Finance Stochast. 7, 403–411 (2003)

Finance and Stochastics

On the closedness of sums of convex cones in L^0 and the robust no-arbitrage property

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Abstract. This note is a natural complement to our previous work where we studied no-arbitrage criteria for markets with efficient friction. We discuss, in our general geometric framework, the recent result of Walter Schachermayer on a necessary and sufficient condition for the existence of strictly consistent price systems and give its quick proof.

Key words: Transaction costs, arbitrage, hedging, solvency

JEL Classification: G13, G11

Mathematics Subject Classification (1991): 60G44

1 Introduction

A theorem by Harrison and Pliska says that in a model of frictionless market with finite discrete time and finite Ω (i.e., finite number of states of the nature) the absence of arbitrage is equivalent to the existence of an equivalent probability measure under which all price processes are martingales. It was discovered by Dalang, Morton, and Willinger that it remains true even without the assumption that Ω is finite. These two theorems triggered a remarkable development referred to as Arbitrage Theory or Fundamental Theorem of Asset Pricing. The interested reader can find relevant information, e.g., in a survey paper [4] containing an extended bibliography.

Our specific interests here are results relevant to market models with proportional transaction costs. Without any doubt, the first serious work in this direction was the paper [2] by Jouini and Kallal who considered an original model based on a bid-ask spread. More traditional approach covering the case of foreign exchange has been developed in a series of articles [3, 1, 5, 7, 4], and, at last, [6].

⋆ Supported by the National Science Foundation of Hungary (OTKA), grant T 032932.
Manuscript received: December 2001; final version received: July 2002

It leads to the following general conclusions:

- in the model with a reference asset (numéraire) portfolios can be expressed either in relative values or in physical quantities;
- the hypothesis of existence of the numéraire is not necessary: in the model of pure exchange portfolios are expressed in physical quantities;
- a framework of controlled linear difference equation with (polyhedral) convex constraints covers both kind of models;
- the quality of agents' results can be evaluated using partial orderings in $\mathbf{R}^d$, e.g., those induced by solvency cones;
- the natural analogs of densities of absolute continuous martingale measures are "consistent price systems" i.e., processes evolving in positive duals to solvency cones while the analogs of densities of equivalent martingale measures seem to be "strictly consistent price systems", i.e., processes evolving in the relative interior of the latter.

The situation of a model with finite Ω was analyzed in details in the paper [7] (see also [4] for some improvements) where criteria for two definitions of no-arbitrage were proved (both coinciding with the Harrison-Pliska theorem for the model with zero transaction costs).

In our paper [6] it was shown that for the model with efficient friction (i.e., non-empty interiors of the positive duals to the solvency cones) the existence of strictly consistent price system is equivalent to the strict no-arbitrage conditions for the whole interval (NA^s). There is an important case studied in [10] where this criterion still holds without assuming the efficient friction. The result of [10] covers models with constant proportional transaction costs Λ and coincides with the Dalang-Morton-Willinger theorem when $\Lambda = 0$.

In the recent preprint [12], Walter Schachermayer proved that the existence of strictly consistent price system (we adopt here his terminology) is equivalent to the newly introduced "robust no-arbitrage property" ensuring the closedness of the set of hedgeable claims. He provided an example showing that NA^s does not guarantee this closedness and, hence, does not coincide with NA^r. The framework of [12] is that of a barter market (i.e., of pure exchange) based on bid-ask matrices but, as it is explained in [12], this is just a matter of language: this class of models coincides with that of currency market (any adapted process evolving in the relative interior of the duals to the solvency cones can be chosen as the price process). However, there are models interesting from the financial point of view having polyhedral solvency cones but not of this class. As examples may serve the model of stock market where the transactions charge the market account, see [9], certain models with homogeneous constraints etc.

In this note we give a synthesis and comparison of available results on no-arbitrage criteria in discrete time using advantages of the more general geometric setting of [6] to give quick proofs.

2 Main results

We shall use the notations and the framework of [6].

Recall that a sequence of set-valued mappings $G = (G_t)$ is a $\mathcal{C}$*-valued process* if there is a countable sequence of adapted $\mathbf{R}^d$-valued processes $X^n = (X^n_t)$ such that for every t and ω only a finite but non-zero number of $X^n_t(\omega)$ is different from zero and $G_t(\omega) := \text{cone}\,\{X^n_t(\omega),\ n \in \mathbf{N}\}$ (i.e., $G_t(\omega)$ is a polyhedral cone generated by the finite set $\{X^n_t(\omega),\ n \in \mathbf{N}\}$).

Let G and $\tilde{G}$ be closed cones. We say that G is *dominated* by $\tilde{G}$ if $G \setminus G^0 \subseteq \text{ri}\,\tilde{G}$ where G^0 is the linear space $G \cap (-G)$. We extend this notion in the obvious way to $\mathcal{C}$-valued processes. It can be formulated in terms of the dual cones because $G \setminus G^0 \subseteq \text{ri}\,\tilde{G}$ iff $\tilde{G}^* \setminus \tilde{G}^{*0} \subseteq \text{ri}\,G^*$. In particular, if G has an interior, $G \setminus G^0 \subseteq \text{int}\,\tilde{G}$ iff $\tilde{G}^* \setminus \{0\} \subseteq \text{ri}\,G^*$.

Let G be a $\mathcal{C}$-valued process, $A_t(G) := -\sum_{s=0}^t L^0(G_s, \mathcal{F}_s)$.

We say that G satisfies:

- the *weak no-arbitrage property* (in brief: NA^w) if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(\partial G_t, \mathcal{F}_t) \qquad \forall\, t \leq T;$$

- the *strict no-arbitrage property* NA^s if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(G^0_t, \mathcal{F}_t) \qquad \forall\, t \leq T;$$

- the *robust no-arbitrage property* NA^r if G is dominated by $\tilde{G}$ satisfying NA^w.

It is an easy exercise to check that if G dominates the constant process $\mathbf{R}^d_+$ then NA^w holds iff $A_T(G) \cap L^0(\mathbf{R}^d_+) = \{0\}$ (see [4]).

We introduce the following conditions:

(a) G satisfies the NA^s-property;
(a') G satisfies the NA^r-property;
(b) there is a martingale Z such that $Z_s \in L^\infty(\text{ri}\,G^*_s, \mathcal{F}_s)$ for each $s \leq T$.
(c) for each $t \leq T$ there is a martingale Z^t such that $Z^t_s \in L^\infty(G^*_s, \mathcal{F}_s)$ for each $s \leq t$ and $Z^t_t \in L^\infty(\text{ri}\,G^*_s, \mathcal{F}_s)$.

Theorem 1 *Assume that G dominates* $\mathbf{R}^d_+$. *Then* $(a') \Leftrightarrow (b)$.

Theorem 2 *Assume that* $L^0(G^0_s, \mathcal{F}_{s-1}) \subseteq L^0(G^0_{s-1}, \mathcal{F}_{s-1})$ *for all* $s \leq T$. *Then* $(a) \Leftrightarrow (b)$.

Theorem 3 *Assume that* $G^0 = \{0\}$. *Then* $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

Remarks 1. By esthetic reasons we formulate the results as equivalences. In fact, the implications $(b) \Rightarrow (a')$ and $(b) \Rightarrow (a)$ hold without extra assumptions, see Lemma 6. One can check also that $(c) \Rightarrow (a)$.

2. Theorem 1 generalizes the criterion established in [12] for a parametric model of pure exchange (barter) market given by the bid-ask matrix process $\Pi = (\pi_{ij})$, see [4], Remark 3.9. Its hypothesis can be slightly relaxed by demanding that G

dominates an increasing $\mathcal{C}$-valued process H such that all H_t have non-empty interiors. In the financial context of [12] a martingale Z satisfying (b) is called the *strictly consistent price system.*

3. Theorem 2 due to Irina Penner [10] is a direct (and beautiful) generalization of the equivalence $(a) \Leftrightarrow (b)$ of Theorem 3 from our paper [6]: the assumption holds trivially for proper cones G_t (notice also that in this case G_t^* has an interior and NA^s means simply that $A_t(G) \cap L^0(G_t, \mathcal{F}_t) = \{0\}$ for all $t \leq T$). The hypothesis of Theorem 2 looks abstract but it is fulfilled in the important case of the model with constant transaction costs when NA^s holds, see Proposition 1 also due to Irina Penner.

4. In financial applications, i.e., for models of security markets with transaction costs, G_t is either the solvency cone in the "physical units domain" (that is $G = \widehat{K}$ in terminology of [4], [6] etc.) or the solvency cone K_t in the "value domain" (with $A_t(K)$ defined as the set of hedgeable claims).

5. For finite Ω the condition (c) is equivalent to NA^s if G dominates $\mathbf{R}_+^d$, see [7]; its role in the general case remains an open question. The following simple example shows that even in the case of Ω consisting of two elementary events ω_1, ω_2 having equal probabilities the condition (c) is weaker than (b) and, hence, NA^s is weaker than NA^r! Indeed, let $\mathcal{F}_0$ be trivial, $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. Take now $G_0^* = G_2^* = \text{cone}\,\{e_1 + e_2\}$ and let $G_1^*(\omega_1) = \text{cone}\,\{e_1 + e_2, e_1 + 2e_2\}$, $G_1^*(\omega_2) = \text{cone}\,\{e_1 + e_2, 2e_1 + e_2\}$. Clearly, one can find martingales Z^t, $t \leq 2$, satisfying (c) but there is no one for which (b) holds.

6. In [12] Schachermayer gave a delicate example of a financial market for which NA^s holds but NA^r not. In fact, his example provides a negative answer to the question whether NA^s implies the closedness of the set of headgeable claims $A_T(G)$

Surprisingly, in the case of a financial market model for which the subspace $F_t = K_t \cap (-K_t)$ is constant over time (e.g., the transaction costs are constant) the properties NA^r and NA^s coincide.

Indeed, let $G_t = \phi_t K_t$ where K is a $\mathcal{C}$-valued process dominating $\mathbf{R}_+^d$,

$$\phi_t(\omega) : (x^1, ..., x^d) \mapsto (x^1/S_t^1(\omega), ..., x^d/S_t^d(\omega)),$$

S_t^i are strictly positive $\mathcal{F}_t$-measurable random variables.

For $J \subseteq \{1, ..., d\}$ we put $\mathbf{1}_J := \sum_{i \in J} e_i$ where $\{e_i\}$ is the canonical basis in $\mathbf{R}^d$.

Proposition 1 *Suppose that there is a partition J_1,...,J_1 of $\{1, ..., d\}$ such that $F_t^{\perp} = \text{span}\,\{\mathbf{1}_{J_1} ... , \mathbf{1}_{J_l}\}$. If G satisfies NA^w-property, then the assumption of Theorem 2 is fulfilled.*

Proof If the claim fails, there is $\xi \in L^0(G_s^0, \mathcal{F}_{s-1})$ such that $\{\xi \notin G_{s-1}^0\}$ is a non-null set. Without loss of generality we may assume that ξ is equal to zero outside it. Necessarily, some set $\{\phi_{s-1}^{-1}\xi\mathbf{1}_{J_k} \neq 0\}$ is non-null. We may assume that $\{\phi_{s-1}^{-1}\xi\mathbf{1}_{J_k} > 0\}$, the random variable ξ is zero outside this set and, moreover, all components of ξ vanish except those corresponding to J_k. Notice that J_k is not a singleton because ϕ_s and ϕ_{s-1} are diagonal operators. Take $i_0 \in J_k$ such that $\xi^{i_0} > 0$ and consider ξ' different of ξ only by the i_0th component

$$\xi'^{i_0} := -\frac{1}{S^{i_0}_{s-1}} \sum_{i \in J_k \setminus \{i_0\}} \xi^i S^i_{s-1}.$$

Clearly, $\xi' \in L^0(G^0_{s-1}, \mathcal{F}_{s-1})$ and $\xi - \xi' = h$ where h is equal to zero except the nontrivial component $h^{i_0} \geq 0$. This violates NA^w property. □

According to [1], in a specific financial context, where K_t is the solvency cones in values (generated by the matrix of transaction costs coefficients Λ_t) and S is the price process, the linear space $F_t^{\perp}$ is always the linear span of the random vectors $\mathbf{1}_{J_1(t)}$...,$\mathbf{1}_{J_l(t)}$ where $J_i(t)$ are the classes of "equivalent" assets (i.e., the assets which can be converted one into another without transaction costs). Of course, in the case of constant transaction costs these vectors do not evolve in time.

3 Proofs

Let N_s be a closed convex cone in $L^0(\mathbf{R}^d, \mathcal{F}_s)$ stable under multiplication by the elements of $L^0(\mathbf{R}_+, \mathcal{F}_s)$, and let $N^0_s := N_s \cap (-N_s)$, $A_t := \sum_{s=0}^t N_s$.

We introduce the following conditions:

(i) $A_T \cap (-N_t) \subseteq N^0_t$ for every $t = 0, ..., T$;

(ii) $A_{t-1} \cap (-N_t) \subseteq N^0_t$ for every $t = 1, ..., T$;

(iii) the relation $\sum_{s=0}^T \xi_s = 0$ with $\xi_s \in N_s$ implies that all $\xi_s \in N^0_s$.

Lemma 1 $(iii) \Rightarrow (i)$.

Proof Suppose that $\sum_{s=0}^T \zeta_s = -\eta$ where $\zeta_s \in N_s$ and $\eta \in N_t$. In virtue of (iii) we have that $\xi_t := \zeta_t + \eta$ is an element of N^0_t. Thus, $\eta = \xi_t - \zeta_t$ is in $-N_t$, i.e., $\eta \in N^0_t$. □

Remark Trivially, $(i) \Rightarrow (ii)$. In general, the implication $(ii) \Rightarrow (iii)$ may fail and this is the difference with the situation considered in [6]: if all $N^0_s = \{0\}$, these three properties are equivalent.

Lemma 2 *If (iii) holds then $A_T = \bar{A}_T$ and hence $A_T \cap L^1(\tilde{P})$ is closed in $L^1(\tilde{P})$ for every $\tilde{P} \sim P$.*

Proof We proceed by induction. For $T = 0$ there is nothing to prove. Suppose that the claim holds up to $T - 1$ periods. Let $\sum_{s=0}^T \xi^n_s \to \xi$ a.s. where $\xi^n_s \in N_s$. The question is whether $\xi = \sum_{s=0}^T \xi_s$ with $\xi_s \in N_s$. Useful observation: if $\Omega_i \in \mathcal{F}_0$, $i \leq N$, form a partition of Ω, we can argue separately with each part as it were the whole Ω, find appropriate representations and "assemble" ξ_s from N pieces (the formal description is obvious).

The case $\Omega = \{\liminf |\xi^n_0| < \infty\}$ is simple: by Lemmas 1 and 2 from [6] we may assume that ξ^n_0 converge to $\xi_0 \in N_0$ and, hence, $\sum_{s=1}^T \xi^n_s$ converge a.s. to a random variable ζ which is in $\sum_{s=1}^T N_s$ by the induction hypothesis.

In the case $\Omega = \{\liminf |\xi_0^n| = \infty\}$ we put $\tilde{\xi}_s^n := \xi_s^n/|\xi_0^n|$ (with the convention $0/0 = 0$). As $|\tilde{\xi}_0^n| \le 1$, we again may assume that $\tilde{\xi}_0^n$ converge to some $\tilde{\xi}_0 \in N_0$. Then $\sum_{s=1}^T \tilde{\xi}_s^n$ converge a.s. to a random variable which can be represented by the induction hypothesis as $\sum_{s=1}^T \tilde{\xi}_s$ where $\tilde{\xi}_s \in N_s$. Since $\xi/|\xi_0^n| \to 0$ a.s., the limit of the whole normalized sum is zero, i.e., $\sum_{s=0}^T \tilde{\xi}_s = 0$. By the assumption all $\tilde{\xi}_s \in N_s^0$. Since $|\tilde{\xi}_0| = 1$ there are disjoint sets $\Gamma_i \in \mathcal{F}_0$ such that $\Omega = \cup_{i=1}^d \Gamma_i$ and $\Gamma_i \subseteq \{\tilde{\xi}_0^i \neq 0\}$.

Put $\bar{\xi}_s^n = \sum_{i=1}^d (\xi_s^n + \beta^{ni}\tilde{\xi}_s) I_{\Gamma_i}$ where $\beta^n = -\xi_0^{ni}/\tilde{\xi}_0^i$. Clearly, $\bar{\xi}_s^n \in N_s$ and $\sum_{s=0}^T \bar{\xi}_s^n$ converge to ξ a.s. The situation is reproduced. It is instructive to represent sequences ξ_0^n and $\bar{\xi}_0^n$ as infinite dimensional matrices with d-dimensional columns. Of course, every zero line of the first matrix remains zero line of the second one. But the second matrix contains one more zero line (namely, the ith for $\omega \in \Gamma_i$). Thus, if the first matrix contains one non-zero line a.s., the proof is accomplished (all $\bar{\xi}_0^n = 0$ and we can use the induction hypothesis). If not, we repeat the whole procedure with the sequence of processes $(\bar{\xi}_s^n)$ etc. □

Remark The above result ("key lemma") generalizes Theorem 2 of [6] where the hypothesis $N_s^0 = \{0\}$ allows to stop the arguments earlier (the properties $\tilde{\xi}_s = 0$ for all s and $|\tilde{\xi}_0| = 1$ are not compatible). In the general case we can conclude only that the "rank" of the sequence $\bar{\xi}_0^n$ is lower and we accomplish the proof in the same way as in [8]. Notice that we opted here for a rather straightforward "algorithmic" description of the Gauss-type elimination procedure rather than to develop a kind of linear algebra with random coefficients as in [11] and [12].

The following assertion is a version of Lemma 5 in [6].

Lemma 3 *Suppose that there is an $\mathbf{R}^d$-valued martingale Z such that:*
1) $Z_s\xi \le 0$ *for every* $\xi \in N_s$, $s \le T$;
2) *the equality* $Z_s\xi = 0$ *where* $\xi \in N_s$ *holds iff* $\xi \in N_s^0$.
Then (iii) *holds.*

Proof Without loss of generality we assume that all random variables $|Z_T||\xi_s|$ are integrable. The trick is standard: we can always replace P by a measure $P' \sim P$ under which they are integrable and Z by $Z' = \rho Z$ where $\rho_t = E(d\tilde{P}/dP'/\mathcal{F}_t)$ choosing $\tilde{P} \sim P'$ such that $d\tilde{P}/dP'$ is bounded and Z is a $\tilde{P}$-martingale (such $\tilde{P}$ exists by the Dalang–Morton–Willinger theorem). With this integrability we obtain, using the martingale property, the relation $\sum_{s=0}^T EZ_s\xi_s = 0$ implying the result. □

Remark The reference to the DMW theorem yields the shortest proof. A direct proof using Lemma 4 from [6] is also simple and left as an exercise.

Lemma 4 *Assume that* (iii) *holds. Then for any* $\zeta \in N_t$, $t \le T$, *there is a bounded* $\mathbf{R}^d$-*valued martingale* Z^ζ *such that:*
1) $Z_s^\zeta\xi \le 0$ *for any* $\xi \in N_s$, $s \le T$;
2) $\zeta I_{\{Z_t^\zeta\zeta = 0\}} \in N_t^0$.

Proof Put $A_T^1 := A_T \cap L^1$, $\mathcal{Z}_T := \{\eta \in L^\infty(\mathbf{R}^d, \mathcal{F}_T) : E\eta\xi \le 0,\ \xi \in A_T^1\}$. With each $\eta \in \mathcal{Z}_T$ we associate the martingale $Z_s := E(\eta|\mathcal{F}_s)$. It satisfies 1): otherwise we would find $\xi \in N_s \cap L^1$ such that the set $\Gamma := \{Z_s\xi > 0\}$ is of positive probability and hence $E\eta(\xi I_\Gamma) = EZ_s(\xi I_\Gamma) > 0$ contradicting the definition of $\mathcal{Z}_T$. Let $a := \sup_{\eta \in \mathcal{Z}_T} P(Z_t\zeta < 0)$. There is $\eta^* = \eta^*(\zeta) \in \mathcal{Z}_T$ such that for the corresponding martingale Z^ζ we have $a = P(Z_t^\zeta \zeta < 0)$. To see this, take $\eta_n \in \mathcal{Z}_T$ with $||\eta_n||_\infty = 1$ such that $P(Z_t^n\zeta < 0) \to a$ and put $\eta^* := \sum 2^{-n}\eta_n$.

If 2) fails, then, for c sufficiently large, $-\zeta I_{\{Z_t^\zeta\zeta=0,\ |\zeta|\le c\}}$ does not belong to N_t^0 and, being in $(-N_t) \cap L^1$, does not belong to the convex cone A_T^1 (Lemma 1) closed in L^1 (Lemma 2). By the Hahn–Banach theorem there is $\eta \in L^\infty(\mathbf{R}^d)$ such that

$$E\eta\xi < -E\eta\zeta I_{\{Z_t^\zeta\zeta=0,\ |\zeta|\le c\}} \quad \forall \xi \in A_T^1.$$

It follows that $E\eta\xi \le 0\ \forall \xi \in A_T^1$ (i.e., $\eta \in \mathcal{Z}_T$) and $E\eta\zeta I_{\{Z_t^\zeta\zeta=0,\ |\zeta|\le c\}} < 0$. Thus, for $\tilde{Z}$ corresponding to $\tilde{\eta} := \eta^* + \eta$ we have

$$P(\tilde{Z}_t\zeta < 0) > P(Z_t^\zeta\zeta < 0) = a.$$

This contradiction shows that 2) holds. □

Corollary 1 *Assume that (iii) holds. Let Γ be a countable set, $\Gamma \subseteq \cup_{s\le T} N_s$. Then there is a bounded $\mathbf{R}^d$-valued martingale Z such that:*

1) $Z_s\xi \le 0$ for any $\xi \in N_s$, $s \le T$;

2′) $\zeta I_{\{Z_s\zeta=0\}} \in N_s^0$ for all $\zeta \in \Gamma$, $s \le T$.

Proof One can take as Z any (countable) convex combination with strictly positive coefficients of all elements of the family $\{Z^\zeta\}_{\zeta\in\Gamma}$ with $|Z_T^\zeta| \le 1$. □

From now on we shall examine the case where $N_t = -L^0(G_t, \mathcal{F}_t)$ and, hence, $N_t^0 = L^0(G_t^0, \mathcal{F}_t)$ and $A_t = A_t(-G)$. Due to the specific structure in the above corollary a countable subset Γ can be replaced by the whole set $\cup_{s\le T} N_s$ and hence the claimed properties mean simply that $Z_s \in L^0(\mathrm{ri}\, G_s^*, \mathcal{F}_s)$, $s \le T$.

Lemma 5 *If G dominates $\mathbf{R}_+^d$ and satisfies NA^r-property, then (iii) holds.*

Proof Let $\tilde{G}$ dominates G and $A_T(\tilde{G}) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$. Assume that in the identity $\sum_{s=0}^T \xi_s = 0$ where $\xi_s \in L^0(-G_s, \mathcal{F}_s)$, a random variable ξ_t does not belong to $L^0(G_t^0, \mathcal{F}_t)$. This means that $\xi_t(\omega) \in \mathrm{int}\,(-\tilde{G}_t(\omega))$ on a set B of positive probability. It follows that there is a random variable $\epsilon \in L^0(\mathbf{R}_+^d, \mathcal{F}_t)$ strictly positive on B such that $\xi_t + \epsilon$ is still in $L^0(-\tilde{G}_t, \mathcal{F}_t)$. The nontrivial random variable $\epsilon = \sum_{s=0}^T \xi_s'$ where $\xi_s' := \xi_s$, $s \ne t$, $\xi_t' := \xi_t + \epsilon$, being in $A_T(\tilde{G}) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T)$, violates NA^w-property of $\tilde{G}$. □

Lemma 6 *If there is a martingale Z such that $Z_s \in L^\infty(\mathrm{ri}\, G_s^*, \mathcal{F}_s)$ for each $s \le T$, then NA^r and NA^s hold.*

Proof Notice that $\tilde{G}_t(\omega) = (\mathbf{R}_+ Z_t(\omega))^*$ is a cone (half-space, in fact) with the interior containing $G_t(\omega) \setminus G_t^0(\omega)$. In virtue of Lemma 3 (applied with $\tilde{G}$) the process $\tilde{G}$ satisfies the property NA^w.

The property NA^s (for G) follows from Lemmas 3 and 1. □

Theorem 1 follows from the above results in an obvious way.

Remark The reader may observe that the novelty with respect to [6] is in Lemmas 5 and 6 based on the enlightening idea of Schachermayer.

The proof of Theorem 2 needs only one change: Lemma 5 has to be replaced by the following one.

Lemma 7 *Suppose that a $\mathcal{C}$-valued process G satisfies NA^s property. If*

$$L^0(G_s^0, \mathcal{F}_{s-1}) \subseteq L^0(G_{s-1}^0, \mathcal{F}_{s-1}) \qquad \forall\, s \le T,$$

then the condition (iii) holds.

Proof This can be shown by induction starting trivially and with an easy step. The equality $\sum_{s=0}^{T-1} \xi_s = -\xi_T$ implies that ξ_T is $\mathcal{F}_{T-1}$-measurable and, in virtue of NA^s-property, belongs to $L^0(G_T^0, \mathcal{F}_T)$. By the assumed inclusion ξ_T belongs also to $L^0(G_{T-1}^0, \mathcal{F}_{T-1})$ and can be combined with ξ_{T-1}, reducing the sum to $T-1$ terms which are elements of $L^0(G_s^0, \mathcal{F}_s)$ (the induction hypothesis). In particular, $\xi_{T-1} + \xi_T$ belongs to $L^0(G_{T-1}^0, \mathcal{F}_{T-1})$ as well as both summands. □

Of course, the lemma remains true for arbitrary sequence of random cones $G = (G_s)$.

Final remarks 1. Theorems 1 and 2 can be augmented by equivalent "neighbor" conditions (e.g., in the spirit of Theorem 1 in [6]). In particular, as it was observed in [12], one can replace in (b) the boundedness of Z by the seemingly weaker property that $|Z_T| \le \xi$ where ξ is a given r.v. with values in $]0,1]$. In fact, starting with an arbitrary Z, we can "improve" its qualities. Indeed, consider the probability $P' = \rho'_T P$ where $\rho'_t := cE(\xi(1+|Z_T|)^{-1}|\mathcal{F}_t)$. By the DMW theorem there is a bounded P'-martingale $\rho > 0$ such that ρZ is a P'-martingale. Then $c^{-1}\rho\rho' Z$ is a needed "better" martingale.

2. The hedging result in [6] (Theorem 4) was established assuming that the cones G_t are proper. Its hypothesis can be replaced by the "major" condition (b) (always implying the closedness of A_T) without changes in the proof.

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