|^2 \int_{\mathcal{O}_1(0)} z^2 \Pi(dz).$$

Due to the choice of Δ we have the bound

$$|\mathcal{I}((f_\varepsilon - v)I_{\mathcal{O}_\Delta(0)\setminus\mathcal{O}_\delta(0)}, x_0)| \le 2M\Pi(\mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0)) \le 2M\varepsilon,$$

where M is the supremum of v on the ball $x_0 + \mathcal{O}_{|x_0|}(0)$.

Since

$$|f_\varepsilon(x_0 + \operatorname{diag} x_0 z) - v(x_0 + \operatorname{diag} x_0 z)| \le \varepsilon |x_0|^2 |z|^2, \quad z \in \Gamma_\delta \setminus \Gamma_0,$$

we get that

$$|\mathcal{I}((f_\varepsilon - v)I_{\mathcal{O}^c_\Delta(0)\cap\Gamma_\delta\setminus\Gamma_0}, x_0)| \le \varepsilon |x_0|^2 \int_{\mathcal{O}_1(0)} |z|^2 \Pi(dz).$$

Also we have

$$|\mathcal{I}((f_\varepsilon - v)I_{\mathcal{O}^c_\Delta(0)\cap\Gamma_\Delta\setminus\Gamma_\delta}, x_0)| \le (2m+\varepsilon)\Pi(\Gamma_\Delta \setminus \Gamma_\delta) \le (2m+\varepsilon)\varepsilon,$$

where m is the supremum of v on the ball $x_0 + \mathcal{O}_1(0)$. Letting ε tend to zero, we obtain that $\mathcal{L}v(x_0) \ge 0$. Arguing similarly with $\varepsilon < 0$, we get the opposite inequality. □

7 Jets

Let f and g be functions defined in a neighborhood of zero. We shall write $f(.) \lessapprox g(.)$ if $f(h) \le g(h) + o(|h|^2)$ as $|h| \to 0$. The notations $f(.) \gtrapprox g(.)$ and $f(.) \approx g(.)$ have the obvious meaning.

For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$ we consider the quadratic function

$$Q_{p,X}(z) := pz + (1/2)\langle Xz, z\rangle\,, \quad z \in \mathbf{R}^d\,,$$

and define the *super*- and *subjets* of a function v at the point x:

$$J^+v(x) := \{(p,X):\ v(x+.) \lessapprox v(x) + Q_{p,X}(.)\},$$
$$J^-v(x) := \{(p,X):\ v(x+.) \gtrapprox v(x) + Q_{p,X}(.)\}.$$

In other words, $J^+v(x)$ (resp. $J^-v(x)$) is the family of coefficients of quadratic functions $v(x) + Q_{p,X}(y - .)$ dominating the function $v(.)$ (resp., dominated by this function) in a neighborhood of the point x with precision up to the second order included and coinciding with $v(.)$ at this point.

In the classical theory developed for differential equations the notion of viscosity solutions admits an equivalent formulation in terms of super- and subjets. Since the latter are "local" concepts, such a characterization is not possible for integro-differential operators. Nevertheless, one can prove the following useful result.

Lemma 7.1 *Let v be a viscosity supersolution of the HJB equation and let $x \in \operatorname{int} K$. Let $(p, X) \in J^{-}v(x)$. Then there is a function $f \in C_1(K) \cap C^2(x)$ such that $f'(x) = p$, $f''(x) = X$, $f(x) = v(x)$, $f \le v$ on K and, hence,*

$$F(X, p, \mathcal{I}(f, x), W(x), x) \le 0.$$

Moreover, this function f can be chosen equal to v outside of an arbitrary small neighborhood of x.

Proof. Take $r > 0$ such that the ball $\mathcal{O}_{2r}(x) = \{y : \ |y - x| \le 2r\}$ lays in the interior of K. By definition,

$$v(x+h) - v(x) - Q_{p,X}(h) \ge |h|^2 \varphi(|h|),$$

where $\varphi(u) \to 0$ as $u \downarrow 0$. We consider on $]0, r[$ the function

$$\delta(u) := \sup_{\{h:\ |h| \le u\}} \frac{1}{|h|^2} (v(x+h) - v(x) - Q_{p,X}(h))^{-} \le \sup_{\{y:\ 0 \le y \le u\}} \varphi^{-}(y)$$

where $a^{-} := (-a) \vee 0$. Obviously, δ is continuous, increasing and $\delta(u) \to 0$ as $u \downarrow 0$. The function

$$\Delta(u) := \frac{2}{3} \int_u^{2u} \int_\eta^{2\eta} \delta(\xi) d\xi d\eta$$

vanishes at zero with its two right derivatives; $u^2 \delta(u) \le \Delta(u) \le u^2 \delta(4u)$. It follows that the function $x \mapsto \Delta(|x|)$ belongs to $C^2(\mathcal{O}_r(0))$, its Hessian vanishes at zero, and

$$v(x+h) - v(x) - Q_{p,X}(h) \ge -|h|^2 \delta(|h|) \ge -\Delta(|h|).$$

Thus, $f(y) := v(x) + Q_{p,X}(y - x) - \Delta(|y - x|)$ is dominated by $v(y)$ in the ball $\mathcal{O}_r(x) = \{y : \ |y - x| \le r\}$. We put $f(y) = v(y)$ outside of the ball $\mathcal{O}_{2r}(x)$. We can extend f continuously to the remaining set $\mathcal{O}_{2r}(x) \setminus \mathcal{O}_r(x)$ preserving the inequality $f \le v$. □

For subsolutions we have a similar result with the inverse inequalities.

8 Supersolutions and Properties of the Bellman Function

8.1 When is the Bellman Function W Finite on K?

First, we present sufficient conditions ensuring that the Bellman function W of the considered maximization problem is finite.

Functions we are interested in are defined in the solvency cone K while the process V which may jump out of the latter. In order to be able to apply later the Itô formula we stop $V = V^{x,\pi}$ at the moment immediately preceding the ruin and define the process

$$\tilde{V} = V^{\theta-} = V I_{[0,\theta[} + V_{\sigma-} I_{[\theta,\infty[},$$

where θ is the exit time of V from the interior of the solvency cone K. This process coincides with V on $[0, \theta[$ but, in contrast to the latter, either always remains in K (due to the stopping at θ if $V_{\theta-} \in \operatorname{int} K$) or exits to the boundary in a continuous way and stays on it at the exit point.

It follows from the definitions (2.2) and (2.6) that

$$\tilde{V}_t = v + \int_0^t I_{[0,\theta]}(s)\operatorname{diag}\tilde{V}_s(\mu_s ds + \Xi dw_s)$$
$$+ \int_0^t \int \operatorname{diag}\tilde{V}_{s-} z I(\tilde{V}_{s-}, z)(p(dz, ds) - q(dz, ds)) + B_t - C_t.$$

Let $\varPhi$ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to the partial ordering $\geq_K$ and such that for every $x \in \operatorname{int} K$ and $\pi \in \mathcal{A}_a^x$ the positive process $X^f = X^{f,x,\pi}$ given by the formula

$$X_t^f := e^{-\beta t} f(\tilde{V}_t) + J_t^\pi, \tag{8.14}$$

where $V = V^{x,\pi}$, is a supermartingale.

The set $\varPhi$ of f with this property is convex and stable under the operation $\wedge$ (recall that the minimum of two supermartingales is a supermartingale). Any continuous function which is a monotone limit (increasing or decreasing) of functions from $\varPhi$ also belongs to $\varPhi$.

Lemma 8.1 *(a) If $f \in \varPhi$, then $W \leq f$;*

(b) if a point $y \in \partial K$ is such that there exists $f \in \varPhi$ such that $f(y) = 0$, then W is continuous at y.

Proof. (a) Using the positivity of f, the supermartingale property of X^f, and, finally, the monotonicity of f we get the following chain of inequalities leading to the required property:

$$EJ_t^\pi \leq EX_t^f \leq f(\tilde{V}_0) = f(V_0) \leq f(V_{0-}) = f(x).$$

(b) The continuity of the function W at the point $y \in \partial K$ follows from the inequalities $0 \leq W \leq f$. □

Remark. Recall that a concave function is locally Lipschitz continuous on the interior of its domain, i.e. on the interior of the set where it is finite. Thus, if W is concave function and $\varPhi$ is not empty, then W is continuous (and even locally Lipschitz continuous) on $\operatorname{int} K$. The concavity of W holds in the case where the price process has no jumps.

Lemma 8.2 *Let $f : K \to \mathbf{R}_+$ be a function in $C_1(K) \cap C^2(\operatorname{int} K)$. If f is a classical supersolution of (5.12), then $f \in \varPhi$, i.e. X^f is a supermartingale.*

Proof. First, notice that a classical supersolution is increasing with respect to the partial ordering $\geq_K$. Indeed, by the finite increments formula we have that for any $x, h \in \operatorname{int} K$

$$f(x + h) - f(x) = f'(x + \vartheta h)h$$

for some $\vartheta \in [0, 1]$. The right-hand side is greater or equal to zero because for the supersolution f we have the inequality $\varSigma_G(f'(y)) \leq 0$ whatever is $y \in \operatorname{int} K$, or, equivalently, $f'(y)h \geq 0$ for every $h \in K$, just by the definition of the support function $\varSigma_G$ and the choice of G as a generator of the cone $-K$. By continuity, $f(x+h)-f(x) \geq 0$ for every $x, h \in K$.

Applying the "standard" Itô formula to $e^{-\beta t}f(\tilde V_t)$ we obtain that

$$\begin{aligned} e^{-\beta t}f(\tilde V_t) &= f(x)+\int_0^t e^{-\beta s}f'(\tilde V_{s-})d\tilde V_s-\beta\int_0^t e^{-\beta s}f(\tilde V_{s-})ds\\ &+\frac{1}{2}\int_0^t e^{-\beta s}\mathrm{tr}\,A(\tilde V_{s-})f''(\tilde V_{s-})ds\\ &+\sum_{s\le t}e^{-\beta s}[f(\tilde V_{s-}+\Delta\tilde V_s)-f(\tilde V_{s-})-f'(\tilde V_{s-})\Delta\tilde V_s]. \end{aligned}$$

Note also that

$$\begin{aligned} &\sum_{s\le t}e^{-\beta s}[f(\tilde V_{s-}+\Delta\tilde V_s)-f(\tilde V_{s-})-f'(\tilde V_{s-})\Delta\tilde V_s]I_{\{\Delta B_s=0\}}\\ &=\int_0^t\int e^{-\beta s}[...]I(\tilde V_{s-},z)I_{\{\Delta B_s=0\}}I_{[0,\theta]}(s)p(dz,ds)\\ &=\int_0^t\int e^{-\beta s}[...]I(\tilde V_{s-},z)I_{[0,\theta]}(s)\Pi(dz)ds\\ &+\int_0^t\int e^{-\beta s}[...]I(\tilde V_{s-},z)I_{\{\Delta B_s=0\}}I_{[0,\theta]}(s)(p(dz,ds)-\Pi(dz)ds), \end{aligned}$$

where we replace in the integrals by dots the lengthy expression

$$f(\tilde V_{s-}+\mathrm{diag}\,\tilde V_{s-}z)-f(\tilde V_{s-})-f'(\tilde V_{s-})\mathrm{diag}\,\tilde V_{s-}z.$$

Using the above formulae we obtain after regrouping terms the following representation for $X_t^f=e^{-\beta t}f(\tilde V_t)+J_t^\pi$:

$$X_t^f=f(x)+\int_0^{t\wedge\theta}e^{-\beta s}[\mathcal{L}_0f(\tilde V_s)-c_sf'(\tilde V_s)+U(c_s)]ds+R_t+m_t, \tag{8.15}$$

where

$$R_t:=\int_0^{t\wedge\theta}e^{-\beta s}f'(V_{s-})dB_s^c+\sum_{s\le t}e^{-\beta s}[f(\tilde V_{s-}+\Delta B_s)-f(\tilde V_{s-})] \tag{8.16}$$

and m is the local martingale

$$\begin{aligned} m_t &= \int_0^{t\wedge\theta}e^{-\beta s}f'(\tilde V_{s-})\mathrm{diag}\,\tilde V_s\Xi dw_s\\ &+\int_0^{t\wedge\theta}\int e^{-\beta s}[f(\tilde V_{s-}+\mathrm{diag}\,\tilde V_{s-}z)-f(\tilde V_{s-})]I(\tilde V_{s-},z)(p(dz,ds)-\Pi(dz)ds). \end{aligned}$$

By definition of a supersolution, for any $x\in\mathrm{int}\,K$,

$$\mathcal{L}_0f(x)\le -U^*(f'(x))\le cf'(x)-U(c)\quad\forall\, c\in\mathcal{C}.$$

Thus, the integral in (8.15) is a decreasing process. The process R is also decreasing. Indeed, the terms of the sum in (8.16) are less or equal to zero in virtue of the monotonicity of f and

$$f'(V_{s-})dB_s^c=I_{\{\Delta B_s=0\}}f'(V_{s-})\dot B_s d||B||_s$$

where $f'(V_{s-})\dot{B}_s \le 0$ since $\dot{B}$ takes values in $-K$. Let σ_n be a localizing sequence for m. Taking into account that $X^f \ge 0$, we obtain from (8.15) that for each n the negative decreasing process $R_{t\wedge\sigma_n}$ dominates an integrable process and so it is integrable. The same conclusion holds for the stopped integral. Being a sum of an integrable decreasing process and a martingale, the process $X^f_{t\wedge\sigma_n}$ is a positive supermartingale and, hence, by the Fatou lemma, X^f is a supermartingale as well. □

Lemma 8.2 implies that the existence of a smooth positive supersolution f of (5.12) ensures the finiteness of W on K. Sometimes, e.g., in the case of power utility function, it is possible to find such a function in a rather explicit form.

Remark. Let $\bar{\mathcal{O}}$ be the closure of an open subset $\mathcal{O}$ of K and let $f : \bar{\mathcal{O}} \to \mathbf{R}_+$ be a classical supersolution in $\bar{\mathcal{O}}$. Let $x \in \mathcal{O}$ and let τ be the exit time of the process $V^{x,\pi}$ from $\bar{\mathcal{O}}$. The above arguments imply that the process $X^f_{t\wedge\tau}$ is a supermartingale and, therefore,

$$E[e^{-\beta(t\wedge\tau)}f(\tilde{V}_{t\wedge\tau}) + J^\pi_{t\wedge\tau}] \le f(x). \tag{8.17}$$

8.2 Strict Local Supersolutions

For the strict supersolution we can get a more precise result which will play the crucial role in deducing from the Dynamic Programming Principle the property of W to be a subsolution of the HJB equation.

We fix a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$ such that the larger ball $\bar{\mathcal{O}}_{2r}(x) \subseteq \operatorname{int} K$ and define $\tau^\pi = \tau^\pi_r$ as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.

$$\tau^\pi := \inf\{t \ge 0 : \ |V^{\pi,x}_t - x| \ge r\}.$$

Lemma 8.3 *Let $f \in C_1(K) \cap C^2(\mathcal{O}_{2r}(x))$ be such that $\mathcal{L}f \le -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exist a constant $\eta = \eta_\varepsilon > 0$ and an interval $]0, t_0]$ such that*

$$\sup_{\pi \in \mathcal{A}^x_a} EX^{f,x,\pi}_{t\wedge\tau^\pi} \le f(x) - \eta t \qquad \forall\, t \in]0, t_0].$$

Proof. We fix a strategy π and omit its symbol in the notations below. In what follows, only the behavior of the processes on $[0,\tau]$ does matter. Note that $|V_\tau - x| \ge r$ on the set $\{\tau < \infty\}$. As in the proof of Lemma 8.2, we apply the Itô formula and obtain the representation

$$\begin{aligned} X^f_{t\wedge\tau} &:= e^{-\beta(t\wedge\tau)}f(\tilde{V}_{t\wedge\tau}) + J^\pi_{t\wedge\tau} \\ &= f(x) + \int_0^{t\wedge\theta\wedge\tau} e^{-\beta s}\mathcal{L}f(\tilde{V}_s)ds \\ &\quad - \int_0^{t\wedge\theta\wedge\tau} e^{-\beta s}[U^*(V_s) + c_s f'(\tilde{V}_s) - U(c_s)]ds + R_{t\wedge\tau} + m_{t\wedge\tau}. \end{aligned}$$

Due to the monotonicity of f we may assume without loss of generality that on the interval $[0,\tau]$ the increment ΔB_t does not exceed the distance of V_s to the boundary of $\mathcal{O}_r(x)$. In other words, if the exit from the ball is due to an action (and not because of a jump of the price process), we can replace this action by a less expensive one, with the jump of the process $\tilde{V}$ in the same direction but a smaller one, ending on the boundary of the ball.

By assumption, for $y \in \mathcal{O}_r(x)$ we have the bounds $\mathcal{L}f(y) \le -\varepsilon$ (helpful to estimate the first integral in the right-hand side) and $\Sigma_G(f'(y)) \le -\varepsilon$. The latter inequality means that $kf'(y) \le -\varepsilon|k|$ for every $k \in -K$ (therefore, we have the inclusion $f'(\bar{\mathcal{O}}_r(x)) \subset \operatorname{int} K^*$). In particular, for $s \in [0,\tau]$

$$f'(V_{s-})\dot{B}_s \le -\varepsilon|\dot{B}_s|, \qquad [f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})] \le -\varepsilon|\Delta B_s|.$$

Since $|\tilde{V}_{s-} - x| \le r$ for $s \in [0,\tau]$, we obtain, using the finite increment formula and the linear growth of f the bound

$$[f(\tilde{V}_{s-} + \operatorname{diag} \tilde{V}_{s-} z) - f(\tilde{V}_{s-})] I(\tilde{V}_{s-}, z) \le \kappa_1 |z|^2 I_{\{|z| \le 1/2\}} + \kappa_2 |z| I_{\{|z|>1/2\}}.$$

It follows that the local martingale $(m_{t\wedge\tau})$ is a martingale with $m_{t\wedge\tau} = 0$.

The above observations imply the inequality

$$EX^{f,x}_{t\wedge\tau} \le f(x) - e^{-\beta t} EN_t,$$

where

$$N_t := \varepsilon\ (t \wedge \tau) + \int_0^{t\wedge\tau} H(c_s, f'(V_s))ds + \varepsilon \int_0^{t\wedge\tau} |\dot{B}_s| d||B||_s$$

with $H(c,p) := U^*(p) + pc - U(c) \ge 0$. It remains to verify that EN_t dominates, on a certain interval $]0, t_0]$, a strictly increasing linear function which is independent of π.

The process N_t looks a bit complicated but we can replace it by another one of a simpler structure. To this end, note that there is a constant κ (“large”, for convenience, $\kappa \ge 1$) such that

$$\inf_{p \in f'(\bar{\mathcal{O}}_r(x))} H(c,p) \ge \kappa^{-1}|c|, \qquad \forall\, c \in \mathcal{C}, \quad |c| \ge \kappa.$$

Indeed, being the image of a closed ball under continuous mapping, the set $f'(\bar{\mathcal{O}}_r(x))$ is a compact in $\operatorname{int} K^*$. The lower bound of the continuous function U^* on $f'(\bar{\mathcal{O}}_r(x))$ is finite. For any p from $f'(\bar{\mathcal{O}}_r(x))$ and $c \in \mathcal{C} \subseteq K$ we have the inequality $(c/|c|)p \ge \varepsilon$. At last, $U(c)/|c| \to 0$ as $c \to \infty$. Combining these facts we infer the claimed inequality. Thus, for the first integral in the definition of N_t we have the bound

$$\int_0^{t\wedge\tau} H(c_s, f'(V_s))ds \ge \kappa^{-1} \int_0^{t\wedge\tau} I_{\{|c_s| \ge \kappa\}} |c_s| ds.$$

The second integral in the definition dominates $\kappa_1 ||B||_{t\wedge\tau}$ for some $\kappa_1 > 0$. To see this, let us consider the absolute norm $|.|_1$ in $\mathbf{R}^d$. In contrast with the total variation $||B||$ calculated with respect to the Euclidean norm $|.|$, the total variation of B with respect to the absolute norm admits a simpler expression $\sum_i \operatorname{Var} B^i$ where $\operatorname{Var} B^i$ is the total variation of the scalar process B^i. Obviously,

$$|\dot{B}|_1 = \sum_i |\dot{B}^i| = \sum_i \left| \frac{dB^i}{d||B||} \right| = \sum_i \left| \frac{dB^i}{d\mathrm{Var}\, B^i} \right| \frac{d\mathrm{Var}\, B^i}{d||B||} = \frac{d \sum_i \mathrm{Var}\, B^i}{d||B||}.$$

But norms in $\mathbf{R}^d$ are equivalent, i.e. $\tilde{\kappa}^{-1}|.| \le |.|_1 \le \tilde{\kappa}|.|$ for some strictly positive constant $\tilde{\kappa}$. The same inequalities relate the corresponding total variation processes. The claimed property follows from here with the constant $\kappa_1 = \tilde{\kappa}^2$.

Summarizing, we conclude that it is sufficient to check the domination property for $E\tilde{N}_t$ with

$$\tilde{N}_t := t \wedge \tau + \int_0^{t\wedge\tau} I_{\{|c_s|\geq\kappa\}}|c_s|ds + ||B||_{t\wedge\tau}. \tag{8.18}$$

These processes $\tilde{N}$ have a transparent dependence on the control. The idea of the concluding reasoning is very simple: on a certain set of strictly positive probability, where one may neglect the random fluctuations, either τ is "large", or the total variation of the control is "large": one can accelerate exit only by an intensive trading or consumption.

The formal arguments are as follows. Using the stochastic Cauchy formula (2.4) and the fact that $\mathcal{E}_{0+}(Y^i) = \mathcal{E}_0(Y^i) = 1$, we get immediately that there exist a number $t_0 > 0$ and a measurable set Γ with $P(\Gamma) > 0$ on which

$$|V^{x,\pi} - x| \leq r/2 + \delta(||B|| + ||C||) \quad \text{on } [0, t_0]$$

whatever is the control $\pi = (B, C)$. Of course, diminishing t_0, we may assume without loss of generality that $\kappa t_0 \leq r/(4\delta)$. For any $t \leq t_0$ we have on the set $\Gamma \cap \{\tau \leq t\}$ the inequality $||B||_\tau + ||C||_\tau \geq r/(2\delta)$ and, hence,

$$\tilde{N}_t \geq ||B||_\tau + ||C||_\tau - \int_0^\tau I_{\{|c_s|<\kappa\}}|c_s|ds \geq \frac{r}{2\delta} - \kappa t_0 \geq \kappa t_0 \geq t_0 \geq t.$$

On the set $\Gamma \cap \{\tau > t\}$ the inequality $\tilde{N}_t \geq t$ is obvious. Thus, $E\tilde{N}_t \geq tP(\Gamma)$ on $[0, t_0]$ and the result is proven. □

9 Dynamic Programming Principle

The aim of this section is to establish the following two assertions which will serve to derive the HJB equation for the Bellman function. For the considered model, they constitute an analog of the classical Dynamic Programming Principle. The latter is usually written in the form of a single identity (see the remark at the end of the section), but for our purpose this form, more precise, is needed.

Lemma 9.1 *Let $\mathcal{T}_f$ be the sets of finite stopping times. Then*

$$W(x) \leq \sup_{\pi\in\mathcal{A}_a^x} \inf_{\tau\in\mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau}W(V_{\tau-}^{x,\pi})I_{\{\tau<\theta\}}\right). \tag{9.19}$$

Lemma 9.2 *Suppose that W is continuous on $\operatorname{int} K$. Then for any $\tau \in \mathcal{T}_f$*

$$W(x) \geq \sup_{\pi\in\mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau}W(V_{\tau-}^{x,\pi})I_{\{\tau\leq\theta\}}\right). \tag{9.20}$$

We work on the canonical filtered space of càdlàg functions equipped with the measure P which is the distribution of the driving Lévy process. The generic point $\omega = \omega_.$ of this space is a d-dimensional càdlàg function on $\mathbf{R}_+$, zero at the origin. Let $\mathcal{F}_t^\circ := \sigma\{\omega_s,\ s \leq t\}$ and $\mathcal{F}_t := \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}^\circ$. We add the superscript P to denote σ-algebras augmented by all P-null sets from Ω. Recall that $\mathcal{F}_t^{\circ,P}$ coincides with $\mathcal{F}_t^P$ (this assertion follows easily from the predictable representation theorem). The Skorohod metric makes Ω a Polish space and its Borel σ-algebra coincides with $\mathcal{F}_\infty$, for details see [16].

Since elements of Ω are paths, we can define such operators as the stopping $\omega_. \mapsto \omega_.^s$, $s \geq 0$, where $\omega_.^s = \omega_{s\wedge .}$ and the translation $\omega_. \mapsto \omega_{s+.} - \omega_s$. Taking Doob's theorem into account, one can describe $\mathcal{F}_s^\circ$-measurable random variables as those of the form $g(w.) = g(w_.^s)$ where g is a measurable function on Ω.

We define also the "concatenation" operator as the measurable mapping

$$g : \mathbf{R}_+ \times \Omega \times \Omega \to \Omega$$

with $g_t(s, \omega_., \tilde{\omega}_.) = \omega_t I_{[0,s[}(t) + (\tilde{\omega}_{t-s} + \omega_s) I_{[s,\infty[}(t)$.

Notice that

$$g_t(s, \omega_.^s, \omega_{.+s} - \omega_s) = \omega_t.$$

Thus, $\pi(\omega) = \pi(g(s, \omega_.^s, \omega_{.+s} - \omega_s))$.

Let π be a fixed strategy from $\mathcal{A}_a^x$ and let $\theta = \theta^{x,\pi}$ be the exit time from $\operatorname{int} K$ for the process $V^{x,\pi}$.

Recall the following general fact on regular conditional distributions.

Let ξ and η be two random variables taking values in Polish spaces X and Y equipped with their Borel σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. Then ξ admits a regular conditional distribution given $\eta = y$ which we shall denote by $p_{\xi|\eta}(\Gamma, y)$. This means that $p_{\xi|\eta}(., y)$ is a probability measure on $\mathcal{X}$, $p_{\xi|\eta}(\Gamma, .)$ is a $\mathcal{Y}$-measurable function, and

$$E(f(\xi,\eta)|\eta) = \int f(x,y) p_{\xi|\eta}(dx, y)\bigg|_{y=\eta} \qquad (a.s.)$$

for any $\mathcal{X} \times \mathcal{Y}$-measurable function $f(x,y) \geq 0$.

We shall apply the above relation to the random variables $\xi = (\omega_{.+\tau} - \omega_\tau)$ and $\eta = (\tau, \omega^\tau)$. It is well-known that the Lévy process starts afresh at stopping times, i.e. the measure $P(.)$ itself (not depending on y) is the regular conditional distribution $p_{\xi|\eta}(., y)$.

At last, for fixed s and w^s, the shifted control $\pi_{.+s}(g(s, \omega_.^s, \tilde{\omega}_.))$ is admissible for the initial condition $V_{s-}^{x,\pi}(\omega)$. Here we denote by $\tilde{\omega}_.$ a generic point of the canonical space.

Proof of Lemma 9.1. For arbitrary $\pi \in \mathcal{A}_a^x$ and $\mathcal{T}_f$ we have that

$$\begin{aligned} EJ_\infty^\pi &= EJ_\tau^\pi + Ee^{-\beta\tau} I_{\{\tau<\theta\}} \int_0^\infty e^{-\beta r} U(c_{r+\tau}) dr \\ &= EJ_\tau^\pi + Ee^{-\beta\tau} I_{\{\tau<\theta\}} E\bigg(\int_0^\infty e^{-\beta r} U(c_{r+\tau}) dr \bigg| (\tau, \omega^\tau) \bigg). \end{aligned}$$

According to the above discussion we can rewrite the second term of the right-hand side as

$$Ee^{-\beta\tau} I_{\{\tau<\theta\}} \int \bigg(\int_0^\infty e^{-\beta r} U(c_{r+\tau}(g(\tau, \omega^\tau, \tilde{\omega}))) dr \bigg) P(d\tilde{\omega})$$

and dominate it by $Ee^{-\beta\tau} I_{\{\tau<\theta\}} W(V_{\tau-}^{x,\pi})$. Thus,

$$EJ_\infty^\pi \leq EJ_\tau^\pi + Ee^{-\beta\tau} I_{\{\tau<\theta\}} W(V_{\tau-}^{x,\pi}).$$

This bound leads directly to the first announced inequality. □

Proof of Lemma 9.2. Fix $\varepsilon > 0$. By hypothesis, the function W is continuous on $\operatorname{int} K$. For each $x \in \operatorname{int} K$ we can find an open ball $\mathcal{O}_r(x) = x + \mathcal{O}_r(0)$ with $r = r(\varepsilon, x) < \varepsilon$

contained in the open set $\{y \in \operatorname{int} K : \ |W(y) - W(x)| < \varepsilon\}$. Moreover, we can find a smaller ball $\mathcal{O}_{\tilde{r}}(x)$ contained in the set $y(x) + K$ with some $y(x) \in \mathcal{O}_r(x)$. Indeed, take an arbitrary $x_0 \in \operatorname{int} K$. Then, for some $\delta > 0$, the ball $x_0 + \mathcal{O}_\delta(0) \subset K$. Since K is a cone, $\lambda x_0 + \mathcal{O}_{\lambda\delta}(0) \subset K$ for every $\lambda > 0$ and this inclusion implies that

$$x + \mathcal{O}_{\lambda\delta}(0) \subset x - \lambda x_0 + K$$

Clearly, the requirement is met for $y(x) = x - \lambda x_0$ and $\tilde{r} = \lambda\delta$ when $\lambda|x_0| < r$ and $\lambda\delta < r$. The family of sets $\mathcal{O}_{\tilde{r}(x)}(x)$, $x \in \operatorname{int} K$, is an open covering of $\operatorname{int} K$. But any open covering of a separable metric space contains a countable subcovering (this is the Lindelöf property; in our case, where $\operatorname{int} K$ is a countable union of compacts, it is obvious). Take a countable subcovering indexed by points x_n. For simplicity, we shall denote its elements by $\mathcal{O}_n$ and $y(x_n)$ by y_n. Put $A_1 = \mathcal{O}_1$, and $A_n = \mathcal{O}_n \setminus \cup_{k<n} \mathcal{O}_k$. The sets A_n are disjoint and their union is $\operatorname{int} K$.

Let $\pi^n = (B^n, C^n) \in \mathcal{A}_a^{y_n}$ be an ε-optimal strategy for the initial point y_n, i.e. such that

$$EJ_\infty^{\pi_n} \geq W(y_n) - \varepsilon.$$

Let $\pi \in \mathcal{A}_a^x$ be an arbitrary strategy. We consider the strategy $\tilde{\pi} \in \mathcal{A}_a^x$ defined by the relation

$$\tilde{\pi} = \pi I_{[0,\tau[} + \sum_{n=1}^{\infty} [(y_n - V_{\tau-}^{x,\pi}, 0) + \bar{\pi}^n] I_{[\tau,\infty[} I_{A_n}(V_{\tau-}^{x,\pi}) I_{\{\tau \leq \theta\}}$$

where $\bar{\pi}^n$ is the translation of the strategy π^n: namely, for a point $\omega_.$ with $\tau(\omega) = s < \infty$ we have

$$\bar{\pi}_t^n(\omega_.) := \pi_{t-s}^n(\omega_{.+s} - \omega_s).$$

In other words, the strategy $\tilde{\pi}$ coincides with π on $[0, \tau[$ and with the shift of π^n on $]\tau, \infty[$ when $V_{\tau-}^{x,\pi}$ is in A_n; the correction term guarantees that in the latter case the trajectory of the control system corresponding to the control $\tilde{\pi}$ passes at time τ through the point y_n.

Now, using the same considerations as in the previous lemma, we have:

$$\begin{aligned}
W(x) \geq EJ_\infty^{\tilde{\pi}} &= EJ_\tau^\pi + \sum_{n=1}^{\infty} EI_{A_n}(V_{\tau-}^{x,\pi}) I_{\{\tau \leq \theta\}} \int_\tau^\infty e^{-\beta s} U(\bar{c}_s^n) ds \\
&\geq EJ_\tau^\pi + \sum_{n=1}^{\infty} EI_{A_n}(V_{\tau-}^{x,\pi}) I_{\{\tau \leq \theta\}} e^{-\beta\tau}(W(y_n) - \varepsilon) \\
&\geq EJ_\tau^\pi + Ee^{-\beta\tau} W(V_{\tau-}^{x,\pi}) I_{\{\tau \leq \theta\}} - 2\varepsilon.
\end{aligned}$$

Since π and ε are arbitrary, the result follows. $\square$

Remark. The previous lemmas imply that for any $\tau \in \mathcal{T}_f$ the following identity holds:

$$W(x) = \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi}) I_{\{\tau \leq \theta\}}\right).$$

It can be considered as a form of the dynamic programming principle but, seemingly, it is not sufficient for our derivation of the HJB equation.

10 The Bellman Function and the HJB Equation

Theorem 10.1 *Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of (5.12).*

Proof. The claim follows from the two lemmas below. □

Lemma 10.2 *If (9.20) holds then W is a viscosity supersolution of (5.12).*

Proof. Let $x \in \operatorname{int} K$. Choose a test function $\phi \in C^1(K) \cap C^2(x)$ such that $\phi(x) = W(x)$ and $W \geq \phi$. Take $r \in]0,1]$ small enough to ensure that the ball $\bar{\mathcal{O}}_{2r}(x) \subset K$ and ϕ is smooth on $O_{2r}(x)$.

At first, we fix an arbitrary point $m \in K$. Let $\varepsilon > 0$ be sufficiently small to guarantee that $x - \varepsilon m \in \mathcal{O}_r(x)$. The function W is increasing with respect to the partial ordering generated by K. Thus,

$$\phi(x) = W(x) \geq W(x - \varepsilon m) \geq \phi(x - \varepsilon m).$$

Taking a limit as $\varepsilon \to 0$, we easily obtain that $-m\phi'(x) \leq 0$ and, hence, $\Sigma_G(\phi'(x)) \leq 0$.

Take now π with $B_t = 0$ and $c_t = c \in \mathcal{C}$. Let $\tau_r = \tau_r^\pi \leq \theta$ be the exit time of the process $V = V^{x,\pi}$ from the ball $\mathcal{O}_r(x)$; obviously, $\tau_r \leq \theta$. The properties of the test function and the inequality (9.20) imply that

$$\begin{aligned} \phi(x) = W(x) &\geq E\left(J^\pi_{t\wedge\tau_r} + e^{-\beta(t\wedge\tau_r)} W(V_{t\wedge\tau_r -})\right) \\ &\geq E\left(J^\pi_{t\wedge\tau_r} + e^{-\beta(t\wedge\tau_r)} \phi(V_{t\wedge\tau_r -})\right). \end{aligned}$$

We get from here using the Itô formula (8.15), that

$$\begin{aligned} 0 &\geq E\left(\int_0^{t\wedge\tau_r} e^{-\beta s} U(c_s)ds + e^{-\beta(t\wedge\tau_r)}\phi(V_{t\wedge\tau_r -})\right) - \phi(x) \\ &\geq E\int_0^{t\wedge\tau_r} e^{-\beta s}[\mathcal{L}_0\phi(V_s) - c\phi'(V_s) + U(c)]ds \\ &\geq \min_{y\in\bar{\mathcal{O}}_r(x)} [\mathcal{L}_0\phi(y) - c\phi'(y) + U(c)] E\left[\frac{1}{\beta}\left(1 - e^{-\beta(t\wedge\tau_r)}\right)\right]. \end{aligned}$$

Dividing the resulting inequality by t and taking successively the limits as t and r converge to zero we infer that $\mathcal{L}_0\phi(x) - c\phi'(x) + U(c) \leq 0$. Maximizing over $c \in \mathcal{C}$ yields the bound $\mathcal{L}_0\phi(x) + U^*(\phi'(x)) \leq 0$ and, therefore, W is a supersolution of the HJB equation. □

Lemma 10.3 *If (9.19) holds then W is a viscosity subsolution of (5.12).*

Proof. Let $x \in \operatorname{int} K$ and let $\phi \in C^1(K) \cap C^2(x)$ be a function such that $\phi(x) = W(x)$ and $W \leq \phi$ on $\mathcal{O}$. Assume that the subsolution inequality for ϕ fails at x. Thus, there exists $\varepsilon > 0$ such that $\mathcal{L}\phi \leq -\varepsilon$ on some ball $\bar{\mathcal{O}}_r(x) \subset \operatorname{int} K$. By virtue of Lemma 8.3 (applied to the function ϕ) there are $t_0 > 0$ and $\eta > 0$ such that on the interval $]0, t_0]$ for any strategy $\pi \in \mathcal{A}^x_a$

$$E\left(J^\pi_{t\wedge\tau^\pi} + e^{-\beta\tau^\pi}\phi(V^{x,\pi}_{t\wedge\tau^\pi})\right) \leq \phi(x) - \eta t,$$

where τ^π is the exit time of the process $V^{x,\pi}$ from the ball $\mathcal{O}_r(x)$. Fix arbitrary $t \in]0, t_0]$. By the second claim of Lemma 9.1) there exists $\pi \in \mathcal{A}_a^x$ such that

$$W(x) \le E\left(J^\pi_{t\wedge\tau} + e^{-\beta\tau} W(V^{x,\pi}_{t\wedge\tau})\right) + \frac{1}{2}\eta t,$$

for every stopping time τ, in particular for τ^π.

Using the inequality $W \le \phi$ and applying Lemma 8.3 we obtain from the above relations that $W(x) \le \phi(x) - (1/2)\eta t$. This is a contradiction because at the point x the values of W and ϕ are the same. □

11 Uniqueness Theorem

Before formulating the uniqueness theorem we recall the Ishii lemma.

Lemma 11.1 *Let v and $\tilde{v}$ be two continuous functions on an open subset $\mathcal{O} \subseteq \mathbf{R}^d$. Consider the function $\Delta(x,y) := v(x) - \tilde{v}(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that*

$$(n(\widehat{x}-\widehat{y}), X) \in \bar{J}^+ v(\widehat{x}), \qquad (n(\widehat{x}-\widehat{y}), Y) \in \bar{J}^- \tilde{v}(\widehat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{11.21}$$

In this statement I is the identity matrix and $\bar{J}^+ v(x)$ and $\bar{J}^- v(x)$ are values of the set-valued mappings whose graphs are closures of graphs of the set-value mappings $J^+ v$ and $J^- v$, respectively.

Of course, if v is smooth, the claim follows directly from the necessary conditions of a local maximum (with $X = v''(\widehat{x})$, $Y = \tilde{v}''(\widehat{y})$ and the constant 1 instead of 3 in inequality (11.21)).

The inequality (11.21) implies the bound

$$\operatorname{tr}(A(x)X - A(y)Y) \le 3n|A|^{1/2}|x-y|^2 \tag{11.22}$$

which will be used in the sequel (for the proof see, e.g., Section 4.2 in [19]).

The following concept plays a crucial role in the proof of the purely analytic result on the uniqueness of the viscosity solution which we establish by a classical method of doubling variables using the Ishii lemma.

Definition. We say that a positive function $\ell \in C_1(K) \cap C^2(\operatorname{int} K)$ is the *Lyapunov function* if the following properties are satisfied:

1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0 \ell(x) \le 0$ for all $x \in \operatorname{int} K$,

2) $\ell(x) \to \infty$ as $|x| \to \infty$.

In other words, ℓ is a classical strict supersolution of the **truncated** equation (without the term U^*), continuous up to the boundary, and increasing to infinity at infinity.

Theorem 11.2 *Assume that the jump measure Π does not charge $(d-1)$-dimensional surfaces. Suppose that there exists a Lyapunov function ℓ. Then the Dirichlet problem (5.12), (5.13) has at most one viscosity solution in the class of continuous functions satisfying the growth condition*

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty. \tag{11.23}$$

Proof. Let W and $\tilde{W}$ be two viscosity solutions of (5.12) coinciding on the boundary ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that

$$W(z) - \tilde{W}(z) - 2\varepsilon\ell(z) > 0.$$

We introduce a family of continuous functions $\Delta_n : K \times K \to \mathbf{R}$ by putting

$$\Delta_n(x,y) := W(x) - \tilde{W}(y) - \frac{1}{2}n|x-y|^2 - \varepsilon[\ell(x) + \ell(y)], \qquad n \geq 0.$$

Note that $\Delta_n(x,x) = \Delta_0(x,x)$ for all $x \in K$ and $\Delta_0(x,x) \leq 0$ when $x \in \partial K$. From the assumption that the function ℓ has a higher growth rate than W we deduce that $\Delta_n(x,y) \to -\infty$ as $|x| + |y| \to \infty$. It follows that the level sets $\{\Delta_n \geq a\}$ are compacts and the function Δ_n attains its maximum. That is, there exists $(x_n, y_n) \in K \times K$ such that

$$\Delta_n(x_n, y_n) = \bar{\Delta}_n := \sup_{(x,y) \in K \times K} \Delta_n(x,y) \geq \bar{\Delta} := \sup_{x \in K} \Delta_0(x,x) > 0.$$

All (x_n, y_n) belong to the compact set $\{(x,y) : \ \Delta_0(x,y) \geq 0\}$. It follows that the sequence $n|x_n - y_n|^2$ is bounded. We continue to argue (without introducing new notations) with a subsequence along which (x_n, y_n) converge to some limit $(\widehat{x}, \widehat{x})$. Necessarily, $n|x_n - y_n|^2 \to 0$ (otherwise we would have $\Delta_0(\widehat{x}, \widehat{x}) > \bar{\Delta}$). It is easily seen that $\bar{\Delta}_n \to \Delta_0(\widehat{x}, \widehat{x}) = \bar{\Delta}$. Thus, $\widehat{x}$ is an interior point of K and so are x_n and y_n for sufficiently large n.

By virtue of the Ishii lemma applied to the functions $v := W - \varepsilon\ell$ and $\tilde{v} := \tilde{W} + \varepsilon\ell$ at the point (x_n, y_n) there exist matrices X^n and Y^n satisfying (11.21) such that

$$(n(x_n - y_n), X^n) \in \bar{J}^+ v(x_n), \qquad (n(x_n - y_n), Y^n) \in \bar{J}^- \tilde{v}(y_n). \tag{11.24}$$

Suppose for a moment that

$$(n(x_n - y_n), X^n) \in J^+ v(x_n), \qquad (n(x_n - y_n), Y^n) \in J^- \tilde{v}(y_n). \tag{11.25}$$

Using the notations $p_n := n(x_n - y_n) + \varepsilon\ell'(x_n)$, $q_n := n(x_n - y_n) - \varepsilon\ell'(y_n)$ and putting $X_n := X^n + \varepsilon\ell''(x_n)$, $Y_n := Y^n - \varepsilon\ell''(y_n)$, we may rewrite the last relations in the following equivalent form:

$$(p_n, X_n) \in J^+ W(x_n), \qquad (q_n, Y_n) \in J^- \tilde{W}(y_n). \tag{11.26}$$

Since W and $\tilde{W}$ are viscosity sub- and supersolutions, one can find, according to Lemma 7.1 the functions $f_n \in C_1(K) \cap C^2(x_n)$ and $\tilde{f}_n \in C_1(K) \cap C^2(y_n)$ such that $f_n'(x_n) = p_n$, $f_n''(x_n) = X_n$, $f_n(x_n) = W(x_n)$, $f_n \leq W$ on K, and $\tilde{f}_n'(y_n) = q_n$, $\tilde{f}_n''(y_n) = Y_n$, $\tilde{f}_n(y_n) = \tilde{W}(y_n)$, $\tilde{f}_n \geq \tilde{W}$ on K,

$$F(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) \geq 0 \geq F(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n).$$

The second inequality implies that $mq_n \leq 0$ for each $m \in G = (-K) \cap \partial\mathcal{O}_1(0)$. But for the Lyapunov function $\ell'(x) \in \operatorname{int} K^*$ when $x \in \operatorname{int} K$ and, therefore,

$$mp_n = mq_n + \varepsilon m(\ell'(x_n) + \ell'(y_n)) < 0.$$

Since G is a compact, $\Sigma_G(p_n) < 0$. It follows that

$$F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) + U^*(p_n) \geq 0,$$
$$F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) + U^*(q_n) \leq 0.$$

Recall that U^* is decreasing with respect to the partial ordering generated by $\mathcal{C}^*$ hence also by K^*. Thus, $U^*(p_n) \le U^*(q_n)$ and we obtain the inequality

$$b_n := F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) - F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) \ge 0.$$

Clearly,

$$\begin{aligned} b_n &= \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij}^n - a^{ij}y_n^i y_n^j Y_{ij}^n) + n\sum_{i=1}^{d}\mu^i(x_n^i - y_n^i)^2 \\ &\quad -\frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) \\ &\quad +\varepsilon(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)). \end{aligned}$$

By virtue of (11.22) the first term in the right-hand side is dominated by a constant multiplied by $n|x_n - y_n|^2$; a similar bound for the second sum is obvious; the last term is negative according to the definition of the Lyapunov function. To complete the proof, it remains to show that

$$\limsup_n \left(\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n)\right) \le 0. \tag{11.27}$$

Indeed, with this we have that $\limsup b_n \le -\beta\bar{\Delta} < 0$, i.e. a contradiction arising from the assumption $W(z) > \tilde{W}(z)$.

Let

$$\begin{aligned} F_n(z) &:= \Big[(f_n - \varepsilon\ell)(x_n + \operatorname{diag} x_n z) - (f_n - \varepsilon\ell)(x_n) \\ &\qquad -\operatorname{diag} x_n z(f_n' - \varepsilon\ell')(x_n)\Big] I(z, x_n), \\ \tilde{F}_n(z) &:= \Big[(\tilde{f}_n + \varepsilon\ell)(y_n + \operatorname{diag} y_n z) - (\tilde{f}_n + \varepsilon\ell)(y_n) \\ &\qquad -\operatorname{diag} y_n z(\tilde{f}_n' + \varepsilon\ell')(y_n)\Big] I(z, y_n). \end{aligned}$$

and $H_n(z) := F_n(z) - \tilde{F}_n(z)$ With this notation

$$\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) = \int H_n(z)\Pi(dz)$$

and the inequality (11.27) will follow from the Fatou lemma if we show that there is a constant C such that for all sufficiently large n

$$H_n(z) \le C(|z| \wedge |z|^2) \quad \text{for all } z \in K \tag{11.28}$$

and

$$\limsup_n H_n(z) \le 0 \quad \Pi\text{-a.s.} \tag{11.29}$$

Using the properties of f_n we get the bound:

$$\begin{aligned} F_n(z) \le &\Big[(W - \varepsilon\ell)(x_n + \operatorname{diag} x_n z) - (W - \varepsilon\ell)(x_n) \\ &-\operatorname{diag} x_n z n(x_n - y_n)\Big] I(z, x_n) \end{aligned}$$

Since the continuous function W and ℓ are of sublinear growth and the sequences x_n and $n(x_n - y_n)$ are converging (hence bounded), absolute value of the function in the right-hand side of this inequality is dominated by a function $c(1+|z|)$. The arguments for $-\tilde{F}_n(z)$ are similar. So, the function H_n is of sublinear growth.

We have the following identity:

$$
\begin{aligned}
H_n(z) = &(\Delta_n(x_n+\operatorname{diag} x_n z, y_n+\operatorname{diag} y_n z)-\Delta_n(x_n,y_n)\\
&+(1/2)n|\operatorname{diag}(x_n-y_n)z|^2)I(z,x_n)I(z,y_n)\\
&+(f_n(x_n+\operatorname{diag} x_n z)-W(x_n+\operatorname{diag} x_n z))I(z,x_n)I(z,y_n)\\
&-(\tilde{f}_n(y_n+\operatorname{diag} y_n z)-\tilde{W}(y_n+\operatorname{diag} y_n z))I(z,x_n)I(z,y_n)\\
&+F_n(z)(1-I(z,y_n))-\tilde{F}_n(z)(1-I(z,x_n)).
\end{aligned}
$$

The function $\Delta(x,y)$ attains its maximum at (x_n,y_n) and $f_n\le W$, $\tilde{f}_n\ge \tilde{W}$. It follows that

$$
H_n(z)\le (1/2)n|x_n-y_n|^2|z|^2+F_n(z)(1-I(z,y_n))-\tilde{F}_n(z)(1-I(z,x_n)).
$$

Let $\delta>0$ be the distance of the point $\widehat{x}$ from the boundary ∂K. Then the points $x_n,y_n\in\mathcal{O}_{\delta/2}(\widehat{x})$ for all sufficiently large n and, hence, the second and the third terms in the right-hand side above are functions vanishing on $\mathcal{O}_1(0)$. It follows that for such n the function H_n is dominated from above on the ball $\mathcal{O}_1(0)$ by $c_n|z|^2$ where $c_n:=(1/2)n|x_n-y_n|^2\to 0$ as $n\to\infty$. Therefore, (11.28) holds. The relation (11.28) also holds because the second and the first terms tends to zero (stationarily) for all z except the set $\{z:\ \widehat{x}+\operatorname{diag}\widehat{x}z\in\partial K\}$. The coordinates of points of $\partial K\setminus\{0\}$ are non-zero. So this set is empty if $\widehat{x}$ has a zero coordinate. If all components $\widehat{x}$ are nonzero, the operator given by the matrix $\operatorname{diag}\widehat{x}$ is non-degenerate and the set in question is of zero measure Π in virtue of our assumption.

However, the reasoning above is based on the assumption (11.25) while we know only (11.24). Fortunately, we can replace the the objects x_n, y_n, X_n, Y_n by objects $\widehat{x}_n$, $\widehat{y}_n$, $\widehat{X}_n$, $\widehat{Y}_n$ approaching rapidly the initial ones and for those (11.25) hold. Repeating the arguments, we get the same contradiction. □

Remark 1. In the case where the cone K is polyhedral, the hypothesis of the theorem can be slightly relaxed. Namely, one can complete the proof using the assumption that the measure Π does not charge hyperplanes.

Remark 2. Note that the definition of the Lyapunov function does not depend on U and hence the uniqueness holds for any utility function U for which U^* is decreasing with respect to the partial ordering induced by K^*. However, to apply the uniqueness theorem one needs to determine the growth rate of W and provide a Lyapunov with a faster growth.

12 Existence of Lyapunov Functions and Classical Supersolutions

In this section we extend results of [18] on the existence of the Lyapunov function to the considered case.

Let $u\in C(\mathbf{R}_+)\cap C^2(\mathbf{R}_+\setminus\{0\})$ be an increasing strictly concave function with $u(0)=0$ and $u(\infty)=\infty$. Introduce the function $R:=-u'^2/(u''u)$. Assume that $\bar{R}:=\sup_{z>0}R(z)<\infty$.

For $p\in K^*\setminus\{0\}$ we define the function $f(x)=f_p(x):=u(px)$ on K. If $y\in K$ and $x\neq 0$, then $yf'(x)=(py)u'(px)\le 0$. The inequality is strict when $p\in\operatorname{int}K^*$.

Recall that $A(x)$ is the matrix with $A^{ij}(x) = a^{ij}x^ix^j$ and the vector $\mu(x)$ has the components $\mu^i x^i$. Suppose that $\langle A(x)p, p\rangle \neq 0$. Isolating the full square we obtain the identity

$$\mathcal{L}_0 f(x) = \frac{1}{2}\left[\langle A(x)p,p\rangle u''(px) + 2\langle \mu(x), p\rangle u'(px) + \frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}\frac{u'^2(px)}{u''(px)}\right]$$
$$+\frac{1}{2}\frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}R(px)u(px) + \mathcal{I}(f,x) - \beta u(px). \tag{12.30}$$

Note that for $x, \operatorname{diag} xz \in \operatorname{int} K$ we have by the Taylor formula that

$$(f(x+\operatorname{diag} xz) - f(x) - \operatorname{diag} xz f'(x)) = \frac{1}{2}u''(x+\vartheta \operatorname{diag} xz)(p\operatorname{diag} xzx)^2,$$

where $\vartheta \in [0,1]$. Since $u'' \le 0$, the expression in the square brackets is negative and so is the whole right-hand side of the above formula if $\beta \ge \eta(p)\bar{R}$ where

$$\eta(p) := \frac{1}{2}\sup_{x\in K}\frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}.$$

Of course, if $\langle A(x)p,p\rangle = 0$ we cannot argue in this way. Nevertheless, if in such a case also $\langle \mu(x), p\rangle = 0$, then for any $\beta \ge 0$ we have the bound $\mathcal{L}_0 f(x) = -\beta u(px) \le 0$. The same conclusion holds (for arbitrary concave function $u \in C^1(\operatorname{int} K)$), if $A = 0$ and $\langle \mu(x), p\rangle \le 0$ for all $x \in \operatorname{int} K$.

These simple observations lead us to the following existence result for Lyapunov functions:

Proposition 12.1 *Let $p \in \operatorname{int} K^*$. Suppose that $\langle \mu(x), p\rangle$ vanishes on the set $\{x \in \operatorname{int} K :\ \langle A(x)p,p\rangle = 0\}$. If $\beta \ge \eta(p)\bar{R}$, then f_p is a Lyapunov function. In the absence of diffusion, f_p is a Lyapunov function for arbitrary concave $u \in C^1(\operatorname{int} K)$ provided that $\langle \mu(x), p\rangle \le 0$ for all $x \in \operatorname{int} K$.*

Let $\bar\eta := \sup_{p\in K^*}\eta(p)$. Note that $\eta(p) = \eta(p/|p|)$. Continuity considerations show that $\bar\eta$ is finite if $\langle A(x)p,p\rangle \neq 0$ for all $x \in K\setminus\{0\}$ and $p \in K^*\setminus\{0\}$. Obviously, if $\beta \ge \bar\eta\bar{R}$, then f_p is a Lyapunov function for $p \in \operatorname{int} K^*$.

The representation (12.30) is useful also in the search of classical supersolutions for the operator $\mathcal{L}$. Since $\mathcal{L}f = \mathcal{L}_0 f + U^*(f')$, it is natural to choose u related to U. For a particular case, where $\mathcal{C} = \mathbf{R}^d_+$ and $U(c) = u(e_1 c)$, with u satisfying the postulated properties (except, maybe, unboundedness) and assuming, moreover, that the inequality

$$u^*(au'(z)) \le g(a)u(z) \tag{12.31}$$

holds, we get, using the homogeneity of $\mathcal{L}_0$, the following result.

Proposition 12.2 *Let $\langle A(x)p,p\rangle \neq 0$ for each $x \in \operatorname{int} K$ and $p \in K^*\setminus\{0\}$. Suppose that (12.31) holds for every $a, z > 0$ with $g(a) = o(a)$ as $a \to \infty$. If $\beta > \bar\eta\bar{R}$, then there exists a_0 such that for every $a \ge a_0$ the function af_p is a classical supersolution of (5.12), whatever is $p \in K^*$ with $p^1 \neq 0$. Moreover, if $p \in \operatorname{int} K^*$, then af_p is a strict supersolution on any compact subset of $\operatorname{int} K$.*

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in]0,1[$, we have

$$R(z) = \gamma/(1-\gamma) = \bar{R},$$

and $u^*(au'(z)) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z)$. Therefore, the inequality (12.31) holds with $g(a) = o(a)$, $a \to 0$.

Let A be a diagonal matrix with $a^{ii} = \sigma^i$. Suppose that $\sigma^1 = 0$, $\mu^1 = 0$ (i.e. the first asset is the *numéraire*) and $\sigma^i \neq 0$ for $i \neq 1$. Then, by the Cauchy–Schwarz inequality applied to $\langle \mu(x), p\rangle$, we have the bound

$$\eta(p) \leq \frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2.$$

The inequality

$$\beta > \frac{1}{2}\frac{\gamma}{1-\gamma}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2 \tag{12.32}$$

(implying the relation $\beta > \bar{\eta}\bar{R}$) is a standing assumption in many studies on the consumption-investment problem under transaction costs, see Akian et al. [1] and Davis and Norman [13].

In particular, for the model with only one risky asset and the power utility function, by virtue of the above computations, we have, for the function $f(x) = au(px)$ given by $p \in K^*$ with $p^1 = 1$, that

$$\mathcal{L}_0 f(x) + U^*(f'(x)) = [...] + \left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu^2}{\sigma^2} - \beta + (1-\gamma)a^{1/(\gamma-1)}\right)f(x)$$

where $[...] \leq 0$. This implies the following conclusion.

Proposition 12.3 *Suppose that in the two-asset model with the power utility function the Merton parameter*

$$\kappa_M := \frac{1}{1-\gamma}\left(\beta - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu^2}{\sigma^2}\right) > 0.$$

Then the function

$$f(x) = \frac{1}{\gamma}\kappa_M^{\gamma-1}(px)^\gamma \tag{12.33}$$

is a classical supersolution of the HJB equation whatever is $p \in K^*$ with $p^1 = 1$.

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