The Dalang–Morton–Willinger Theorem Under Delayed and Restricted Information

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Summary. We extend the classical no-arbitrage criteria to the case of a model where the investor's decisions are based on a partial information (e.g., because of delay or round-off errors), that is the portfolio strategies are predictable with respect to a subfiltration. Our main result is a ramification of the famous Dalang–Morton–Willinger theorem: the model is arbitrage-free if and only if there exists an equivalent probability measure $\tilde{P}$ such that the optional projection of the price process with respect to $\tilde{P}$ is a $\tilde{P}$-martingale.

Key words: no-arbitrage criteria, martingale measure, optional projection.

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1. Introduction. The Dalang–Morton–Willinger theorem asserts, for the standard discrete-time finite horizon model of security market, that there is no arbitrage if and only if the price process is a martingale with respect to an equivalent probability measure. This remarkable result sometimes is referred to as the Fundamental Theorem of Arbitrage (or Asset) Pricing (FTAP) or simply the First Fundamental Theorem, [5]. Its various aspects have been thoroughly investigated, the theorem has been augmented by additional equivalent conditions and extended in many directions, see, e.g. [2].

In this note we deal with the same model modified in only one aspect: the agent's decision are based on a restricted information flow described by a filtration which can be smaller than a filtration generated by the price process. Apparently, such situations may arise if the information arrives with a delay or based on quantified prices and so on. We show that there is no-arbitrage under partial information iff there exists an equivalent probability $\tilde{P}$ such that the optional projection with respect to $\tilde{P}$ is a $\tilde{P}$-martingale. Surprisingly, this natural generalization was not studied previously and there is a certain explanation for this. Almost all available proofs use a reduction to the one-period model. This reduction is possible because in the standard setting, as was observed already in [1], the NA property is equivalent to the absence of arbitrage on each step. Due to this, all efforts are concentrated to construct a martingale density for the one-step model; proofs are accomplished by assembling the required density process from the one-step densities using the procedure suggested in [1]. Unfortunately, attempts to follow the same strategy of proof for the partial information case cannot be fruitful: as we show below, the equivalence between the "global" NA and the collection of one-step NA properties fails in general. However, the proof of NA criteria given in [3] and reproduced in [2] (to our knowledge, the unique one which does not rely upon the reduction to the one-step model) works well and requires only minor changes.

2. No-arbitrage criteria under delayed information. Let $(\Omega, \mathcal{G}, P)$ be a probability space equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)$, $t = 0, 1, \dots, T$, with $\mathcal{F}_T \subseteq \mathcal{G}$. We are given a d-dimensional process $S = (S_t)$ which is not necessarily adapted. Let

$$R_T := \{\xi : \ \xi = H \cdot S_T, \ H \in \mathcal{P}\}$$

where $\mathcal{P}$ is the set of all predictable d-dimensional processes with respect to $\mathbf{F}$ (i.e. H_t is $\mathcal{F}_{t-1}$-measurable) and

$$H \cdot S_T := \sum_{t=1}^{T} H_t \Delta S_t, \qquad \Delta S_t := S_t - S_{t-1}.$$

Put $A_T := R_T - L^0_+$; $\bar{A}_T$ is the closure of A_T in probability, L^0_+ is the set of non-negative random variables. In the context of mathematical finance the process S describes the discounted price process of d assets. The assumption that the strategy H is predictable with respect to a filtration $\mathbf{F}$ to which S may not be adapted, covers, in particular, the situation where the investor has no access to the full information contained in the price process (for instance, he may observe the price process after some delay).

We formulate our main result in the same manner as in [3].

Theorem 1. *The following conditions are equivalent:*

(a) $A_T \cap L^0_+ = \{0\}$;
(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$;
(c) $\bar{A}_T \cap L^0_+ = \{0\}$;
(d) *there is a probability* $\tilde{P} \sim P$ *with* $\mathrm{d}\tilde{P}/\mathrm{d}P \in L^\infty$ *such that all* S_t *are* $\tilde{P}$*-integrable and* $\tilde{E}(S_{t+1}|\mathcal{F}_t) = \tilde{E}(S_t|\mathcal{F}_t)$ *for* $t = 0, \dots, T-1$.

The last condition means that the $(\mathbf{F}, \tilde{P})$-optional projection $\tilde{S}$ is an $(\mathcal{F}, \tilde{P})$-martingale (in discrete time $\tilde{S}_n = \tilde{E}(S_n|\mathcal{F}_n)$ by definition).

Condition (a) is interpreted as the absence of arbitrage; it can be written in the obviously equivalent form $R_T \cap L^0_+ = \{0\}$ (or $H \cdot S_T \geqslant 0 \Rightarrow H \cdot S_T = 0$). When S is adapted to $\mathbf{F}$, (a) is equivalent to condition:

(a') $H_t \Delta S_t \geqslant 0 \Rightarrow H_t \Delta S_t = 0 \quad$ for all $t = 1, \dots, T$.

This is no longer true when S is not adapted. Consider the following simple example where $T = 2$, $\mathcal{F}_0 = \mathcal{F}_1 = \{\varnothing, \Omega\}$ but there is $A \in \mathcal{G}$ such that $0 < P(A) < 1$. Put

$$\Delta S_1 := I_A - \frac{1}{2} I_{A^c}, \qquad \Delta S_2 := -\frac{1}{2} I_A + I_{A^c}.$$

There is no arbitrage on each of two steps but the constant process with $H_1 = H_2 = 1$ is an arbitrage strategy for the two-step model.

3. Proof of Theorem 1. $(a) \Rightarrow (b)$ For the sake of completeness and the reader's convenience we repeat the arguments from [3] which are based on the following observation due to H.-J. Engelbert and H. von Weizsäcker (see [3] for a proof).

Lemma 1. *Let $\eta^n \in L^0(\mathbf{R}^d)$ be such that $\liminf |\eta^n| < \infty$. Then there are $\tilde{\eta}^k \in L^0(\mathbf{R}^d)$ such that for all ω the sequence of $\tilde{\eta}^k(\omega)$ is a convergent subsequence of the sequence of $\eta^n(\omega)$.*

To show that A_T is closed we proceed by induction. Let $T = 1$. Suppose that $H_1^n \Delta S_1 - r^n \to \zeta$ a.s. where H_1^n is $\mathcal{F}_0$-measurable and $r^n \in L^0_+$. It is sufficient to find $\mathcal{F}_0$-measurable random variables $\tilde{H}_1^k$ which are a.s. convergent and $\tilde{r}^k \in L^0_+$ such that $\tilde{H}_1^k \Delta S_1 - \tilde{r}^k \to \zeta$ a.s.

Suppose that certain sets $\Omega_i \in \mathcal{F}_0$ form a finite partition of Ω. Obviously, we may argue on each Ω_i separately as on an autonomous measure space (considering the restrictions of random variables and traces of σ-algebras).

Let $\underline{H}_1 := \liminf |H_1^n|$. On the set $\Omega_1 := \{\underline{H}_1 < \infty\}$ we can take, using Lemma 1, $\mathcal{F}_0$-measurable $\tilde{H}_1^k$ such that $\tilde{H}_1^k(\omega)$ is a convergent subsequence of $H_1^n(\omega)$ for every ω; $\tilde{r}^k$ are defined correspondingly. Thus, if Ω_1 is of full measure, the goal is achieved.

On $\Omega_2 := \{\underline{H}_1 = \infty\}$ we put $G_1^n := H_1^n/|H_1^n|$ and $h_1^n := r^n/|H_1^n|$ and observe that $G_1^n \Delta S_1 - h_1^n \to 0$ a.s. By Lemma 1 we find $\mathcal{F}_0$-measurable $\tilde{G}_1^k$ such that $\tilde{G}_1^k(\omega)$ is a convergent subsequence of $G_1^n(\omega)$ for every ω. Denoting the limit by $\tilde{G}_1$, we obtain that $\tilde{G}_1 \Delta S_1 = \tilde{h}_1$ where $\tilde{h}_1$ is non-negative, hence, in virtue of (a), $\tilde{G}_1 \Delta S_1 = 0$.

As $\tilde{G}_1(\omega) \neq 0$, there exists a partition of Ω_2 into d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that the ith coordinate $\tilde{G}_1^i \neq 0$ on Ω_2^i. Define $\bar{H}_1^n := H_1^n - \beta^n \tilde{G}_1$ where $\beta^n := H_1^{ni}/\tilde{G}_1^i$ on Ω_2^i. Then $\bar{H}_1^n \Delta S_1 = H_1^n \Delta S_1$ on Ω_2. As it was mentioned above we may consider as isolated the set $\Omega_2^i \in \mathcal{F}_0$. The replacement of the sequence (H_1^n) by the $(\bar{H}_1^n)$ does not change the limits. Represent these sequences as infinite matrices with infinitely many columns, H_1^n and $\bar{H}_1^n$, respectively. The difference is that in the second matrix the ith row is zero and if the first matrix already has null rows, they remain null in the second one. We restart the entire procedure on Ω_2^i with the sequence $\bar{H}_1^n$ such that $\bar{H}_1^{ni} = 0$ for all n. Since at each step the number of zero lines increases, the process stops after a finite number of steps. The induction step from $T - 1$

to T can be done exactly in the same way, considering always the sequence of (H_1^n) to construct a partition of Ω and making the needed operations also with (H_t^n) for $t \geqslant 2$: this is legitimate since they do not destroy measurability.

$(b) \Rightarrow (c)$ Trivial.

$(c) \Rightarrow (d)$ Notice that for any random variable η there is an equivalent probability P' with bounded density such that $\eta \in L^1(P')$ (e.g., one can take $P' = C\mathbf{E}^{-|\eta|}P$). Property (c) (as well as (a) and (b)) is invariant under equivalent change of probability. This consideration allows us to assume from the very beginning that all S_t are integrable. The convex set $A_T^1 := \bar{A}_T \cap L^1$ is closed in L^1 and hence satisfies the hypotheses of the well-known result due to Kreps and Yan, [4], [6] (its proof can also be found in [3] or [2]).

Lemma 2. *Let $K \supseteq -L_+^1$ be a closed convex cone in L^1 with $K \cap L_+^1 = \{0\}$. Then there is a probability $\tilde{P} \sim P$ with $\mathrm{d}\tilde{P}/\mathrm{d}P \in L^\infty$ such that $\tilde{E}\,\xi \leqslant 0$ for all $\xi \in K$.*

This lemma ensures the existence of $\tilde{P} \sim P$ with bounded density and such that $\tilde{E}\,\xi \leqslant 0$ for all $\xi \in A_T^1$, in particular, for $\xi = \pm H_t \Delta S_t$ where H_t is bounded and $\mathcal{F}_{t-1}$-measurable. Thus, $\tilde{E}(\Delta S_t|\mathcal{F}_{t-1}) = 0$.

$(d) \Rightarrow (a)$ Let $\xi \in A_T \cap L_+^0$, i.e. $0 \leqslant \xi \leqslant H \cdot S_T$. As $\tilde{E}(H_t \Delta S_t|\mathcal{F}_{t-1}) = 0$, we obtain by conditioning that $\tilde{E}\,H \cdot S_T = 0$. Thus, $\xi = 0$.

4. Optional projection. It may happen that the $(\mathbf{F}, P)$-optional projection of S does not satisfy our NA condition although S does. Indeed, consider again the two-step model with $\mathcal{F}_0 = \mathcal{F}_1 = \{\varnothing, \Omega\}$. Let ΔS_1, ΔS_2 be independent random variables uniformly distributed on $[-1, 3]$. Then $E(\Delta S_i|\mathcal{F}_i) = 1$ for $i = 1, 2$ but for any point $(H_1, H_2) \in \mathbf{R}^2$ different from the origin the distribution of the random variable $H_1 \Delta S_1 + H_2 \Delta S_2$ charges both $]-\infty, 0[$ and $]0, \infty[$ and, therefore, S has the NA property with restricted information.

5. Comment on continuous-time models. The following example illustrates that in the continuous-time setting where $t \in \mathbf{R}_+$, the question of absence of arbitrage under partial information can be posed even if the price process is not a semimartingale.

Let $B = (B_s)$ be a standard Brownian motion with respect to a filtration $(\mathcal{H}_t)$ satisfying the usual conditions. Let $\mathbf{F}$ be the trivial filtration formed by the σ-algebras $\mathcal{F}_t$ generated by the null sets from $\mathcal{H}_\infty$.

Put $\phi(x) := \pi + \arctan x$, $\theta(t) := \mathbf{E}^t$, and $S_t := \int_0^t \phi(B_s)\,\mathrm{d}B_s + \int_0^t B_{\theta(s)}\,\mathrm{d}s$. Let $\mathbf{G}$ be any filtration satisfying the usual conditions for which the process S is $\mathbf{G}$-adapted. If the diameters of the partitions $\pi_n := \{0 = t_0 \leqslant \ldots \leqslant t_n = t\}$ converge to zero, $\sum_{\pi_n} (S_{t_{i+1}} - S_{t_i})^2 \to [S, S]_t$ in probability. So the process $[S, S]_t = \int_0^t \phi^2(B_s)\,\mathrm{d}s$ is $\mathbf{G}$-adapted. It follows that (B_t) is $\mathbf{G}$-adapted and hence also $(B_{\theta(t)})$. Therefore, (B_t) cannot be a $\mathbf{G}$-semimartingale. Thus, (S_t) is not a $\mathbf{G}$-semimartingale. Nevertheless, we can define the stochastic integral $h \cdot S_t$ for any bounded Borel function and, moreover, for this integral $E\,h \cdot S_t = 0$ and, hence, the $\mathbf{F}$-optional projection of S is a martingale.

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