Remarks on the true no-arbitrage property

Yuri Kabanov[1,2] and Christophe Stricker[1]

[1] UMR 6623, Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray, F-25030 Besançon Cedex, France
kabanov@math.univ-fcomte.fr, stricker@math.univ-fcomte.fr
[2] Central Economics and Mathematics Institute, Moscow, Russia

Summary. We discuss conditions of absence of arbitrage in the classical sense (the "true" NA property) for the model given by a family of continuous value processes. In particular, we obtain a criterion for the NA property in a market model with countably many securities with continuous price processes. This result generalizes the well-known criteria due to Levental–Skorohod and Delbaen–Schachermayer.

1 Introduction

In the paper [7] Levental and Skorohod proved a criterion for the absence of arbitrage in a model of frictionless financial market with diffusion price processes. In the publication [2] Delbaen and Schachermayer suggested a necessary condition for the absence of arbitrage in a more general model where the price process is a continuous $\mathbf{R}^d$-valued semimartingale S: if the property NA holds then there is a probability measure $Q \ll P$ such that S is a local martingale with respect to Q. We analyze their proof and show that the arguments allow to conclude that there exists Q with an extra property: $Q|\mathcal{F}_0 \sim P|\mathcal{F}_0$. Now let σ runs the set of all stopping times. Since the NA property of S implies the NA property for each process $I_{]\sigma,\infty]} \cdot S$, this implies the existence of local martingale measures ${}^{\sigma}Q \ll P$ for the processes $I_{]\sigma,\infty]} \cdot S$ such that ${}^{\sigma}Q|\mathcal{F}_\sigma \sim P|\mathcal{F}_\sigma$. It turns out that this property is a necessary and sufficient condition for NA, cf. with [9].

In this note we establish a necessary condition for the absence of arbitrage in the framework where the model is given by a set of value processes and the price process even is not specified and the concept of the absolutely continuous martingale measure is replaced by that of absolutely continuous separating measure (ACSM). For the model with a continuous price process S the latter is a local martingale measure. We use intensively ideas of Delbaen and Schachermayer. In particular, we deduce the existence of ACSM from a suitable criterion for the NFLVR property. In contrast to [2], we use the fundamental theorem from [5] (a ramification of the corresponding result from [1]) and make explicit the notion of supermartingale density as a supermartingale $Y \geq 0$ such that $YX + Y$ is supermartingale for every value process $X \geq -1$. The suggested approach allows us to avoid vector integrals and work exclusively with scalar processes and standard facts of stochastic calculus.

As usual, the difficult part is "NA $\Rightarrow$...". For this we use Theorem 4 involving "technical" hypotheses. One of them, **H**, requires the existence of a supermartingale density Y and a process $\bar{X} \geq -1$ coinciding locally, up to the explosion time, with value processes and exploding on the set where Y hits zero. For the model generated by a scalar continuous semimartingale S the absence of immediate arbitrage (a property which is weaker than NA) implies **H** (with $\bar{X} = Y^{-1} - 1$). This can be easily verified following the same lines as in [2] (for the reader's convenience we provide a proof of Theorem 5 which is version of Theorem 3.7 from [2]).

The passage to the multidimensional case reveals an advantage to formulate the conditions of Theorem 4 in terms of value processes. If the latter are generated by a finite or countable family of scalar continuous semimartingales $\{S^i\}$ with orthogonal martingale components, then a required supermartingale density can be assembled from the semimartingale densities constructed individually for each S^i. An orthogonalization procedure reduces the general case to the considered above. In this way we obtain a NA criterion for the model spanned by countably many securities. This result seems to be of interest for bond market models where the prices of zero coupon bonds are driven by countably many Wiener processes.

Notice that in our definition the set of value processes corresponding to the family $\{S^i\}$ is the closed linear space generated by the integrals with respect to each S^i. That is why we are not concerned by the particular structure of this space, i.e., by the question whether this is the space of vector integrals. The positive answer to this question is well-known for stock markets but for bond markets (with a continuum of securities) a suitable integration theory is still not available.

2 Preliminaries and general results

In our setting a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ satisfying the usual conditions as well as a finite time horizon T are fixed. For the notational convenience we extend the filtration and all processes stationary after the date T.

To work comfortably within the standard framework of stochastic calculus under a measure $\tilde{P} \ll P$ we shall consider the "customized" stochastic basis $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbf{F}}, P)$ where $\tilde{\mathcal{F}}$ which is a $\tilde{P}$-completion of $\mathcal{F}$ and the filtration $\tilde{\mathbf{F}}$ is formed by the σ-algebras $\tilde{\mathcal{F}}_t$ generated by $\mathcal{F}_t$ and the $\tilde{P}$-null sets. For any $\xi \in \tilde{\mathcal{F}}_t$ there is $\xi' \in \mathcal{F}_t$ different from ξ only on a $\tilde{P}$-null set. With this remark the right-continuity of the new filtration is obvious. The processes $\xi I_{[t,\infty[}$ and $\xi' I_{[t,\infty[}$ coincide $\tilde{P}$-a.s. The monotone class argument implies that for any $\tilde{\mathbf{F}}$-predictable process H there exists a $\mathbf{F}$-predictable process H' which is $\tilde{P}$-indistinguishable from H, see details in [4].

We denote by $\mathcal{S} = \mathcal{S}(P)$ the linear space of scalar semimartingales starting from zero equipped with the Emery topology, generated, e.g., by the quasi-norm $\mathbf{D}(X) = \sup_H E|H \cdot X_T| \wedge 1$ where sup is taken over all predictable processes H with $|H| \le 1$.

Let $\mathcal{X} \subseteq \mathcal{S}$ be a convex set of bounded from below semimartingales stable under the concatenation in the following sense: for any $X^1, X^2 \in \mathcal{X}$ and any bounded predictable processes H^1, H^2 with $H^1H^2 = 0$, the sum of stochastic integrals $H^1 \cdot X^1 + H^2 \cdot X^2$, if bounded from below, belongs to $\mathcal{X}$. Obviously, $\mathcal{X}$ is a cone. For any $X \in \mathcal{X}$ and any stopping time τ the process $X^\tau = I_{[0,\tau]} \cdot X$ belongs to $\mathcal{X}$.

In the context of financial modeling the elements of $\mathcal{X}$ are interpreted as value processes; those for which $0 \le X_T \ne 0$ are called arbitrage opportunities. Let $\mathcal{X}^a := \{X \in \mathcal{X} : X \ge -a\}$. We introduce the sets of attainable "gains" or "results" $R := \{X_T : \ X \in \mathcal{X}\}$ and $R^a := \{X_T : \ X \in \mathcal{X}^a\}$ and define also $C := (R - L^0_+) \cap L^\infty$, the set of claims hedgeable from the zero initial endowment.

The NA property of $\mathcal{X}$ means that $R \cap L^0_+ = \{0\}$ (or $C \cap L^\infty_+ = \{0\}$). A stronger property, NFLVR (no free lunch with vanishing risk), means that $\bar{C} \cap L^\infty_+ = \{0\}$ where $\bar{C}$ is the norm closure of C in L^∞. There is the following simple assertion relating them (Lemma 2.2 in [5] which proof is the same as of the corresponding result in [1]).

Lemma 1 *NFLVR holds iff NA holds and R^1 is bounded in L^0.*

Remark 1. Note that R^0 is a cone in R^1. If R^1 is bounded in L^0, then necessarily $R^0 = \{0\}$ and for any arbitrage opportunity X' there are $t < T$ and $\varepsilon > 0$ such that the set $\Gamma := \{X'_t \le -\varepsilon\}$ is non-null. In this case the process $X := I_{\Gamma \times]t,\infty[} \cdot X'$ is an arbitrage opportunity with

$$\{X_T > 0\} = \{X_T \ge \varepsilon\} = \Gamma \in \mathcal{F}_t.$$

We say that $\mathcal{X}$ admits an equivalent separating measure (briefly: the ESM property holds) if there exists $\tilde{P} \sim P$ such that $\tilde{E} X_T \le 0$ for all $X \in \mathcal{X}$.

Now we recall also one of the central (and difficult) results of the theory in the abstract formulation of [5], Th. 1.1 and 1.2 (cf. with that of the original paper [1] where the value processes are stochastic integrals).

Theorem 2 *Suppose that $\mathcal{X}^1$ is closed in $\mathcal{S}$. Then NFLVR holds iff ESM holds.*

We say that a supermartingale $Y \ge 0$ with $EY_0 = 1$ is a *supermartingale density* if $Y(X+1)$ is a supermartingale for each $X \in \mathcal{X}^1$.

The following statement indicates that criteria for the NA property can be obtained from those for the NFLVR.

Lemma 3 *Let Y be a supermartingale density such that $Y_T > 0$ $\tilde{P}$-a.s. where $\tilde{P} \ll P$. Then the set R^1 is bounded in $L^0(\tilde{P})$.*

Proof. Let $X \in \mathcal{X}^1$. Since $EY(X+1) \le 1$, the set

$$Y_T R^1 := \{Y_T X_T : \ X \in \mathcal{X}^1\}$$

is bounded in $L^1(P)$, hence, it is bounded in $L^0(P)$. The absolute continuous change of measure as well as the multiplication by a finite random variable preserve the boundedness in probability. Thus, the set $R^1 = Y_T^{-1}(Y_T R^1)$ is bounded in $L^0(\tilde{P})$. □

We need the following condition.

H. There exist a supermartingale density Y and a càdlàg process $\bar{X}$ with values in $[-1,\infty]$, having ∞ as an absorbing state, and such that $\bar{X}^{\theta_n} \in \mathcal{X}^1$ for every stopping time $\theta_n := \inf\{t : \ \bar{X}_t \ge n\}$ and $\{\bar{X}_T < \infty\} \subseteq \{Y_T > 0\}$ a.s.

Theorem 4 *Suppose that $\mathcal{X}^1$ is closed in $\mathcal{S}$ and the hypothesis* **H** *is satisfied. If NA holds then there exists an ACSM Q such that $Q|\mathcal{F}_0 \sim P|\mathcal{F}_0$.*

Proof. Clearly, $c := P(\bar{X}_T < \infty) > 0$ (otherwise $\bar{X}^{\theta_1}$ violates NA) and we can define the martingale $Z_t := c^{-1}E(I_{\{\bar{X}_T<\infty\}}|\mathcal{F}_t)$ and the probability measure $\tilde{P} := Z_T P$, the trace of P on $\{\bar{X}_T < \infty\}$.

The NA property implies that $I_{\{Z>0\}} \ge I_{\{\bar{X}<\infty\}}$, i.e. Z does not hit zero before the explosion of $\bar{X}$. Indeed, in the opposite case

$$B_t^N := \{\sup_{s\le t} \bar{X}_s \le N, \ Z_t = 0\}$$

is not a null-set for some $t < T$ and $N < \infty$. Since zero is the absorbing point for Z, $B_t^N \subseteq \{Z_T = 0\} = \{\bar{X}_T = \infty\}$ (a.s.). The process $I_{B_t^N \times]0,\infty[} \cdot \bar{X}$, bounded from below by $-N-1$, is nontrivial only on B_t^N where it explodes. This violates the NA.

In particular, $Z_{\theta_n} > 0$, i.e. $\tilde{P}|\mathcal{F}_{\theta_n} \sim P|\mathcal{F}_{\theta_n}$. Since $\tilde{P}(\bar{X}_T < \infty) = 1$, the assumed existence of a supermartingale density ensures the boundedness of R^1 in $L^0(\tilde{P})$.

Let $\tilde{\mathcal{X}}^1$ be the closure of $\mathcal{X}^1$ in $\mathcal{S}(\tilde{P})$ and let $\tilde{\mathcal{X}} := \operatorname{cone} \tilde{\mathcal{X}}^1$. Recall that the elements of $\mathcal{S}(\tilde{P})$, a space over the stochastic basis $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbf{F}}, \tilde{P})$, are, in fact, not processes but classes of equivalence. Notice that for any $\tilde{X} \in \tilde{\mathcal{X}}^1$ there is a process X such that $X^{\theta_n} \in \mathcal{X}^1$ and $\tilde{X}^{\theta_n} = X^{\theta_n}$ $\tilde{P}$-a.s. for every n.

One can verify that $\tilde{\mathcal{X}}$ is stable under concatenation.

From the definition of the Emery topology it follows that the set $\tilde{R}^1$ formed by the terminal values of processes from $\tilde{\mathcal{X}}^1$ is a part of the closure of R^1 in $L^0(\tilde{P})$ and, hence, $\tilde{R}^1$ is bounded in this space.

If the set $\tilde{\mathcal{X}}$ does not satisfy NA under $\tilde{P}$, we can find, according to Remark 1, a process $X = I_{\Gamma\times]t,T]} \cdot X \in \tilde{\mathcal{X}}^1$ such that the set $\Gamma \in \tilde{\mathcal{F}}_t$, $\tilde{P}(\Gamma) > 0$, and

$\{X_T \geq \varepsilon\} = \Gamma$ $\tilde{P}$-a.s. Choosing appropriate representatives we may assume without loss of generality that $\Gamma \in \mathcal{F}_t$ and $X^{\theta_n} \in \mathcal{X}^1$ for every n. On the stochastic interval $[0,\theta|$ the process $\bar{X}^\varepsilon := H \cdot \bar{X}$ with

$$H := (\varepsilon/2)(2 + \bar{X}_t)^{-1} I_{\Gamma \times]t,T[}$$

is well-defined. On $[0,\theta|$ the process $X + \bar{X}^\varepsilon \geq -1 - \varepsilon/2$; at θ it explodes to infinity in a continuous way on the set $\Gamma \cap \{\theta \leq T\}$ and has a finite positive limit bigger than $\varepsilon/2$ on the set $\Gamma \cap \{\theta > T\}$. Since $P(\Gamma) > 0$, an appropriate stopping yields a process in $\mathcal{X}$ which is an arbitrage opportunity. The obtained contradiction shows that $\tilde{\mathcal{X}}$ under $\tilde{P}$ satisfies NA and, by virtue of Lemma 1, also the property NFLVR.

The result follows because by Theorem 2 there exists a measure $Q \sim \tilde{P}$ separating $\tilde{K}$ and $L^0_+(\tilde{P})$. $\square$

3 Semimartingales with the structure property

By definition, the structure property of $X \in \mathcal{S}$ means that $X = M + h \cdot \langle M \rangle$ where $M \in \mathcal{M}^2_{loc}$ and h is a predictable process such that $|h| \cdot \langle M \rangle_T < \infty$.

The next result is a version of Theorem 3.7 from [2] and its proof is given for the reader's convenience.

Theorem 5 *Let X be a continuous semimartingale with the structure property. Then there exists an integrand H such that $H \cdot X \geq 0$ and*

$$\{H \cdot X_t > 0 \ \forall t \in]0,T]\} = \{h^2 \cdot \langle M \rangle_{0+} = \infty\}.$$

Proof. Without loss of generality we may assume that $\Gamma := \{h^2 \cdot \langle M \rangle_{0+} = \infty\}$ is of full measure (replacing, if necessary, P by its trace on Γ). With this assumption the main ingredient of the proof is the following assertion:

Lemma 6 *Suppose that $h^2 \cdot \langle M \rangle_{0+} = \infty$ a.s. Then for any $\varepsilon > 0$, $\eta \in]0,1]$ there exist $\delta > 0$ arbitrarily close to zero and a bounded integrand $H = HI_{]\delta,\varepsilon]}$ such that*

(i) $H \cdot X \geq -1$;
(ii) $|Hh| \cdot \langle M \rangle_T + H^2 \cdot \langle M \rangle_T < 3$;
(iii) $P(H \cdot X_T \leq 1) \leq \eta$.

Proof. Let $R = 32/\eta$. Since $h^2 \cdot \langle M \rangle_{0+} = \infty$, for sufficiently small δ

$$P(h^2 I_{]\delta,\varepsilon]} I_{\{|h| \leq 1/\delta\}} \cdot \langle M \rangle_T \geq R) \geq 1 - \eta/2.$$

Let

$$\tau := \inf\{t \geq 0 : \ h^2 I_{]\delta,\varepsilon]} I_{\{|h| \leq 1/\delta\}} \cdot \langle M \rangle_t \geq R\} \wedge \varepsilon.$$

For the integrand $\tilde{H} := 2R^{-1} h I_{]\delta,\tau]} I_{\{|h| \leq 1/\delta\}}$ we have that $|\tilde{H}h| \cdot \langle M \rangle_T \leq 2$ with $P(|\tilde{H}h| \cdot \langle M \rangle_T < 2) \leq \eta/2$. Also,

$$\tilde{H}^2 \cdot \langle M \rangle_T \le 4R^{-1} < 1$$

and, by the Chebyshev and Doob inequalities,

$$P(\sup_{s \le T} |\tilde{H} \cdot M_s| \ge 1) \le 4E(\tilde{H} \cdot M_T)^2 = 4E\tilde{H}^2 \cdot \langle M \rangle_T \le 16R^{-1} \le \eta/2.$$

Thus, $P(\tau_1 \le T) \le \eta/2$ for the stopping time $\tau_1 := \inf\{t \ge 0 : \ \tilde{H} \cdot M_t \le -1\}$. The integrand $H := \tilde{H} I_{[0,\tau_1]}$ obviously meets the requirements (i) and (ii). At last, because of the inclusion $\{H \cdot X_T \le 1, \ \tau_1 > T\} \subseteq \{|Hh| \cdot \langle M \rangle_T < 2\}$, we obtain that

$$P(H \cdot X_T \le 1) = P(H \cdot X_T \le 1, \ \tau_1 \le T) + P(H \cdot X_T \le 1, \ \tau_1 > T) \le \eta/2 + \eta/2$$

and (iii) holds. □

Using this lemma, we construct, starting, e.g., with $\varepsilon_0 = T$, a sequence of positive numbers $\varepsilon_n \downarrow 0$ and a sequence of integrands $H_n = H_n I_{]\varepsilon_{n+1}, \varepsilon_n]}$ such that the conditions (i) – (iii) hold with $\eta_n = 2^{-n}$. The properties (i) and (ii) ensure that the process $G := \sum_n 3^{-n} H_n$ is integrable and $G \cdot X$ is bounded from below. By the Borel–Cantelli lemma for every ω outside a null set there is $n_0(\omega)$ such that $H_k \cdot X_{\varepsilon_k}(\omega) > 1$ for all $k > n_0(\omega)$. For $t \in]\varepsilon_{n+1}, \varepsilon_n]$ and any $n > n_0(\omega)$ we have

$$G \cdot X_t(\omega) = \sum_{k > n} 3^{-k} H_k \cdot X_{\varepsilon_k}(\omega) + 3^{-n} H_n \cdot X_t(\omega) \ge \sum_{k > n} 3^{-k} - 3^{-n} = \frac{1}{2} 3^{-n} > 0.$$

Thus, $\sigma := \inf\{t > 0 : \ G \cdot X_t = 0\} > 0$ a.s. It follows that for the integrand $H := \sum 2^{-n} I_{[0, \sigma \wedge n^{-1}]} G$ the process $H \cdot X$ is strictly positive on $]0, T]$. □

4 Models based on a continuous price process

Let S be a continuous $\mathbf{R}^d$-valued semimartingale, $L(S)$ be the set of predictable processes integrable with respect to S, and $\mathcal{A}$ be the set of integrands H for which the process $H \cdot S$ is bounded from below.

We consider the model where $\mathcal{X} = \mathcal{X}(S) := \{H \cdot S : \ H \in \mathcal{A}\}$. Mémin's theorem [8] says that $\{H \cdot S, \ H \in L(S)\}$ is a closed subspace of $\mathcal{S}$. It follows immediately that $\mathcal{X}^1$ is also closed.

First, we look at the case $d = 1$. Replacing, if necessary, the generating process by a suitable integral, we may assume without loss of generality that S is a bounded continuous semimartingale starting from zero (hence, an element of $\mathcal{X}$) and even that in its canonical decomposition $S = M + A$ the martingale M and total variation of the predictable process A are both bounded.

Recall the following simple fact:

Lemma 7 *Suppose that $R^0 = \{0\}$. Then $S = M + A$ with $A = h \cdot \langle M \rangle$.*

Proof. If the claim fails, we can find, using the Lebesgue decomposition, a predictable process $H = H^2$ for which $H \cdot \langle M \rangle = 0$ and $H \cdot \mathrm{Var}\, A_T \neq 0$. Let the process G be defined as the sign of the (predictable) process $dA/d\mathrm{Var}\, A$. Since $(GH) \cdot S = H \cdot \mathrm{Var}\, A$, we obtain a contradiction with the assumption that $R^0 = \{0\}$. □

The assumption $R^0 = \{0\}$ (no immediate arbitrage in the terminology of [2]) implies, by Theorem 5, that $h^2 \cdot \langle M \rangle_{0+}$ is finite as well as $h^2 I_{]\sigma,\infty[} \cdot \langle M \rangle_{\sigma+}$ whatever is the stopping time σ (by the same theorem applied to the process $S_{.+\sigma} - S_\sigma$ adapted to the shifted filtration $(\mathcal{F}_{t+\sigma})$). Put

$$\tau := \inf\{t \geq 0 : \; h^2 \cdot \langle M \rangle_t = \infty\},$$
$$\tau_n := \inf\{t \geq 0 : \; h^2 \cdot \langle M \rangle_t \geq n\}.$$

It follows that $h^2 \cdot \langle M \rangle_{\tau_-} = \infty$ (a.s.) on the set $\{\tau \leq T\}$ (i.e. no jump to infinity). This allows us to define the process

$$Y := e^{-h \cdot M - (1/2) h^2 \cdot \langle M \rangle} I_{[0,\tau[}.$$

It follows from the law of large numbers for continuous local martingales (see Remark 2 below) that $\{Y_{\tau-} = 0\} = \{h^2 \cdot \langle M \rangle_\tau = \infty\}$ a.s., i.e. Y hits zero not by a jump. For every stopping time τ_n the stochastic exponential $Y^{\tau_n} = \mathcal{E}(-h \cdot M^{\tau_n})$ is a positive martingale and, hence, by the Fatou lemma, $Y^\tau = Y$ is a supermartingale. By the Ito formula

$$Y^{\tau_n}(H \cdot S^{\tau_n}) = Y^{\tau_n} \cdot (H \cdot M^{\tau_n}) + (H \cdot S^{\tau_n}) \cdot Y^{\tau_n}$$

Thus, for any $X \in \mathcal{X}^1$ the process $Y^{\tau_n}(X^{\tau_n} + 1)$ is a local martingale and, again by the Fatou lemma, $Y^\tau(X^\tau + 1) = Y(X + 1)$ is a supermartingale.

At last, put $\bar{X} = Y^{-1} - 1$. Then $\{Y_T = 0\} = \{\bar{X}_T = \infty\}$ and by the Ito formula

$$\bar{X}^{\theta_n} = I_{[0,\theta_n]} Y^{-1} h \cdot S.$$

Summarizing, we come to the following:

Proposition 8 *Suppose that $R^0 = \{0\}$. Then the condition* **H** *holds.*

Remark 2. If $N \in \mathcal{M}^c_{loc}$ and $c > 0$, then

$$\{\lim_{t \to \infty} (N_t - c\langle N \rangle_t) \to -\infty\} = \{\langle N \rangle_\infty = \infty\} \text{ a.s.},$$

see, e.g., [6], Lemma 6.5.6. The needed extension to the case where $N^{\tau_n} \in \mathcal{M}^c$ and $\tau_n \to \tau$ can be proved in the same way.

Remark 3. Though we established the above proposition only for the case $d = 1$, the extension of the arguments to the vector case when the components $S^i = M^i + A^i$ are such that $\langle M^i, M^j \rangle = 0$, $i \neq j$, is obvious: consider $Y = \mathcal{E}(-\sum_i h^i \cdot M^i)$. But without loss of generality we may always assume that S satisfies this property. It is sufficient to notice that

$\mathcal{X}(S) = \mathcal{X}(\tilde{S})$ for some continuous semimartingale $\tilde{S}$ with orthogonal martingale components. This semimartingale can be constructed recursively. Namely, suppose that the orthogonality holds up to the index $n-1$. Let $M^n = \sum_{i\leq n-1} H^i \cdot M^i + \tilde{M}^n$ be the Kunita–Watanabe decomposition. One can take $\tilde{S}^n = \tilde{M}^n + A^n - \sum_{i\leq n-1} H^i \cdot A^i$. Of course, to ensure the existence of $H^i \cdot A^i$ it may be necessary to replace first S^n by $G \cdot S^n$ with a suitable integrand G taking values in $]0,1]$. This orthogonalization procedure works well also for a countable family $\{S^i\}_{i\in\mathbf{N}}$. Moreover, we can find bounded $\tilde{S}^i$ such that $\sum_i \tilde{S}^i$ converges in $\mathcal{S}$ to a bounded semimartingale $\tilde{S}$.

Theorem 9 *Suppose that $\mathcal{X} = \mathcal{X}(S)$ where S is a continuous $\mathbf{R}^d$-valued semimartingale. Then the NA property holds iff for any stopping time σ there exists a probability measure ${}^\sigma Q \ll P$ with ${}^\sigma Q|\mathcal{F}_\sigma \sim P|\mathcal{F}_\sigma$ such that the process $I_{]\sigma,\infty]} \cdot S \in \mathcal{M}^c_{loc}({}^\sigma Q)$.*

Proof. Necessity follows from Theorem 4 and Proposition 8 applied to the process $S_{.+\sigma} - S_\sigma$ adapted to the shifted filtration $(\mathcal{F}_{t+\sigma})$. As usual, the sufficiency is almost obvious. Indeed, if the claim fails, there exists a bounded process $X \in \mathcal{X}^1$ such that for the stopping time $\sigma := \inf\{t > 0 : \ X_t \neq 0\}$ we have $P(\sigma < T) > 0$. But then ${}^\sigma X := I_{]\sigma,\infty]} \cdot X$ is in $\mathcal{M}^c({}^\sigma Q)$ or, equivalently, ${}^\sigma XZ$ is a martingale with respect to P. It starts from zero and hence is zero. The density process Z of ${}^\sigma Q$ with respect to P is equal to one at σ and, being right-continuous, remains strictly positive on a certain stochastic interval on which ${}^\sigma X$ should be zero. This contradicts to the assumption that $P(\sigma < T) > 0$. $\square$

Remark 4. Let B be a Brownian motion, $\sigma := \inf\{t \geq 0 : \ B_t = -1\}$, and $Z_t = 1 + B_{t\wedge\sigma}$. Take $S_t = B_t - B_{t\wedge\sigma} + \sqrt{(t-\sigma)^+}$. Then $S_t Z_t = 0$ and, therefore, $S \in \mathcal{M}^c_{loc}(\tilde{P})$ where $\tilde{P} := Z_T P$. Nevertheless, according to Theorem 5 there is an immediate arbitrage at σ.

In virtue of Remark 3 we obtain in the same way the following

Theorem 10 *Suppose that $\mathcal{X}$ consists of all processes bounded from below and belonging to the closed linear subspace of $\mathcal{S}$ generated by $\mathcal{X}(S^i)$, $i \in \mathbf{N}$, where S^i are continuous semimartingales. Then the NA property holds iff for any stopping time σ there exists a probability measure ${}^\sigma Q \ll P$ with the restriction ${}^\sigma Q|\mathcal{F}_\sigma \sim P|\mathcal{F}_\sigma$ such that $I_{]\sigma,\infty]} \cdot S^i \in \mathcal{M}^c_{loc}({}^\sigma Q)$ for every $i \in \mathbf{N}$.*

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