On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper

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Abstract

This note contains ramifications of results of Delbaen et al. (2002). Assuming that the price process is locally bounded and admits an equivalent local martingale measure with finite entropy, we show, without further assumption, that in the case of exponential utility the optimal portfolio process is a martingale with respect to each local martingale measure with finite entropy. Moreover, the optimal value always can be attained on a sequence of uniformly bounded portfolios.

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1 Prototype result

To explain the questions discussed in this note we start with the classical "toy" model of financial market given by a **finite** probability space $(\Omega, \mathcal{F}, P)$ with a d-dimensional discrete-time price process $S = (S_t)$ adapted to a filtration $(\mathcal{F}_t)$, $t = 0, 1, ..., T$. We consider the set R of final wealth of self-financing portfolios with zero initial endowment, i.e. the collection of random variables $H \cdot S_T := \sum_{t \le T} H_t \Delta S_t$ where H_t is $\mathcal{F}_{t-1}$-measurable. Suppose that an investor maximizes the expectation of exponential utility over R. We are interested in the value J^o of the maximization problem

$$E(1 - e^{-\eta}) \to \max \quad \text{on } R \tag{1}$$

and, of course, in the structure of the optimal portfolio.

Let $\mathcal{Z}$ (resp. $\mathcal{Z}^e$) be the set of positive (resp., strictly positive) random variables ξ with $E\xi = 1$ such that $E\xi\eta = 0$ for all $\eta \in R$. Let $\mathcal{Q}$ and $\mathcal{Q}^e$ be the corresponding sets of probabilities $Q := \xi P$.

The space of random variables L^0 with the scalar product $\langle \xi, \eta \rangle = E\xi\eta$ is a finite-dimensional Euclidean space and the elements of $\mathcal{Z}$ can be interpreted as functionals separating R and L^0_+. This justifies the terminology "separating measures" for probabilities from $\mathcal{Q}$. For finite Ω the set $\mathcal{Q}^e := \{Q : \ Q = \xi P, \ \xi \in \mathcal{Z}^e\}$ is exactly the set of equivalent martingale measures.

Assume that $\mathcal{Z} \neq \emptyset$. Let $\underline{J}$ be the value of the minimization problem

$$E\xi \ln \xi \to \min \quad \text{on } \mathcal{Z}. \tag{2}$$

The set $\mathcal{Z}$ being compact, the minimum is attained; moreover, it is attained on the element of $\mathcal{Z}^e$ if the latter set is non-empty (see Proposition 3.1).

Proposition 1.1 $J^o = 1 - e^{-\underline{J}}$.

Proof. We introduce the convex function $U(x) := e^x - 1$. Its Fenchel conjugate

$$U^*(y) := \sup_x (yx - U(x)) = \begin{cases} y \ln y - y + 1, & y \ge 0, \\ \infty, & y < 0. \end{cases}$$

Let consider the minimization problem

$$f(\eta) + g(\eta) \to \min \quad \text{on } L^0 \tag{3}$$

where $f(\eta) := EU(\eta)$ and $g = \delta_R$, the indicator function (in the sense of convex analysis) which is equal to zero on R and infinity on the complement. Clearly, $f^*(\xi) = EU^*(\xi)$ and $g^* = \delta_{R^\circ}$. In our case the polar R° is just $R^\perp$, the subspace orthogonal to R. The conditions of Fenchel's theorem, see Aubin (1993) p. 38, are obviously fulfilled. Thus, J^o coincides with the (attained) value of the dual problem

$$f^*(-\xi) + g^*(\xi) \to \min \quad \text{on } L^0,$$

i.e. J^o is equal to the minimum of f^* on the set $R^\perp \cap L^0_+ = \mathbf{R}_+\mathcal{Z}$. Since

$$\inf_{\xi\in\mathcal{Z}}\inf_{t\geq 0} E(t\xi\ln t\xi - t\xi + 1) = \inf_{\xi\in\mathcal{Z}}(1-e^{-E\xi\ln\xi}) = 1-e^{-\underline{J}},$$

we get the result. □

Remarks. 1. If $\mathcal{Z}^e \neq \emptyset$, Fenchel's theorem ensures the existence of the solution to the primal problem. This is not the case if $\mathcal{Z}^e = \emptyset$: there is an arbitrage strategy H^a with $\eta^a := H^a \cdot S_T \geq 0$ and $\eta^a \neq 0$. For any $\eta \in R$ the value of the functional in (1) on $\eta^a + \eta$ is strictly larger than on η.

2. In the more interesting case of the utility function $1 - e^{-r(x+c-\zeta)}$, where c is a constant (an initial endowment) and ζ is a random variable (a contingent claim),

$$\sup_{\eta\in R} E(1-e^{-r(\eta+c-\zeta)}) = 1 - e^{-\underline{J}'} \tag{4}$$

where

$$J' := \min_{\xi\in\mathcal{Z}}(E\xi\ln\xi + rc - rE\xi\zeta).$$

This is an easy corollary of Proposition 1.1: the passage to the equivalent probability $P' := \rho P$ with $\rho := e^{-r(c-\zeta)}/Ee^{-r(c-\zeta)}$ reduces the problem to the treated above.

3. The use of martingale measures, although common, is restricted to models of frictionless market. It is clear from the proof, that the natural dual variables are unnormalized densities. Their appropriate generalizations are indispensable in models with transaction costs, see, e.g., Kabanov (1999) and Bouchard et al. (2001).

4. Proposition 1.1 can be easily extended to the situation where R is a closed convex cone arising when there are convex constrains on portfolios.

2 Ramifications

So, for the "toy" model Proposition 1.1 is just an exercise on Fenchel's theorem in its simplest case but the general probability space and, especially, continuous-time models pose some mathematical problems. At the moment, there are two approaches to obtain generalizations. The first one is to exploit further the convex duality, using, e.g., more general versions of Fenchel's theorem. This idea is developed in Bellini and Frittelli (2000) where the class of strategies with value processes bounded from below is considered, see, also Kramkov and Schachermayer (1999) and Kramkov and Schachermayer (1999). The paper Delbaen et al. (2002) follows another approach based on the following result which appeared in the thesis Rheinländer (1998). Assume that the price process S is a locally bounded semimartingale and that there exists an equivalent local martingale measure with finite entropy. Then the optimal density in (2) is of the form $\underline{\xi} = e^{\underline{J} - H^o\cdot S_T}$ where the integral $H^o \cdot S$ is a true martingale with respect to the optimal measure $\underline{P}$. This fact implies immediately that H^o is the optimal strategy for (1), at least, in the class of strategies for which the process $H \cdot S$ is a $\underline{P}$-martingale, and ensures the corresponding duality relations. The main results of Delbaen et al. (2002) are Theorems 2 and 3 asserting that under a certain additional assumption ("reverse Hölder inequality") the process $H^o \cdot S$ is a true martingale with respect to **every** local martingale measure Q with finite entropy and that the optimal value in the primal problem can be attained on a sequence of strategies with bounded value processes.

The aim of our note is to remove the additional hypothesis from these two theorems. We give complete proofs and recollect also, for future references, some basic results in a slightly more general form than that needed for this purpose. In particular, we prove in the appendix a continuous-time version of Barron's inequalities, see Barron (1985).

We consider the standard **continuous-time setting** with a semimartingale price process S given on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \le T}, P)$. Assume that S is **locally bounded**. Let $\mathcal{Z}$ the set of local martingale densities which is closed in this case. Assume also that $\mathcal{Z}^e \neq \emptyset$ and $\underline{J} < \infty$. Thus, there exists $\underline{\xi} \in \mathcal{Z}^e$ solving (2), see the next section. Put $\underline{P} = \underline{\xi} P$.

Let $\mathcal{A}$ (resp., $\mathcal{A}_b$) be the set of all integrands such that the process $H \cdot S$ is a $\underline{P}$-martingale (resp., is bounded) and let $\mathcal{R} := \{\eta : \ \eta = H \cdot S_T, \ H \in \mathcal{A}\}$.

Theorem 2.1 *(a) We have $\underline{\xi} = e^{\underline{J} - H^o \cdot S_T}$ where $H^o \in \mathcal{A}$,*

$$J^o := \sup_{\eta \in \mathcal{R}} E(1 - e^{-\eta}) = 1 - e^{-\underline{J}},$$

and the supremum is attained at the (unique) point $\eta^o = H^o \cdot S_T$;

(b) $H^o \cdot S$ is a Q-martingale for all $Q = \xi P$ such that $\xi \in \mathcal{Z}$ and $E\xi \ln \xi < \infty$;

(c) there are $H^n \in \mathcal{A}_b$ such that $E(1 - e^{-H^n \cdot S_T}) \to J^o$.

As in Remark 2, one can easily extend this result, by equivalent change of probability, to cover the more general (and useful) setting of Delbaen et al. (2002). So, the parts (b) and (c) are generalizations of Theorem 2 and 3 while the part (a) is just Theorem 1 ibid.

In our proof of (c) we shall use the following related result, Bellini and Frittelli (2000), Corollary 3:

Theorem 2.2 *Let $\mathcal{R}_{bb}$ be the set of random variables $H \cdot S$ where H is an integrand for which the process $H \cdot S$ is bounded from below. Then*

$$\sup_{\eta \in \mathcal{R}_{bb}} E(1 - e^{-\eta}) = 1 - e^{-\underline{J}}. \tag{5}$$

3 Minimization of convex functionals on $\mathcal{Z}$

Let $\underline{J} := \inf_{\xi \in \mathcal{Z}} E\phi(\xi)$ be the value of the minimization problem

$$E\phi(\xi) \to \min \quad \text{on } \mathcal{Z} \tag{6}$$

where $\mathcal{Z}$ is a non-empty convex set of probability densities and $\phi : I\!R_+ \to I\!R$ is a convex function such that $\phi \geq -c$ and $\phi(x)/x \to \infty$ as $x \to \infty$.

Proposition 3.1 *Let $\phi'(0) = -\infty$. If there exists $\xi \in \mathcal{Z}$ which is strictly positive and such that $E\phi(\xi) < \infty$, then any minimizer $\underline{\xi}$ is also strictly positive.*

Proof. Let $f_t := \phi(t\xi + (1-t)\underline{\xi})$ and $F_t := Ef_t$, $t \in [0,1]$. Since f is convex, the ratio $(f_t - f_0)/t$ decreases to f_0' as $t \downarrow 0$. Clearly, $f_0' = a(\underline{\xi}, \xi)(\xi - \underline{\xi})$ where $a(\underline{\xi}, \xi)$ is the right or left derivative of ϕ at the point $\underline{\xi}$ in correspondence to the sign of $\xi - \underline{\xi}$. Also $f_1 - f_0 = \phi(\xi) - \phi(\underline{\xi})$. Thus, f_0' is dominated by an integrable random variable and, by the monotone convergence, $F_0' = Ef_0'$. As F attains its minimum at zero, $F_0' \geq 0$. It follows that f_0' is integrable and the probability of the set $\{\underline{\xi} = 0\}$ is zero, because on this set $f_0' = \phi'(0)\xi = -\infty$. □

Proposition 3.2 *If $\mathcal{Z}$ is closed in L^1 and $\underline{J} < \infty$, then $\underline{J} = E\phi(\underline{\xi})$ for some $\underline{\xi} \in \mathcal{Z}$.*

Proof. Take a sequence $\xi_j \in \mathcal{Z}$ such that $E\phi(\xi_j) \to \underline{J}$. Since $\xi_j \geq 0$ and $E\xi_j = 1$, in virtue of the Komlós theorem there is a subsequence j_k such that $\tilde{\xi}_n := n^{-1}\sum_{k=1}^n \xi_{j_k}$ converge a.s. to a certain $\underline{\xi} \in L^1$. Due to the Fatou lemma and convexity of ϕ

$$E\phi(\underline{\xi}) = E\lim \phi(\tilde{\xi}_n) \leq \liminf E\phi(\tilde{\xi}_n) \leq \lim \frac{1}{n}\sum_{k=1}^n E\phi(\xi_{j_k}) = \underline{J}.$$

The de la Vallée-Poussin criterion ensures that $(\tilde{\xi}_n)$ are uniformly integrable and, hence, converge also in L^1. Thus, $\underline{\xi} \in \mathcal{Z}$. □

If ϕ is strictly convex, then, obviously, the minimizer is unique.

Suppose now that ϕ is strictly convex and differentiable on $]0,\infty[$. Let ψ be the inverse of ϕ' and let

$$\mathcal{C} := \{\eta : E\xi|\eta| < \infty \text{ and } E\xi\eta \leq 0 \quad \forall\, \xi \in \mathcal{Z}_\phi\}.$$

Here $\mathcal{Z}_\phi := \{\xi \in \mathcal{Z} : \; E\phi(\xi) < \infty\}$; $\mathcal{Q}_\phi$ denotes the set of corresponding measures.

Proposition 3.3 *Assume that $|\phi'(x)x| \leq c_1(\phi^+(x) + 1)$ for all $x \geq 0$ and $\underline{J} < \infty$. Let $\underline{\xi}$ be a strictly positive random variable. Then $\underline{\xi}$ is a minimizer iff $\underline{\xi} = \psi(c_0 - \eta)$ where $c_0 := E\phi'(\underline{\xi})\underline{\xi}$ and $\eta \in \mathcal{C}$ with $E\underline{\xi}\eta = 0$.*

Proof. The "if" part is obvious. Indeed, the convexity of ϕ implies that

$$E\phi(\xi) \geq E\phi(\underline{\xi}) + E\phi'(\underline{\xi})(\xi - \underline{\xi}) = E\phi(\underline{\xi}) - E\eta(\xi - \underline{\xi}) \geq E\phi(\underline{\xi}).$$

Conversely, let $\underline{\xi} > 0$ be a minimizer. Define $\eta := c_0 - \phi'(\underline{\xi})$. Take $\xi \in \mathcal{Z}_\phi$. As in the proof of Proposition 3.1, f_0' is integrable and $F_0' \geq 0$, i.e. $\phi'(\underline{\xi})(\xi - \underline{\xi})$ is integrable and

$$0 \leq E\phi'(\underline{\xi})(\xi - \underline{\xi}) = -E\eta\xi \leq E\phi(\xi) - E\phi(\underline{\xi}) < \infty.$$

Thus, $\eta\xi$ is integrable and $E\eta\xi \le 0$. $\square$

In particular, for $\phi(x) = x\ln x$ (with $\phi(0) := 0$) the functional attains its (finite) minimum at $\underline{\xi} > 0$ iff $\underline{\xi} = e^{\underline{J}-\eta}$ where $\eta \in \mathcal{C}$ and $E\underline{\xi}\eta = 0$. Note that in this case $E\xi|\ln\underline{\xi}| < \infty$ for every $\xi \in \mathcal{Z}$ with $E\xi\ln\xi < \infty$.

Let us consider a more specific setting where $\mathcal{Z}$ is the set of normalized annulators of a linear subspace R in L^∞, i.e. $\mathcal{Z} = \{\xi \ge 0 : \ E\xi = 1, \ E\xi\eta = 0 \ \forall \eta \in R\}$. Let $A := R - L^\infty_+$ and let $\bar{A}^Q$ denote the closure of A in $L^1(Q)$.

Proposition 3.4 *Let ϕ be such that for every $c, x > 0$*

$$\phi^+(cx) \le r_1(c)\phi^+(x) + r_2(c)(x+1), \qquad (7)$$

where $r_i \ge 0$ are increasing functions. Assume that $\mathcal{Z}_\phi \ne \emptyset$. Then $\mathcal{C} = \cap_{Q\in\mathcal{Q}_\phi}\bar{A}^Q$.

Proof. The inclusion $\mathcal{C} \supseteq \cap_{Q\in\mathcal{Q}_\phi}\bar{A}^Q$ is obvious. Assume that the inverse inclusion does not hold, i.e. there exists $\eta \in \mathcal{C}$ such that $\eta \notin \bar{A}^Q$ for some $Q = \xi P$, $\xi \in \mathcal{Z}_\phi$. Applying the Hahn–Banach theorem we find $\xi' \in L^\infty(Q)$, $\xi' \ne 0$, such that

$$E_Q\xi'\zeta < E_Q\xi'\eta \quad \forall \zeta \in \bar{A}^Q.$$

Since $-L^\infty \subseteq A$ and A is a cone, $\xi' \ge 0$ Q-a.s. and $E_Q\xi'\eta > 0$. Without loss of generality we may assume that $0 \le \xi' \le c$ and $E_Q\xi' = 1$. Noticing that $E_Q\xi'\zeta = 0$ when ζ belongs to the linear space R, we infer that $\tilde{\xi} := \xi\xi'$ is in $\mathcal{Z}_\phi$ because

$$E\phi^+(\tilde{\xi}) \le r_1(c)E\phi^+(\xi) + 2r_2(c) < \infty.$$

The inequality $E\tilde{\xi}\eta > 0$ contradicts $\eta \in \mathcal{C}$. $\square$

Theorem 3.5 *Let $\phi(x) = x\ln x$ and $\mathcal{Z}^e_\phi \ne \emptyset$. Then the problem (6) has a solution $\underline{\xi} = e^{\underline{J}-\eta}$ where $\eta \in \bar{R}^{\underline{P}}$ and $E\underline{\xi}\eta = 0$. Moreover,*

$$\max_{\zeta\in\bar{R}^{\underline{P}}} E(1-e^\zeta) = E(1-e^{-\eta}) = 1 - e^{-\underline{J}}. \qquad (8)$$

Proof. We know yet that the problem (6) admits a unique solution $\underline{\xi} = e^{\underline{J}-\eta}$ where $\eta \in \mathcal{C}$ and $\underline{E}\eta = 0$. Proposition 3.4 ensures that $\eta \in \bar{R}^{\underline{P}}$. Indeed, for some $\eta^n \in R$ and $\alpha^n \in L^\infty$ we have $\eta^n - \alpha^n \to \eta$ in $L^1(\underline{P})$. But $\underline{E}\eta^n = 0$ and $\underline{E}\eta = 0$. Thus, $\underline{E}\alpha^n \to 0$ and $\eta^n \to \eta$ in $L^1(\underline{P})$.

Plugging into the Fenchel inequality $U(x) + U^*(y) \ge xy$ for $U(x) = e^x - 1$ an arbitrary $\zeta \in \bar{R}^{\underline{P}}$ and $\underline{\xi}e^{-\underline{J}}$ and taking the expectation we get the bound

$$E(1-e^\zeta) \le 1 - e^{-\underline{J}}.$$

It remains to recall that we have the equality here when $\zeta = -\eta$. $\square$

Comment. All the results of this section can be found in various versions in the literature. To our knowledge, the observation that the assertion of Proposition 3.2 follows immediately from the Komlós theorem and the Fatou lemma is due to Evstigneev, see, e.g., the book Arkin and Evstigneev (1979). This existence result was discussed in more specific "financial" setting in the recent works Frittelli (2000) and Kramkov and Schachermayer (1999). Proposition 3.1 goes back to Csiszár. Proposition 3.3 and 3.4 are minor extensions of Theorem 3 and 4 in Frittelli (2000) where the case $\phi(x) = x \ln x$ is considered, see also Goll and Rüschendorf (2001).

4 Density processes

In the case where we are given not a probability space but a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$, we associate with $\mathcal{Z}$ the set $\mathcal{D} = \{Z : \ Z_t = E(\xi|\mathcal{F}_t), \ \xi \in \mathcal{Z}\}$ of density processes and consider $\underline{Z}_t = E(\underline{\xi}|\mathcal{F}_t)$ corresponding to a minimizer in the problem (6).

Let Π^Z denote the optional projection of the process $\phi(Z_T/Z)$, the unique right-continuous adapted process such that $\Pi^Z_\tau = E(\phi(Z_T/Z_\tau)|\mathcal{F}_\tau)$ for every τ from the set $\mathcal{T}_T$ of stopping times with values in $[0,T]$. Notice that $\Pi^Z \geq \phi(1)$ and $\Pi^Z_T = \phi(1)$.

We say that the set $\mathcal{D}$ is *stable under concatenation* if for each its elements Z^1 and Z^2 and every $\tau \in \mathcal{T}_T$ it contains also the process

$$\tilde{Z} := Z^1 I_{[0,\tau[} + Z^2 (Z^1_\tau / Z^2_\tau) I_{[\tau,T]}.$$

Proposition 4.1 *Let $\phi(x) = x^p$, $p > 1$, or $\phi(x) = x \ln x$. Assume that $\underline{J} < \infty$ and $\mathcal{D}$ is stable under concatenation. Then $\Pi^{\underline{Z}} \leq \Pi^Z$ for all $Z \in \mathcal{D}$.*

Proof. If the assertion fails then there exists $\varepsilon > 0$ such that for the stopping time $\tau := \inf\{t : \Pi^{\underline{Z}}_t \geq \Pi^Z_t + \varepsilon\} \wedge T$ we shall have $P(\tau < T) > 0$. Put

$$\tilde{Z} := \underline{Z} I_{[0,\tau[} + Z(\underline{Z}_\tau / Z_\tau) I_{[\tau,T]}.$$

Since $Z_T = \underline{Z}_\tau Z_T / Z_\tau$ and $\Pi^Z_\tau \leq \Pi^{\underline{Z}}_\tau - \varepsilon$ on the set $\{\tau < T\}$, we have for $\phi = x^p$:

$$\begin{aligned} E\phi(\tilde{Z}_T) I_{\{\tau<T\}} &= E\underline{Z}^p_\tau (Z_T/Z_\tau)^p I_{\{\tau<T\}} \\ &= E\underline{Z}^p_\tau \Pi^Z_\tau I_{\{\tau<T\}} \\ &\leq E\underline{Z}^p_\tau \Pi^{\underline{Z}}_\tau I_{\{\tau<T\}} - \varepsilon E\underline{Z}^p_\tau I_{\{\tau<T\}} \\ &= E\underline{Z}^p_\tau (\underline{Z}_T/\underline{Z}_\tau)^p I_{\{\tau<T\}} - \varepsilon E\underline{Z}^p_\tau I_{\{\tau<T\}} \\ &= E\phi(\underline{Z}_T) I_{\{\tau<T\}} - \varepsilon E\underline{Z}^p_\tau I_{\{\tau<T\}}. \end{aligned}$$

For $\phi(x) = x \ln x$ we have in a similar way that

$$\begin{aligned} E\phi(\tilde{Z}_T) I_{\{\tau<T\}} &= E\{[\underline{Z}_\tau (Z_T/Z_\tau)[\ln \underline{Z}_\tau + \ln(Z_T/Z_\tau)] I_{\{\tau<T\}}\} \\ &= E\underline{Z}_\tau \ln \underline{Z}_\tau I_{\{\tau<T\}} + E\underline{Z}_\tau \Pi^Z_\tau I_{\{\tau<T\}} \end{aligned}$$

$$
\begin{aligned}
&\leq \; E\underline{Z}_\tau \ln \underline{Z}_\tau I_{\{\tau<T\}} + E\underline{Z}_\tau \Pi_\tau^{\underline{Z}} I_{\{\tau<T\}} - \varepsilon E\underline{Z}_\tau I_{\{\tau<T\}} \\
&= \; E\underline{Z}_\tau \ln \underline{Z}_\tau I_{\{\tau<T\}} + E\underline{Z}_T \ln(\underline{Z}_T/\underline{Z}_\tau) I_{\{\tau<T\}} - \varepsilon E\underline{Z}_\tau I_{\{\tau<T\}} \\
&= \; E\phi(\underline{Z}_T) I_{\{\tau<T\}} - \varepsilon E\underline{Z}_\tau I_{\{\tau<T\}}.
\end{aligned}
$$

As $\tilde{Z}_T = \underline{\xi}$ on the set $\{\tau = T\}$ and $\underline{Z}_\tau > 0$ on $\{\tau < T\}$, we obtain in both cases the inequality $E\phi(\tilde{Z}_T) < E\phi(\underline{\xi})$ contradicting to the minimality of $\underline{\xi} = \underline{Z}_T$. □

From now on and until the end of this section we shall work with $\phi(x) = x \ln x$ assuming that $\mathcal{Q}_\phi^e \neq \emptyset$ and using $\underline{E}$ to denote the expectation with respect to the measure $\underline{P} := \underline{Z}_T P$ equivalent to P in virtue of Proposition 3.1.

Let $\bar{Z}_t := \exp\{\underline{E}(\ln \underline{Z}_T | \mathcal{F}_t)\}$. Clearly, $\ln \bar{Z} = \Pi^{\underline{Z}} + \ln \underline{Z}$ because

$$
\ln \bar{Z}_t = E\left((\underline{Z}_T/\underline{Z}_t) \ln(\underline{Z}_T/\underline{Z}_t) + (\underline{Z}_T/\underline{Z}_t) \ln \underline{Z}_t | \mathcal{F}_t\right).
$$

In the sequel X^* denotes the maximal function of a process X, i.e. $X_t^* = \sup_{s\leq t} |X_s|$.

Lemma 4.2 *Let $Q := Z_T P$ and $E\phi(Z_T) < \infty$. Then $E_Q(\ln \underline{Z})_T^* < \infty$. Moreover, if $\mathcal{D}$ is stable under concatenation, then the family $\{\ln \bar{Z}_\tau\}_{\tau\in\mathcal{T}_T}$ is Q-uniformly integrable.*

Proof. We may assume that $Q \sim P$ (the general case follows by considering the measure $(Q + \underline{P})/2$). Put $Z' := Z/\underline{Z}$. Then $Z'_T = dQ/d\underline{P}$ and Z' is the density process of Q with respect to $\underline{P}$. According to Proposition 3.3, $EZ_T|\ln \underline{Z}_T| < \infty$ and hence $\underline{E}Z'_T \ln Z'_T = EZ_T \ln(Z_T/\underline{Z}_T) < \infty$. Applying Barron's inequality (A.2) for Z (with respect to P) and Z' (with respect to $\underline{P}$) we get that $E_Q(\ln Z)_T^*$ and $E_Q(\ln Z')_T^*$ are finite. Since $\underline{Z} = Z/Z'$, this implies that $E_Q(\ln \underline{Z})_T^*$ is finite. It follows that the families $\{\ln \bar{Z}_\tau\}_{\tau\in\mathcal{T}_T}$ and $\{\Pi_\tau^{\underline{Z}}\}_{\tau\in\mathcal{T}_T}$ are Q-uniformly integrable simultaneously. But under the concatenation hypothesis the second family is Q-uniformly integrable. Indeed, for every $\tau \in \mathcal{T}_T$

$$
Q(\Pi_\tau^Z \geq N) \leq \frac{1}{N} E_Q \Pi_\tau^Z \leq \frac{1}{N} E(Z_T \ln Z_T + 1/e)
$$

(recall that $\Pi^Z \geq 1 \ln 1 = 0$ and $x \ln x \geq -1/e$). In virtue of Proposition 4.1,

$$
E_Q \Pi_\tau^{\underline{Z}} I_{\{\Pi_\tau^{\underline{Z}} \geq N\}} \leq E_Q \Pi_\tau^Z I_{\{\Pi_\tau^Z \geq N\}} \leq E(Z_T \ln Z_T + 1/e) I_{\{\Pi_\tau^Z \geq N\}}.
$$

The Q-uniform integrability of $\{\Pi_\tau^{\underline{Z}}\}_{\tau\in\mathcal{T}_T}$ follows easily from here. □

Comment. For prototypes of the above results see Lemmas 4-5 in Delbaen et al. (2002).

5 Proof of Theorem 2.1

(a) Let us consider the linear subspace R formed by all random variables $H\cdot S_T \in L^\infty$, H is a bounded predictable process of the form $H = \sum \xi_i I_{]t_i,t_{i+1}]}$, $\xi_i \in L^\infty(\mathcal{F}_{t_i})$. The assertion (a) follows now from Theorem 3.5 because, in virtue of the Yor theorem, Delbaen and Schachermayer (1999), $\bar{R}^{\underline{P}} = \mathcal{R}$.

(b) The set $\mathcal{Z}$ of local martingale measures is stable under concatenation. Let $Q \in \mathcal{Q}_\phi$ where $\phi(x) = x \ln x$. Since $\ln \bar{Z} = \underline{J} - H^o \cdot S$, we have in virtue of Lemma 4.2 that the family $H^o \cdot S_\tau$, $\tau \in \mathcal{T}_T$, is uniformly integrable with respect to Q. But this property is a necessary and sufficient condition ensuring that $H^o \cdot S$ is a Q-martingale, see Chou et al. (1980).

(c) We need the following extension of lemma 9 in Delbaen et al. (2002) :

Lemma 5.1 *Let $U(x)$ be an non-decreasing continuous function bounded from above and let the process $H \cdot S$ be bounded from below by a constant a. Then there exist bounded integrands $H^n \in \mathcal{A}_b$ such that $U(H^n \cdot S_T) \to U(H \cdot S_T)$ in L^1.*

Proof. Since the process $U(H \cdot S)$ is bounded and for any stopping time τ

$$HI_{[0,\tau]} \cdot S = (H \cdot S)^\tau = H \cdot S^\tau,$$

we may assume without loss of generality that S is not only locally bounded but bounded, say, by a constant c. Let $\tilde{H}^n := HI_{\{|H| \le n]\}}$, $\tau_n := \inf\{t : H \cdot S_t \ge n\}$. In virtue of the construction of stochastic integrals $((H - \tilde{H}^n) \cdot S)^*_T \to 0$ in probability as $n \to \infty$. For the sequence of stopping times

$$\sigma_n := \inf\{t : \ ((H - \tilde{H}^n) \cdot S)^*_t \ge 1\} \wedge T$$

we have that $P(\sigma_n = T) \to 1$. The important observation is that $\tilde{H}^n \cdot S \ge a - 1$, because the jump of $\tilde{H}^n \cdot S$ at σ_n is either zero, or equal to the jump of $H \cdot S$. Let $H^n := \tilde{H}^n I_{[0,\sigma_n \wedge \tau_n]}$. Then $H^n \cdot S_T \to H \cdot S_T$ in probability and

$$a - 1 \le H^n \cdot S_T \le n + 1 + 2nc.$$

The bounded sequence $U(H^n \cdot S_T)$, converging in probability to $U(H \cdot S_T)$, converges also in L^1. □

With the above lemma the assertion (c) follows from Theorem 2.2. □

A Barron's inequalities

Lemma A.1 *For any non-negative supermartingale Y*

$$E(\ln^- Y)^*_T \le e + e \sup_{t \le T} E \ln^- Y_t. \tag{A.1}$$

Proof. For $r > 1$ the function $\psi(x) := 1 \vee (\ln^- x)^{1/r}$ is convex and non-increasing and hence the process $X := \psi(Y)$ is a submartingale. By the Doob inequality

$$E(X^*_T)^r \le \left(\frac{r}{r-1}\right)^r \sup_{t \le T} E X^r_t.$$

It follows that

$$E(\ln^- Y)^*_T \le \left(\frac{r}{r-1}\right)^r \sup_{t\le T}\left(1+E\ln^- Y_t\right).$$

Taking the limit as $r\to\infty$ we get (A.1). □

Proposition A.2 *Let $Z=(Z_t)_{t\le T}$ be a density process. Then*

$$EZ_T(\ln Z)^*_T \le 2+e+eEZ_T\ln Z_T. \tag{A.2}$$

Proof. Let $Q=Z_TP$. A process X is a Q-supermartingale if and only if XZ is a supermartingale. Thus, $Y:=1/ZI_{\{Z>0\}}$ is a Q-supermartingale: $I_{\{Z>0\}}$ is a supermartingale since the function $I_{\{x>0\}}$ is concave on $I\!R_+$. Notice that Z is strictly positive Q-a.s. Using Lemma A.1, the bound $x\ln^+ x\le x\ln x+1/e$, and the fact that $Z\ln Z$ is a submartingale we obtain that

$$E_Q(\ln^+ Z)^*_T \le e+e\sup_{t\le T}E_Q\ln^+ Z_T \le 1+e+eEZ_T\ln Z_T.$$

It remains to combine this with the Ionescu Tulcea inequality

$$E_Q(\ln^- Z)^*_T \le 1. \tag{A.3}$$

The proof of the latter is simple. Indeed,

$$E_Q(\ln^- Z)^*_T = \int_0^\infty Q((\ln^- Z)^*_T\ge t)dt$$

Put $\tau_t:=\inf\{s:\ \ln(1/Z_s)\ge t\}$ and note that $E_Q(1/Z_{\tau_t})\le 1$. Then for $t>0$

$$Q((\ln^- Z)^*_T\ge t) = Q\left(\sup_{s\le T}\ln(1/Z_s)\ge t\right) = Q(\ln(1/Z_{\tau_t})\ge t) = Q(1/Z_{\tau_t}\ge e^t)\le e^{-t}$$

by the Chebyshev inequality. Integrating yields (A.3). □

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