Arbitrage Theory

Kabanov Yu. M.
Laboratoire de Mathématiques, Université de Franche-Comté
16 Route de Gray, F-25030 Besançon Cedex, FRANCE
and
Central Economics and Mathematics Institute
of the Russian Academy of Sciences, Moscow*

*The research of this paper was partially supported by the NATO Grant CRG-972115.

1 Introduction

We shall consider models where an investor, acting on a financial market with random price movements and having T as his time horizon, transforms the initial endowment ξ into a certain resulting wealth; let R_T^ξ denote the set of all final wealth corresponding to possible investment strategies. The natural question is, whether the investor has arbitrage opportunities, i.e. whether he can get non-risky profits.

Let us "hide" in a "black box" the interior dynamics on the time-interval $[0, T]$ (i.e. the price process specification, market regulations, description of admissible strategies) and examine only the set R_T^ξ.

At this level of generality, the answer, as well as the hypotheses, should be formulated only in terms of properties of the sets R_T^ξ. E.g., in the simplest situation of frictionless market without constraints, R_T^0 is a linear subspace in the space L^0 of (scalar) random variables and $R_T^\xi = \xi + R_T^0$. The absence of arbitrage opportunities can be formalized by saying that the intersection of R_T^0 with the set L_+^0 of non-negative random variables contains only zero. If the underlying probability space is finite, i.e. if we assume in our model only a finite number of states of the nature, it is easy to prove that there is no arbitrage if and only if there exists an equivalent "separating" probability measure with respect to which every element of R_T^0 has zero mean. Close look at this result shows that this assertion is nothing but the Stiemke lemma [62] of 1915 which is well-known in the theory of linear inequalities and linear programming as an example of the so-called alternative (or transposition) theorems, see historical comments in [61]; notice that the earliest alternative theorem due to Gordan [21] (of 1873) can be also interpreted as a no-arbitrage criterion.

The one-step model can be generalized (or specialized, depending on the point of view) in many directions giving rise to what is called arbitrage theory. The reader should not be confused by using "general" and "special" in this context: obviously, one-step models are particular cases of N-period models, but quite often the main difficulties in the analysis of models with a detailed ("specialized") structure of the "black box" consist in verifying hypotheses of theorems corresponding to the one-step case. The geometric essence of these results is a separation of convex sets with a subsequent identification of the separating functional as a probability measure; the properties of the latter in connection with the price process are of particular interest.

To the date one can find in the literature dozens of models of financial markets together with a plethora of definitions of arbitrage opportunities. These models can be classified using the following scheme.

1. Finite probability space.

Assuming only a finite number of states of the nature is popular in the literature on economics. Of course, the hypothesis is not adequate to the basic paradigm of stochastic modeling because random variables with continuous distributions cannot "live" on finite probability spaces. The advantage of working under this assumption is that a very restricted set of mathematical tools (basically, elementary finite-dimensional geometry) is required. Results obtained in this simplified setting have an important educational value and quite often may serve as the starting point for a deeper development.

2. General probability space.

In contrast to the case of finite probability space, the straightforward separation arguments, which are the main instruments to obtain no-arbitrage criteria, fail to be applied without further topological assumptions on R^0_T. In many particular cases, especially in the theory of continuous trading, they are not fulfilled. This circumstance led Kreps (1981) to a more sophisticated "no-arbitrage" concept, namely, that of "no free lunch" (NFL). However, certain no-arbitrage criteria are of the same form as for the models with finite probability space Ω.

3. Discrete-time multi-period models.

Even for the case of finite probability space Ω, these models are important because they allow us to describe the intertemporal behavior of investors in financial markets, i.e. to penetrate into the structure of the "black box" using concepts of random processes. One of the most interesting features is that in the simplest model without constraints the value processes of the investor's portfolios are martingales with respect to separating measures and the same property holds for the underlying price process; this explains the terminology "equivalent martingale measures". Models based on the infinite Ω posed challenging mathematical questions, e.g., whether the absence of arbitrage is still equivalent to the existence of equivalent martingale measure. For frictionless market the affirmative answer has been given by Dalang, Morton, and Willinger in 1990. Their work, together with the earlier paper of Kreps, stimulated further research in geometric functional analysis and stochastic calculus, involving rather advanced mathematics.

4. Continuous trading.

Although the continuous-time stochastic processes were used for modeling from the very beginning of mathematical finance (one can say that they were even invented exactly for this purpose, having in mind the Bachelier thesis "Théorie de la spéculation" where Brownian motion appeared for the first time), their "golden age" began in 1973 when the famous Black–Scholes formula was published. Subsequent studies revealed the role of the uniqueness of the equivalent martingale measure for pricing of derivative securities via replication. The importance of no-arbitrage criteria seems to be overestimated in financial literature: the unfortunate alias FTAP — Fundamental Theorem of Asset (or Arbitrage) Pricing, ambitious and misleading, is still widely used. If there are many equivalent martingale measures, the idea of "pricing by replication" fails: a contingent claim may not belong to R^x_T whatever x is, or may belong to many R^x_T. In the latter case it is not clear which martingale measure can be used for pricing and this is the central problem of current studies on incomplete markets. However, as to mathematics, the no-arbitrage criteria for general semimartingale models are considered among the top achievements of the theory.

In 1980 Harrison and Pliska noticed that stochastic calculus, i.e. the integration theory for semimartingales, developed by P.-A. Meyer in a purely abstract way, is "tailor-made" for financial modeling. In 1994 Delbaen and Schachermayer confirmed this conclusion by proving that the absence of arbitrage in the class of elementary, "practically admissible" strategies implies the semimartingale property of the price process. In a series of papers they provided a profound analysis of the various concepts culminating in a result that the Kreps NFL condition (equivalent to a whole series of properties with easier economic interpretation) holds if and only if the price process is a σ-martingale under some $\tilde{P} \sim P$. There is another justification of the increasing interest in semimartingales in financial modeling: mathematical statistics sends alarming signals that in many cases empirical data for financial time series are not compatible with the hypothesis that they are generated by processes with continuous sample paths. Thus, diffusions should be viewed only as strongly stylized models of financial data; it has been revealed that Lévy processes give much better fit.

5. Large financial markets.

This particular group, including the so-called Arbitrage Pricing Model (or Theory), abbreviated to APM (or APT), due to Ross and Huberman (for the one-period case), has the following specific feature. In contrast with the conventional approach of describing a security market by a single probabilistic model, a sequence of stochastic bases with an increasing but always finite number of assets is considered. One can think that the agent wants to concentrate his activity on smaller portfolios because of his physical limitations but larger portfolios in this market may have better performance. The arbitrage is understood in an asymptotic sense. Its absence implies relationships between model parameters which can be verified empirically. This circumstance makes such models especially attractive. The weak side of APM is the use of the quadratic risk measure. This means that gains are punished together with losses in symmetric ways which is unrealistic. Luckily, the conclusion of APM, the Ross–Huberman boundedness condition, seems to be sufficiently "robust" with respect to the risk measure and the variation of certain model parameters.

In the recent papers [36] and [37], where the theory of large financial markets was extended to the general semimartingale framework, the concept of asymptotic arbitrage is developed for an "absolutely" risk-averse agent. In spite of completely different approach, the absence of asymptotic arbitrage implies, for various particular models, relations similar to the Ross–Huberman condition.

6. Models with transaction costs.

In the majority of models discussed in mathematical finance, the investor's wealth is scalar, i.e. all positions are measured in units of a single asset (money, bond, bank account, etc.). However, in certain cases, e.g., in models with constraints and, especially, in those taking transaction costs into account, it is quite natural to consider, as the primary object, the whole vector-valued process of current positions, either in physical quantities or in units of values measured by a certain numéraire. It happens that this approach allows not only for a more detailed and realistic description of the portfolio dynamics but also opens new perspectives for further mathematical development, in particular, for an extensive use of ideas from theory of partially ordered spaces, utility theory, optimal control, and mathematical economics. Until now only a few results are available in this new branch of arbitrage theory. Recent studies [34] and [41] show that the basic concept of arbitrage theory, that of the equivalent martingale measure, should be modified and generalized in an appropriate way. There are various approaches to the problem which will be discussed here. Notice that models with transaction costs quite often were considered as completely different from those of a frictionless market and the classical results could not be obtained as corollaries when transaction costs vanish. The modern trend in the theory is to work in the framework which covers the latter as a special case.

Arbitrage theory includes another, even more important subject, namely, hedging theorems, closely related with the no-arbitrage criteria. These results, discussed in the present survey in a sketchy way, give answers to whether a contingent claim can be replicated in an appropriate sense by a terminal value of a self-financing portfolio or whether a given initial endowment is sufficient to start a portfolio replicating the contingent claim. Other related problems such as market completeness or models with continuum securities, arising in the theory of bond markets, are not touched here.

The books [52], [57], and [29] may serve as references in convex analysis, probability, and stochastic calculus.

2 Discrete-time models

2.1 General setting

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$ be a stochastic basis (i.e. filtered probability space), $t = 0, 1, ..., T$. We assume that each σ-algebra $\mathcal{F}_t$ is complete.

We are given:

- convex cones $R_t^0 \subseteq L^0(\mathbf{R}^d, \mathcal{F}_t)$;

- closed convex cones $\mathcal{K}_t \subseteq L^0(\mathbf{R}^d, \mathcal{F}_t)$.

The notation $L^0(K_t, \mathcal{F}_t)$ is used for the set of all $\mathcal{F}_t$-measurable random variables with values in the set K_t (or $\mathcal{F}_t$-measurable selectors of K_t if K_t depends on ω).

The usual financial interpretation: R_t^0 is the set of portfolio values at the date t corresponding to the zero initial endowment, i.e. all imaginable results that can be obtained by the investor to the date t.

The cones $\mathcal{K}_t$ induce the partial orderings in the sets $L^0(\mathbf{R}^d, \mathcal{F}_t)$:

$$\xi \geq_t \eta \quad \Leftrightarrow \quad \xi - \eta \in \mathcal{K}_t.$$

The partial orderings $\geq_t$ allow us to compare current results.

As a rule, they are obtained by "lifting" partial orderings from $\mathbf{R}^d$ to the space of random variables.

A typical example: $\mathcal{K}_t = L^0(K, \mathcal{F}_t)$ where K is a closed cone in $\mathbf{R}^d$ (which may depend on ω and t). In particular, the "standard" ordering $\geq_t$ is induced by $K_t = \mathbf{R}^d_+$ when $\xi \geq_t \eta$ if $\xi^i \geq \eta^i$ (a.s.) for all $i \leq d$; for the case $d = 1$ it is the usual linear ordering of the real line. However, we do not exclude other partial orderings.

In the theory of frictionless market, usually, $d = 1$; for models with transaction costs d is the number of assets in the portfolio.

We define also the set $A^0_T := R^0_T - \mathcal{K}_T$. The elements of A^0_T are interpreted as contingent claims which can be hedged (or super-replicated) by the terminal values of portfolios starting from zero.

The linear space $\mathcal{L}_T := \mathcal{K}_T \cap (-\mathcal{K}_T)$ describes the positions ξ such that $\xi \geq_T 0$ and $\xi \leq_T 0$, which are "financially equivalent to zero". The comparison of results can be done modulo this equivalence, i.e. in the quotient space $L^0/\mathcal{L}_T$ equipped with the ordering induced by the proper cone $\tilde{\mathcal{K}}_T := \pi_T \mathcal{K}_T$ where $\pi_T : L^0 \to L^0/\mathcal{L}_T$ is the natural projection.

2.2 No-arbitrage criteria for finite Ω

The most intuitive formulation of the property that the market has no arbitrage opportunities for the investors without initial capital is the following:

NA. $\mathcal{K}_T \cap R^0_T \subseteq \mathcal{L}_T$.

In the particular case when $\mathcal{K}_T$ is a proper cone we have

NA′. $\mathcal{K}_T \cap R^0_T \subseteq \{0\}$ (with equality if R^0_T is closed).

The first no-arbitrage criteria has the following form.

Theorem 2.1 *Let Ω be finite. Assume that R^0_T is closed. Then* **NA** *holds if and only if there exists $\eta \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ such that*

$$E\eta\zeta > 0 \qquad \forall \zeta \in \mathcal{K}_T \setminus \mathcal{L}_T$$

and

$$E\eta\zeta \leq 0 \qquad \forall \zeta \in R^0_T.$$

Because L^0 is a finite-dimensional space, this result is a reformulation of Theorem A.2 on separation of convex cones.

It is easy to verify that $\mathcal{K}_T \cap R^0_T \subseteq \mathcal{L}_T$ if and only if $\mathcal{K}_T \cap A^0_T \subseteq \mathcal{L}_T$. Hence, in this theorem one can replace R^0_T by A^0_T.

The above criterion can be classified as a result for the one-step model where T stands for "terminal". It has important corollaries for multi-period models where the sets R^0_T have a particular structure.

3 Multi-step models

3.1 Notations

For $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ we define $X_- := (X_{t-1})$ (various conventions for X_{-1} can be used), $\Delta X_t := X_t - X_{t-1}$, and, at last,

$$X \cdot Y_t := \sum_{k=0}^{t} X_k \Delta Y_k,$$

for the discrete-time integral. Here X and Y can be scalar or vector-valued. In the latter case sometimes we shall use the abbreviation $X \bullet Y$ for the vector process formed by the pairwise integrals of the component

$$X \bullet Y := (X^1 \cdot Y^1, ..., X^d \cdot Y^d).$$

Though in the discrete-time case the dynamics can be expressed exclusively in terms of differences, "integral" formulae are often instructive for continuous-time extensions.

For finite Ω, if X is a predictable process (i.e., X_t is $\mathcal{F}_{t-1}$-measurable) and Y belongs to the space $\mathcal{M}$ of martingales, then $X \cdot Y$ is also a martingale.

The product formula

$$\Delta(XY) = X\Delta Y + Y_-\Delta X$$

is obvious.

3.2 Example 1. Model of frictionless market

The model being classical, we do not give details and financial interpretations: they are widely available in many textbooks.

Let $S = (S_t)$, $t = 0, 1, ..., T$, be a fixed n-dimensional process adapted to a discrete-time filtration $\mathbf{F} = (\mathcal{F}_t)$. Here T is a finite integer and, for simplicity, the σ-algebra $\mathcal{F}_0$ assumed to be trivial. The convention $S_{-1} = S_0$ is used. Define R^0_T as the linear space of all scalar random variables of the form $N \cdot S_T$ where N is an n-dimensional predictable process. For $x \in \mathbf{R}$ we put $R^x_T = x + R^0_T$. We take $\mathcal{K}_0 := \mathbf{R}_+$ and $\mathcal{K}_T := L^0(\mathbf{R}_+, \mathcal{F}_T)$.

The components S^i describe the price evolution of n risky securities, N^i is the portfolio strategy which is self-financing, and V is the value process. In this specification it is tacitly assumed that there is a traded asset with the constant unit price, i.e. this asset is the *numéraire*.

Remark 3.1 One should take care that there is another specification where the numéraire is not necessarily a traded asset. A possible confusion may arise because the formula for the value process looks similar but the integrand and the integrator are in the latter case d-dimensional processes with $d = n + 1$. The increments of a self-financing portfolio strategy are explicitly constrained by the relation

$$S_{t-1}\Delta N_t = 0.$$

If the numéraire ("cash" or "bond") is traded, the integral with respect to the latter vanishes but, of course, holdings in "cash" are not arbitrary but defined from the above relation.

For finite Ω we have, in virtue of Theorem 2.1, that the model has no-arbitrage if and only if there is a strictly positive random variable η such that $E\eta\zeta = 0$ for all $\zeta \in R^0_T$. Without loss of generality we may assume that $E\eta = 1$ and define the probability measure $\tilde{P} = \eta P$. Clearly, $\tilde{E}\zeta = 0$ for all $\zeta \in R^0_T$ (i.e., $\tilde{E}N \cdot S_T = 0$ for all predictable N) if and only if S is a martingale. With this remark we get the Harrison–Pliska theorem:

Theorem 3.2 *Assume that Ω is finite. Then the following conditions are equivalent:*

(a) $R^0_T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ *(no-arbitrage);*

(b) *there exists a measure $\tilde{P} \sim P$ such that $S \in \mathcal{M}(\tilde{P})$.*

Let $\rho_t := d\tilde{P}_t/dP_t$ be the density corresponding to the restrictions of $\tilde{P}$ and P to $\mathcal{F}_t$. Recall that the density process $\rho = (\rho_t)$ is a martingale $\rho_t = E(\rho_T|\mathcal{F}_t)$. Since

$$S \in \mathcal{M}(\tilde{P}) \iff S\rho \in \mathcal{M}(P),$$

we can add to the conditions of the above theorem the following one:

(b') *there is a strictly positive martingale ρ such that $\rho S \in \mathcal{M}$.*

Notice that the equivalence of (b) and (b') is a general fact which holds for arbitrary Ω and even in the continuous-time setting.

Though the property (b') can be considered simply as a reformulation of (b), it is more adapted to various extensions. The advantage of (b) is in the interpretation of $\tilde{P}$ as a "risk-neutral" probability.

3.3 Example 2. Model with transaction costs

Now we describe a discrete-time version of a multi-currency model with proportional transaction costs introduced in [34] and studied in the papers [11] and [41].

It is assumed that the components of an adapted process $S = (S^1_t, \dots, S^d_t)$, $t = 0, 1, ..., T$, describing the dynamics of prices of certain assets, e.g., currencies quoted in a certain reference asset (say, “euro”), are strictly positive. It is convenient to choose the scales to have $S_0^i = 1$ for all i. We do not suppose that the numéraire is a traded security.

The transaction costs coefficients are given by an adapted process $\Lambda = (\lambda^{ij})$ taking values in the set $\mathbf{M}_+^d$ of non-negative $d \times d$-matrices with zero diagonal.

The agent's portfolio at time t can be described either by a vector of “physical” quantities $\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d)$ or by a vector $V = (V_t^1, \dots, V_t^d)$ of values invested in each asset. The relation

$$\widehat{V}_t^i = V_t^i / S_t^i, \quad i \le d,$$

is obvious. Introducing the diagonal operator

$$\phi_t(\omega) : (x^1, ..., x^d) \mapsto (x^1/S_t^1(\omega), ..., x^d/S_t^d(\omega)). \tag{1}$$

we may write that

$$\widehat{V}_t = \phi_t V_t.$$

The increments of portfolio values are

$$\Delta V_t^i = \widehat{V}_t^i \Delta S_t^i + b_t^i \tag{2}$$

with

$$b_t^i = \sum_{j=1}^d \alpha_t^{ji} - \sum_{j=1}^d (1 + \lambda^{ij}) \alpha_t^{ij},$$

where $\alpha_t^{ji} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ represents the net amount transferred from the position j to the position i at the date t.

The first term in the right-hand side of (2) is due to the price increment while the second corresponds to the agent's actions (made after the revealing of new prices). Notice that these actions are charged by the amount

$$-\sum_{i=1}^d b_t^i = \sum_{i=1}^d \sum_{j=1}^d \lambda^{ij} \alpha_t^{ij}$$

diminishing the total portfolio value.

With every $\mathbf{M}_+^d$-valued process (α_t) and any initial endowment

$$v = V_{-1} \in \mathbf{R}^d$$

we associate, using recursively the formula (2), a value process $V = (V_t)$, $t = 0, ..., T$. The terminal values of these processes form the set R_T^v.

Remark 3.3 In the literature one can find other specifications for transaction costs coefficients. To explain the situation, let us define $\tilde{\alpha}^{ij} := (1 + \lambda^{ij}) \alpha^{ij}$. The increment of value of the i-th position can be written as

$$b^i = \sum_{j=1}^d \mu^{ji} \tilde{\alpha}_t^{ji} - \sum_{j=1}^d \tilde{\alpha}_t^{ij},$$

where $\mu^{ji} := 1/(1+\lambda^{ji}) \in]0,1]$. The matrix (μ^{ij}) can be specified as the matrix of the transaction costs coefficients. In models with a traded numéraire, i.e. a non-risky asset, a mixture of both specifications is used quite often.

Before analyzing the model, we write it in a more convenient way reducing the dimension of the action space.

To this aim we define, for every (ω, t), the convex cone

$$M_t(\omega) := \Big\{x \in \mathbf{R}^d : \exists\, a \in \mathbf{M}^d_+ \text{ such that } x^i = \sum_{i=1}^d [(1+\lambda_t^{ij}(\omega))a^{ij} - a^{ji}],\ i \le d\Big\},$$

which is a polyhedral one as it is the image of the polyhedral cone $\mathbf{M}^d_+$ under a linear mapping. Its dual positive cone

$$M_t^*(\omega) := \{w \in \mathbf{R}^d : \inf_{x \in M_t(\omega)} wx \ge 0\}$$

can be easily described by linear homogeneous inequalities. Specifically,

$$M_t^*(\omega) = \{w \in \mathbf{R}^d : w^j - (1+\lambda_t^{ij}(\omega))w^i \le 0,\ 1 \le i, j \le d\}.$$

We introduce also the solvency cone (in values)

$$K_t(\omega) := \Big\{x \in \mathbf{R}^d : \exists\, a \in \mathbf{M}^d_+ \text{ such that } x^i + \sum_{i=1}^d [a^{ji} - (1+\lambda_t^{ij}(\omega))a^{ij}] \ge 0,\ i \le d\Big\},$$

i.e. $K_t(\omega) = M_t(\omega) + \mathbf{R}^d_+$. The negative holdings of a position vector in $K_t(\omega)$ can be liquidated (under transaction costs given by $(\lambda_t^{ij}(\omega))$ to get a position vector in $\mathbf{R}^d_+$.

Let $\mathcal{B}$ be the set of all processes $B = (B_t)$ with $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$. It is an easy exercise on measurable selection to check that ΔB_t can be represented using a certain $\mathcal{F}_t$-measurable transfer matrix α_t. Thus, the set of portfolio process in the "value domain" coincides with the set of processes $V = V^{v,B}$, $B \in \mathcal{B}$, given by the system of linear difference equations

$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta B_t^i, \quad V_{-1}^i = v^i, \tag{3}$$

with

$$\Delta Y_t^i = \frac{\Delta S_t^i}{S_{t-1}^i}, \quad Y_0^i = 1. \tag{4}$$

Remark 3.4 Using the notations introduced at the beginning of this section, we can rewrite these equations in the integral form

$$V = v + V_- \bullet Y + B, \tag{5}$$

with

$$Y^i = 1 + (1/S_-^i) \cdot S^i, \tag{6}$$

which remains the same also for the continuous-time version but with a different meaning of the symbols, see [34], [39].

It is easier to study no-arbitrage properties of the model working in the "physical domain" where portfolio evolves only because of the agent's action. Indeed, the dynamics of $\widehat{V}$ is simpler:

$$\Delta V_t^i = \frac{\Delta B_t^i}{S_t^i}.$$

This equation is obvious because of its financial interpretation but one can check it formally (e.g., using the product formula).

Put $\widehat{M}_t(\omega) := \phi_t(\omega) M_t(\omega)$ and introduce the solvency cone (in physical units)

$$\widehat{K}_t(\omega) := \phi_t K_t(\omega) = \widehat{M}_t(\omega) + \mathbf{R}^d.$$

Every process $\widehat{b}$ with $\widehat{b}_t \in L^0(-\widehat{M}_t, \mathcal{F}_t)$, $0 \le t \le T$, defines a portfolio process $\widehat{V}$ with $\Delta V = \widehat{b}$ and the zero initial endowment. All portfolio processes (in physical units) can be obtained in this way.

The notations R_T^0 and $\widehat{R}_T^0$ are obvious.

Lemma 3.5 *The following conditions are equivalent:*
(*a*) $R_T^0 \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(\partial K_T, \mathcal{F}_T)$;
(*b*) $R_T^0 \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$;
(*c*) $\widehat{R}_T^0 \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$.

Proof. The equivalence of (b) and (c) is obvious. The implication $(a) \Rightarrow (b)$ holds because $\mathbf{R}_+^d \setminus \{0\}$ is a subset of $\operatorname{int} K_T$. To prove the remaining implication $(b) \Rightarrow (a)$ we notice that if $V_T^B \in L^0(K_T, \mathcal{F}_T)$ where $B \in \mathcal{B}$ then there exists $B' \in \mathcal{B}$ such that $V_T^{B'} \in L^0(\mathbf{R}_+^d, \mathcal{F}_T)$ and $V_T^{B'}(\omega) \neq 0$ on the set $V_T^B(\omega) \notin \partial K_T(\omega)$. To construct such B', it is sufficient to modify only ΔB_T by combining the last transfer with the liquidation of the negative positions. $\square$

In accordance with [41] we shall say that the market has *weak no-arbitrage property at the date T* ($\mathbf{NA}_T^w$) if one of the equivalent conditions of the above lemma is fulfilled. Apparently, $\mathbf{NA}_T^w$ implies $\mathbf{NA}_t^w$ for all $t \le T$.

Lemma 3.6 *Assume that Ω is finite. Then $\widehat{R}_T^0 \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$ if and only if there exists a d-dimensional martingale Z with strictly positive components such that $Z_t \in L^0(\widehat{M}^*, \mathcal{F}_t)$.*

Proof. The cone $\widehat{R}_T^0$ is polyhedral. In virtue of Theorem 2.1 the first condition is equivalent to the existence of a strictly positive random variable η such that $E\eta\zeta \le 0$ for all $\zeta \in \widehat{R}_T^0$. Let $Z_t = E(\eta|\mathcal{F}_t)$. Since $L^0(-\widehat{M}_t, \mathcal{F}_T) \subseteq \widehat{R}_T^0$, the inequality $EZ_t\zeta \ge 0$ holds for all $\zeta \in L^0(\widehat{M}_t, \mathcal{F}_t)$ implying that $Z_t \in L^0(\widehat{M}_t^*, \mathcal{F}_t)$. If the second condition of the lemma is fulfilled, we can take $\eta = Z_T$. $\square$

Let $\mathcal{D}_T$ be the set of martingales $Z = (Z_t)$ such that $\widehat{Z}_t \in L^0(K_t^*, \mathcal{F}_t)$. The following result from [41] is a simple corollary of the above criteria:

Theorem 3.7 *Assume that Ω is finite. Then $\mathbf{NA}^w_T$ holds if and only if there exists a process $Z \in \mathcal{D}$ with strictly positive components.*

This result contains the Harrison–Pliska theorem. Indeed, in the case where all $\lambda^{ij} = 0$, the cone $K = \tilde{K} := \{x \in \mathbf{R}^d : \ x\mathbf{1} \geq 0\}$ and $K^* = \mathbf{R}_+\mathbf{1}$. Thus, for $Z \in D$ all components of the process $\widehat{Z}$ are equal. If, e.g., the first asset is the numéraire, then $\widehat{Z}^1 = Z^1$ is a martingale as well as the processes $S^i Z^1$, $i = 2, ..., d$, i.e. Z^1 is a martingale density.

Remark 3.8 For models with transaction costs other types of arbitrage may be of interest. E.g., it is quite natural to consider the ordering induced by the cone $\tilde{K} := \{x \in \mathbf{R}^d : \ x\mathbf{1} \geq 0\}$ (corresponding to the absence of transaction costs), see a criterion in [41] which can be obtained along the same lines as above.

Remark 3.9 It is easily seen that

$$\widehat{M}_t(\omega) := \Big\{ y \in \mathbf{R}^d : \ \exists\, c \in \mathbf{M}^d_+ \text{ such that } y^i = \sum_{j=1}^{d} [\pi^{ij}_t(\omega) c^{ij} - c^{ji}], \ i \leq d \Big\}, \tag{7}$$

where

$$\pi^{ij}_t := (1 + \lambda^{ij}_t) S^j_t / S^i_t, \qquad 1 \leq i, j \leq d. \tag{8}$$

One can start the modeling by specifying instead of the process (λ^{ij}_t) the process (π^{ij}_t) with values in the set of non-negative matrices with units on the diagonal. Defining directly the set of processes $\widehat{V}$ with $\Delta \widehat{V}_t \in L^0(-\widehat{M}_t, \mathcal{F}_t)$ and the set of "results" $\widehat{R}^0_T$, one can get Lemma 3.6 immediately. The advantage of this approach is that the existence of the reference asset (i.e., of the price process S) is not assumed and we have a model of "pure exchange". A question arises when such a model can be reduced to a transaction costs model with a reference asset, i.e. under what conditions on the matrix (π^{ij}) one can find a matrix (λ^{ij}) with positive entries and a vector S with strictly positive entries satisfying the relation (8).

3.4 Dalang–Morton–Willinger theorem

Let us consider again the classical model of the frictionless market but now without any assumption on the stochastic basis.

Theorem 3.10 *The following conditions are equivalent:*

(a) $R^0_T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ *(no-arbitrage);*

(b) $A^0_T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$*;*

(c) $A^0_T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ *and* $A^0_T = \bar{A}^0_T$*, the closure in* L^0*;*

(d) $\bar{A}^0_T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$*;*

(e) for every probability measure $P' \sim P$ *there is a measure* $\tilde{P} \sim P$ *such that* $d\tilde{P}/dP' \leq \text{const}$ *and* $S \in \mathcal{M}(\tilde{P})$*;*

(f) *there is a probability measure* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}(\tilde{P})$.

(g) *there is a probability measure* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}_{loc}(\tilde{P})$.

It seems that these equivalent conditions (among many others) are the most essential ones to be collected in a single theorem. The equivalence of (a), (e), and (f) relating a "financial property" of absence of arbitrage with important "probabilistic" properties is due to Dalang, Morton, and Willinger [8]. Their approach is based on a reduction to a one-stage problem which is very simple for the case of trivial initial σ-algebra; regular conditional distributions and measurable selection theorem allow us to extend the arguments to treat the general case, see [53], [29], and [58] for other implementations of the same idea. Formally, the equivalence (a) $\Leftrightarrow$ (f) is exactly the same as the Harrison–Pliska theorem and one could think that it is just the same result under the relaxed hypothesis on Ω. In fact, such a conclusion seems to be superficial: the equivalent "functional-analytic property" (c), discovered by Schachermayer in [56], shows clearly the profound difference between these two situations. Schachermayer's condition opens the door to an extensive use of geometric functional analysis in the discrete-time setting which was reserved previously only for continuous-time models. It is quite interesting to notice that the set R^0_T is always closed while A^0_T is not.

The condition (d) introduced by Stricker in [60] also gives a hint on an appropriate use of separation arguments. Specifically, the Kreps–Yan theorem (see the Appendix) can be applied to separate $A_0^T \cap L^1(P')$ from $L^1_+(P') = L^1(\mathbf{R}_+, P')$ where the measure $P' \sim P$ can be chosen arbitrarily: this freedom allows us to obtain an "equivalent separating measure" with a desired property.

Notice that the crucial implication (b) $\Rightarrow$ (d) seems to be easier to prove than (a) $\Rightarrow$ (c), see [36] where a kind of "linear algebra" with random coefficients was suggested.

The literature provides a variety of other equivalent conditions complementing the list of the above theorem. Some of them are interesting and non-trivial. A family of conditions is related with various classes of admissible strategies $\mathcal{B}$ (which is the set of all predictable process in our formulation). Since the sets R^0_T and A^0_T depend on this class, so does the no-arbitrage property. It happens, however, that the latter is quite "robust": e.g., it remains the same if we consider as admissible only the strategies with non-negative value processes. The problem of admissibility is not of great importance since we assume a finite time horizon. The situation is radically different for continuous-time models where one must work out the doubling strategies which allow us to win even betting on a martingale.

Proof of Theorem 3.10. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious as well as the chain (e) $\Rightarrow$ (f) $\Rightarrow$ (g).

To prove the implication (d) $\Rightarrow$ (e) we observe that the two properties are invariant under the equivalent change of measure. Thus, we may assume that $P' = P$ and, moreover, by passing to the measure $ce^{-\eta}P$ with $\eta = \sup_{t \le T} |S_t|$, that all S_t are integrable. The set $\bar{A}^1_0 \cap L^1$ is closed in L^1 and intersects with L^1_+ only at zero.

By the Kreps–Yan theorem there is a $\tilde{P}$ with $d\tilde{P}/dP \in L^\infty$ such that $\tilde{E}\xi \le 0$ for all $\xi \in \bar{A}_0^1 \cap L^1$. Taking $\xi = \pm H_t \Delta S_t$ where H_t is bounded and $\mathcal{F}_{t-1}$-measurable, we conclude that S is a martingale.

The implication $(g) \Rightarrow (a)$ is also easy. If $H \cdot S_t \ge 0$ for all $t \le T$, then, by the Fatou lemma, the local $\tilde{P}$-martingale $H \cdot S$ is a $\tilde{P}$-supermartingale and, therefore, $\tilde{E} H \cdot S_T \le 0$, i.e. $H \cdot S_T = 0$. In other words, there is no arbitrage in the class of strategies with non-negative value processes. This implies (a) since for any arbitrage opportunity H there is an arbitrage opportunity H' with non-negative value process. Indeed, if $P(H \cdot S_s \le -b) > 0$ for some $s < T$ and $b > 0$, then one can take $H' = I_{]s,T] \times \{H \cdot S_s \le -b\}} H$.

In the proof of the "difficult" implication $(b) \Rightarrow (c)$ we follow [42].

Lemma 3.11 *Let $\eta^n \in L^0(\mathbf{R}^d)$ be such that $\underline{\eta} := \liminf |\eta^n| < \infty$. Then there are $\tilde{\eta}^k \in L^0(\mathbf{R}^d)$ such that for all ω the sequence of $\tilde{\eta}^k(\omega)$ is a convergent subsequence of the sequence of $\eta^n(\omega)$.*

Proof. Let $\tau_0 := 0$ and $\tau_k := \inf\{n > \tau_{k-1} : \ ||\eta^n| - \underline{\eta}| \le 1/k\}$. Then $\tilde{\eta}_0^k := \eta^{\tau_k}$ is in $L^0(\mathbf{R}^d)$ and $\sup_k |\tilde{\eta}_0^k| < \infty$. Working further with the sequence of $\tilde{\eta}_0^n$ we construct, applying the above procedure to the first component, a sequence of $\tilde{\eta}_1^k$ with the convergent first component and such that for all ω the sequence of $\tilde{\eta}_1^k(\omega)$ is a subsequence of the sequence of $\tilde{\eta}_0^n(\omega)$. Passing on each step to the newly created sequence of random variables and to the next component we arrive at a sequence with the desired properties. □

To show that A_T^0 is closed we proceed by induction. Let $T = 1$. Suppose that $H_1^n \Delta S_1 - r^n \to \zeta$ a.s., where H_1^n is $\mathcal{F}_0$-measurable and $r^n \in L_+^0$. It is sufficient to find $\mathcal{F}_0$-measurable random variables $\tilde{H}_1^k$ convergent a.s. and $\tilde{r}^k \in L_+^0$ such that $\tilde{H}_1^k \Delta S_1 - \tilde{r}^k \to \zeta$ a.s.

Let $\Omega_i \in \mathcal{F}_0$ form a finite partition of Ω. Obviously, we may argue on each Ω_i separately as on an autonomous measure space (considering the restrictions of random variables and traces of σ-algebras).

Let $\underline{H}_1 := \liminf |H_1^n|$. On $\Omega_1 := \{\underline{H}_1 < \infty\}$ we take, using Lemma 3.11, $\mathcal{F}_0$-measurable $\tilde{H}_1^k$ such that $\tilde{H}_1^k(\omega)$ is a convergent subsequence of $H_1^n(\omega)$ for every ω; $\tilde{r}^k$ are defined correspondingly. Thus, if Ω_1 is of full measure, the goal is achieved.

On $\Omega_2 := \{\underline{H}_1 = \infty\}$ we put $G_1^n := H_1^n/|H_1^n|$ and $h_1^n := r_1^n/|H_1^n|$ and observe that $G_1^n \Delta S_1 - h_1^n \to 0$ a.s. By Lemma 3.11 we find $\mathcal{F}_0$-measurable $\tilde{G}_1^k$ such that $\tilde{G}_1^k(\omega)$ is a convergent subsequence of $G_1^n(\omega)$ for every ω. Denoting the limit by $\tilde{G}_1$, we obtain that $\tilde{G}_1 \Delta S_1 = \tilde{h}_1$ where $\tilde{h}_1$ is non-negative, hence, in virtue of (b), $\tilde{G}_1 \Delta S_1 = 0$.

As $\tilde{G}_1(\omega) \ne 0$, there exists a partition of Ω_2 into d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $\tilde{G}_1^i \ne 0$ on Ω_2^i. Define $\bar{H}_1^n := H_1^n - \beta^n \tilde{G}_1$ where $\beta^n := H_1^{ni}/\tilde{G}_1^i$ on Ω_2^i. Then $\bar{H}_1^n \Delta S_1 = H_1^n \Delta S_1$ on Ω_2. We repeat the procedure on each Ω_2^i with the sequence $\bar{H}_1^n$ knowing that $\bar{H}_1^{ni} = 0$ for all n. Apparently, after a finite number of steps we construct the desired sequence.

Let the claim be true for $T-1$ and let $\sum_{t=1}^{T} H_t^n \Delta S_t - r^n \to \zeta$ a.s., where H_t^n are $\mathcal{F}_{t-1}$-measurable and $r^n \in L^0_+$. By the same arguments based on the elimination of non-zero components of the sequence H_1^n and using the induction hypothesis we replace H_t^n and r^n by $\tilde{H}_t^k$ and $\tilde{r}^k$ such that $\tilde{H}_1^k$ converges a.s. This means that the problem is reduced to the one with $T-1$ steps. □

4 No-arbitrage criteria in continuous time

Nowadays, in the era of electronic trading, there are no doubts that continuous-time models are much more important than their discrete-time relatives. As a theoretical tool, differential equations (eventually, stochastic) show enormous advantage with respect to difference equations. Easy to analyze, they provide very precise description of various phenomena and, quite often, allow for tractable closed-form solutions. As we mentioned already, the mathematical finance started from a continuous-time model. The unprecedented success of the Black–Scholes formula confirmed that such models are adequate tools to describe financial market phenomena. The current trend is to go beyond the Black–Scholes world. Statistical tests for financial data reject the hypothesis that prices evolve as processes with continuous sample paths. Much better approximation can be obtained by stable or other types of Lévy processes. Apparently, semimartingales provide a natural framework for discussion of general concepts of financial theory like arbitrage and hedging problems. Though more general processes are also tried, yet a very weak form of absence of arbitrage (namely, the NFLVR-property for simple integrands) in the case of a locally bounded price process implies that it is a semimartingale, see Theorem 7.2 in [12].

4.1 No Free Lunch and separating measure

In this subsection we explain relations between the No Free Lunch (NFL) condition due to Kreps, No Free Lunch with Bounded Risk (NFLBR) due to Delbaen, and No Free Lunch with Vanishing Risk (NFLVR) introduced by Delbaen and Schachermayer (see, [48], [10], [12]).

Let us assume that in a one-step model of frictionless market admissible strategies are such that the convex cone R_T^0 (the set of final portfolio values corresponding to zero initial endowment) contains only (scalar) random variables bounded from below. As usual, let $A_T^0 := R_T^0 - L^0(\mathbf{R}_+)$. Define the set $C := A_T^0 \cap L^\infty$. We denote by $\bar{C}$, $\tilde{C}^*$, and $\bar{C}^*$ the norm closure, the union of weak* closures of denumerable subsets, and the weak* closure of C in L^∞; $C_+ := C \cap L^\infty_+$ etc.

The properties NA, NFLVR, NFLBR, and NFL mean that $C_+ = \{0\}$, $\bar{C}_+ = \{0\}$, $\tilde{C}^*_+ = \{0\}$, and $\bar{C}^*_+ = \{0\}$, respectively. Consecutive inclusions induce the hierarchy of these properties:

$$\begin{array}{ccccccc} C & \subseteq & \bar{C} & \subseteq & \tilde{C}^* & \subseteq & \bar{C}^* \\ \text{NA} & \Leftarrow & \text{NFLVR} & \Leftarrow & \text{NFLBR} & \Leftarrow & \text{NFL}. \end{array}$$

Define the ESM (Equivalent Separating Measure) property as follows: *there exists* $\tilde{P} \sim P$ *such that* $\tilde{E}\xi \le 0$ *for all* $\xi \in R^0_T$.

The following criterion for the NFL-property was established by Kreps.

Theorem 4.1 *NFL* $\Leftrightarrow$ *ESM.*

Proof. ($\Leftarrow$) Let $\xi \in \bar{C}^* \cap L^\infty_+$. Since $d\tilde{P}/dP \in L^1$, there are $\xi_n \in C$ with $\tilde{E}\xi_n \to \tilde{E}\xi$. By definition, $\xi_n \le \zeta^n$ where $\zeta^n \in R^0_T$. Thus, $\tilde{E}\xi_n \le 0$ implying that $\tilde{E}\xi \le 0$ and $\xi = 0$.

($\Rightarrow$) Since $\bar{C}^* \cap L^\infty_+ = \{0\}$, the Kreps–Yan separation theorem given in the Appendix provides $\tilde{P} \sim P$ such that $\tilde{E}\xi \le 0$ for all $\xi \in C$, hence, for all $\xi \in R^0_T$. $\square$

4.2 Semimartingale model

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$ be a stochastic basis, i.e., a probability space equipped with a filtration $\mathbf{F}$ satisfying the "usual conditions". Assume for simplicity that the initial σ-algebra is trivial, the time horizon T is finite, and $\mathcal{F}_T = \mathcal{F}$.

A process $X = (X_t)_{t\in[0,T]}$ (right-continuous and with left limits) is a semimartingale if it can be represented as a sum of a local martingale and a process of bounded variation. Let $\mathcal{U}_1$ be the set of all predictable processes h taking values in the interval $[-1, 1]$. We denote by $h \cdot S$ the stochastic integral of a predictable process h with respect to a semimartingale. The definition of this integral in its full generality, especially for vector processes (necessary for financial application), is rather complicated and we send the reader to textbooks on stochastic calculus.

The linear space $\mathcal{S}$ of semimartingales starting from zero is a Frechet space with the quasinorm

$$\mathbf{D}(X) := \sup_{h\in\mathcal{U}_1} E(1 \wedge |h \cdot X_T|)$$

which induces the Émery topology, [17].

We fix in $\mathcal{S}$ a closed convex subset $\mathcal{X}^1$ of processes $X \ge -1$ which contains 0 and satisfies the following condition: for any $X, Y \in \mathcal{X}^1$ and for any non-negative bounded predictable processes H, G with $HG = 0$ the process $Z := H \cdot X + G \cdot Y$ belongs to $\mathcal{X}^1$ if $Z \ge -1$.

Put $\mathcal{X} := \operatorname{cone} \mathcal{X}^1$. The set $\mathcal{X}$ is interpreted as the set of value processes.

Put $R^0_T := \{X_T : \ X \in \mathcal{X}\}$.

In this rather general semimartingale model we have NFLVR $\Leftrightarrow$ NFLBR $\Leftrightarrow$ NFL in virtue of the following:

Theorem 4.2 *Under NFLVR* $C = \bar{C}^*$.

The proof of this theorem given in [34] follows closely the arguments of the Delbaen–Schachermayer paper [12]. Their setting is based on a n-dimensional price process S, the admissible strategies H are predictable $\mathbf{R}^n$-valued processes for which stochastic integrals $H \cdot S$ are defined and bounded from below. The set $\mathcal{X}^1$ of all value process $H \cdot S \geq -1$ is closed in virtue of the Mémin theorem on closedness in $\mathcal{S}$ of the space of stochastic integrals [50]. If S is bounded then the process $H = \xi I_{]s,t]}$ is admissible for arbitrary $\xi \in L^\infty(\mathbf{R}^n, \mathcal{F}_t)$, and hence $\tilde{E}\xi(S_t - S_s) \leq 0$ for any separating measure $\tilde{P}$. In fact, there is equality here because one can change the sign of ξ. Thus, if S is bounded then it is a martingale with respect to any separating measure $\tilde{P}$. It is an easy exercise to check that if S is locally bounded (i.e. if there exists a sequence of stopping times τ_k increasing to infinity such that the stopped processes S^{τ_k} are bounded) then S is a local martingale with respect to $\tilde{P}$. The case of arbitrary, not necessarily bounded S is of a special interest because the semimartingale model includes the classical discrete-time model as a particular case. The corresponding theorem, also due to Delbaen–Schachermayer [14], involves the notions of a σ-martingale and an equivalent σ-martingale measure.

A semimartingale S is a σ-martingale (notation: $S \in \Sigma_m$) if $G \cdot S \in \mathcal{M}_{loc}$ for some G with values in $]0,1]$. The property EσMM means that there is $Q \sim P$ such that $S \in \Sigma_m(Q)$.

Theorem 4.3 *Let $\mathcal{X}^1$ be the set of stochastic integrals $H \cdot S \geq -1$. Then*

$$NFLVR \Leftrightarrow NFLBR \Leftrightarrow NFL \Leftrightarrow ESM \Leftrightarrow E\sigma MM.$$

The remaining nontrivial implication ESM $\Rightarrow$ EσMM follows from

Theorem 4.4 *Let $\tilde{P}$ be a separating measure. Then for any $\varepsilon > 0$ there is $Q \sim \tilde{P}$ with* $\mathrm{Var}\,(\tilde{P} - Q) \leq \varepsilon$ *such that S is a σ-martingale under Q.*

A brief account of the Delbaen–Schachermayer theory including a short proof of the above theorem based on the inequality for the total variation distance from [40] is given in [33].

4.3 Hedging theorem and optional decomposition

Let us consider the semimartingale model based on an n-dimensional price process S. Let C be a scalar random variable bounded from below and let

$$\Gamma := \{x \in \mathbf{R} : \ \exists \text{ admissible } H \text{ such that } x + H \cdot S_T \geq C\}.$$

In other words, Γ is the set of initial endowments for which one can find an admissible strategy such that the terminal value of the corresponding portfolio dominates (super-replicates) the contingent claim C. "Admissible" means that the portfolio process is bounded from below by a constant.

Obviously, if non-empty, Γ is a semi-infinite interval. The following "hedging" theorem gives its characterization.

Let $\mathcal{Q}$ be the set of probability measures $Q \sim P$ with respect to which S is a local martingale.

Theorem 4.5 *Assume that $\mathcal{Q} \neq \emptyset$. Then $\Gamma = [x_*, \infty[$ where*

$$x_* = \sup_{Q \in \mathcal{Q}} E_Q C.$$

This general formulation is due to Kramkov [47] who noticed that the assertion is a simple corollary of the following two results.

Theorem 4.6 *Assume that $\mathcal{Q} \neq \emptyset$. Let X be a process bounded from below which is a supermartingale with respect to any $Q \in \mathcal{Q}$. Then there is an admissible strategy H and an increasing process A such that $X = X_0 + H \cdot S - A$.*

The process $H \cdot S$, being bounded from below, is a local martingale with respect to every $Q \in \mathcal{Q}$ (the property that an integral with respect to a local martingale is also a local martingale if it is one-side bounded is due to Émery for the scalar case and to Ansel–Stricker [1] for the vector case). Thus, this decomposition resembles that of Doob–Meyer but it holds simultaneously for the whole set $\mathcal{Q}$; in general, it is non-unique and A may not be predictable but only adapted, hence, A, being right-continuous, is optional. This explains why the above result is usually referred to as the optional decomposition theorem. It was proved in [47] for the case where S is locally bounded; this assumption was removed in the paper [18]. The proof in [18] is probabilistic and provides an interpretation of the integrand H as the Lagrange multiplier. Alternative proofs with intensive use of functional analysis can be found in [13]. For an optional decomposition with constraints see [20], an extended discussion of the problem is given [19]. In [43] it is shown that if $P \in \mathcal{Q}$ then the subset of $\mathcal{Q}$ formed by the measures with bounded densities is dense in $\mathcal{Q}$; this result implies, in particular, that, without any hypothesis, the subset of (local) martingale measures with bounded entropy is dense in $\mathcal{Q}$.

Proposition 4.7 *Assume that C is such that $\sup_{Q \in \mathcal{Q}} E_Q C < \infty$. Then there exists a process X which is a supermartingale with respect to every $Q \in \mathcal{Q}$ such that*

$$X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}} E_Q(C|\mathcal{F}_t).$$

This result is due to El Karoui and Quenez [16]; its proof also can be found in [47].

Proof of Theorem 4.5. The inclusion $\Gamma \subseteq [x_*, \infty[$ is obvious: if $x + H \cdot S_T \geq C$ then $x \geq E_Q C$ for every $Q \in \mathcal{Q}$. To show the opposite inclusion we may suppose that $\sup_{Q \in \mathcal{Q}} E_Q H < \infty$ (otherwise both sets are empty). Applying the optional decomposition theorem to the process

$$X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}} E_Q(C|\mathcal{F}_t)$$

we get that $X = x_* + H \cdot S - A$. Since $x_* + H \cdot S_T \geq X_T = C$, the result follows. $\square$

4.4 Semimartingale model with transaction costs

In this model it is assumed that the price process is a semimartingale S with non-negative components. The dynamics of the value process $V = V^{v,B}$ is given by the linear stochastic equation

$$V = v + V_- \bullet Y + B$$

where $Y^i = (1/S^i_-) \cdot S^i$,

$$B^i := \sum_{j=1}^d L^{ji} - \sum_{j=1}^d (1+\lambda^{ij}) L^{ij},$$

and L^{ij} is an increasing right-continuous process representing the accumulated net wealth "arriving" at a position i from the position j.

At this level of generality, criteria of absence of arbitrage are still not available but the paper of Jouini–Kallal [30] is an important contribution to the subject. It provides an NFL criterion for the model of stock market with a bid–ask spread where, instead of transaction costs coefficients, two process are given, $\underline{S}$ and $\overline{S}$, describing the evolution of the selling and buying prices. It is shown that a certain (specifically formulated) NFL property holds if and only if there exist a probability measure $\tilde{P} \sim P$ and a process S whose components evolve between the corresponding components of $\underline{S}$ and $\overline{S}$ such that S is a martingale with respect to $\tilde{P}$. This result is consistent with the NA criteria for finite Ω, see [41]. Apparently, the approach of Jouini and Kallal can be easily extended to the case of currency markets. However, one should take care that the setting of [30] is that of the L^2-theory. The limitations of the latter in the context of financial modeling are well-known; in contrast with engineering where energy constraints are welcome, they do not admit an economical interpretation. We attract the reader's attention to the recent paper [32] of the same authors where problems of equilibrium and viability (closely related to absence of arbitrage) are discussed; see also [31] for models with short-sell constrains.

The situation with the hedging theorem is slightly better. Its first versions in [6] (for two-asset model) and in [34] were established within the L^2-framework. In the preprint [38] an attempt was made to work with the class of strategies for which the value process is bounded from below in the sense of partial ordering induced by the solvency cone. This class of strategies corresponds precisely to the usual definition of admissibility in the case of frictionless market. However, the result was proved only for bounded price processes. To avoid difficulties one can look for other reasonable classes of admissible strategies. This approach was exploited in the paper [39] which contains the following hedging theorem.

It is assumed that the matrix Λ of transaction costs coefficients is constant, the first asset is the numéraire, and there exists a probability measure $\tilde{P}$ such that S is a (true) martingale with respect to $\tilde{P}$.

Let $\mathcal{B}_b$ be the class of strategies B such that the corresponding value processes are bounded from below by a price process multiplied by (negative) constants (this definition resembles that used by Sin in the frictionless case, [55]). In particular, it is admissible to keep short a finite number of units of assets.

Let $\mathcal{D}$ be the set of martingales Z such that $\widehat{Z}$ takes values in K^*. Notice that $\{Z : \ \widehat{Z} = w\rho, \ w \in K^*\} \subseteq \mathcal{D}$ where $\rho_t := E(d\tilde{P}/dP|\mathcal{F}_t)$. Moreover, $Z \in \mathcal{D}$ and we have $\widehat{Z}^1 = Z^1$; since the transaction costs are constant, it follows from the inequalities defining K^* that $|\widehat{Z}| \leq \kappa Z^1$ for a certain fixed constant κ. With these remarks it is easy to conclude that $\widehat{Z}V^{v,B}$ is always a supermartingale whatever $Z \in \mathcal{D}$ and $B \in \mathcal{B}_b$ are.

Define the convex set of hedging endowments

$$\Gamma = \Gamma(\mathcal{B}_b) := \{v \in \mathbf{R}^d : \ \exists B \in \mathcal{B}_b \text{ such that } \ V_T^{v,B} \geq_K C\}$$

and the closed convex set

$$D := \{v \in \mathbf{R}^d : \ \widehat{Z}_0 v \geq E\widehat{Z}_T C \ \ \forall Z \in \mathcal{D}\}.$$

Theorem 4.8 *Assume that S is a continuous process and the solvency cone K is proper. Then $\Gamma = D$.*

The "easy" inclusion $\Gamma \subseteq D$ holds in virtue of the supermartingale property of $\widehat{Z}V^{v,B}$ even without extra assumptions. The proof of the opposite inclusion given in [39] is based on a bipolar theorem in the space $L^0(\mathbf{R}^d, \mathcal{F}_T)$ equipped with a partial ordering. The hypotheses of the theorem and the structure of admissible strategies are used heavily in this proof. The assumption that K is proper, i.e. the interior of K^* is non-empty, is essential (otherwise, Γ may not be closed). However, the assertion $\bar{\Gamma} = D$ can be established for arbitrary K. How to remove or relax the assumptions on continuity of S to make the result adequate to the hedging theorem without friction remains an open problem.

Remark 4.9 It is important to note that the set of hedging endowments depends on the chosen class of admissible strategies. Let $\mathcal{B}^0$ be the class of buy-and-hold strategies with a single revision of the portfolio, namely, at time zero when the investor enters the market. It happens that in the most popular two-asset model under transaction costs with the price dynamics given by the geometric Brownian motion where the problem is to hedge a European call option (or, more generally, a contingent claim $C = g(S_T)$) we have $\Gamma(\mathcal{B}_b) = \Gamma(\mathcal{B}^0)$. This astonishing property was conjectured by Clark and Davis [9] and proved independently in [49] and [59], see also [7] and [2] for further generalizations. More precisely, in the mentioned papers it was shown that the investor having the initial endowment in money which is a minimal one to hedge the contingent claim C, can hedge it using buy-and-hold strategy from $\mathcal{B}^0$. In other words, the conclusion was that the point with zero ordinate lying on the boundary of $\Gamma(\mathcal{B}_b)$ belongs also to the boundary of a smaller set $\Gamma(\mathcal{B}^0)$. In fact, one can extend the arguments and proof that both sets coincide.

5 Large financial markets

5.1 Ross–Huberman APM

The main conclusion of the Capital Asset Pricing Model (CAPM) by Lintner and Sharp is the following:

the mean excess return on an asset is a linear function of its "beta", a measure of risk associated with this asset.

More precisely, we have the following result. Assume for simplicity that the riskless asset pays no interest. Suppose that the return on the i-th asset has mean μ_i and variance σ_i^2, the market portfolio return has mean μ_0 and variance σ_0^2. Let γ_i be the correlation coefficient between the returns on the i-th asset and the market portfolio. Then $\mu_i = \mu_0\beta_i$ where $\beta_i := \gamma_i\sigma_i/\sigma_0$.

Unfortunately, the theoretical assumptions of CAPM are difficult to justify and its empirical content is dubious. One can expect that the empirical values of (β_i, μ_i) form a cloud around the so-called security market line but this phenomenon is observed only for certain data sets. The alternative approach, the Arbitrage Pricing Model (APM) suggested by Ross in [54] and placed on a solid mathematical basis by Huberman, results in a conclusion that there exists a relation between model parameters, which can be viewed as "approximately linear", giving much better consistency with empirical data. Based on the idea of asymptotic arbitrage, it attracted considerable attention, see, e.g., [3], [4], [26], [27]; sometimes it is referred to as the Arbitrage Pricing Theory (APT). An important reference is the note by Huberman [25] who gave a rigorous definition of the asymptotic arbitrage together with a short and transparent proof of the fundamental result of Ross. The idea of Huberman is to consider a sequence of classical one-step finite-asset models instead of a single one with infinite number of securities (in the latter case an unpleasant phenomenon may arise similar to that of doubling strategies for models with infinite time horizon). When the number of assets increases to infinity, this sequence of models can be considered as a description of a large financial market.

A general specification of the n-th model M^n is as follows. We are given a stochastic basis $(\Omega^n, \mathcal{F}^n, \mathbf{F}^n, P^n)$ with a convex cone R_T^{0n} of square integrable (scalar) random variables. Assume for simplicity that the initial σ-algebra is trivial, $\mathcal{F}_T = \mathcal{F}$. Here T stands for "terminal" and can be replaced by 1. As usual, the elements of R_T^{0n} are interpreted as the terminal values of portfolios.

By definition, a sequence $\xi^n \in R_T^{0n}$ realizes an *asymptotic arbitrage opportunity* (AAO) if the following two conditions are fulfilled (E^n and D^n denote the mean and variance with respect to P^n):

(a) $\lim_n E^n\xi^n = \infty$;

(b) $\lim_n D^n\xi^n = \lim_n E^n(\xi^n - E^n\xi^n)^2 = 0$.

Roughly speaking, if AAO exists, then, working with large portfolios, the investor can become infinitely rich (in the mean sense) with vanishing quadratic risk.

We say that the large financial market has *NAA property* if there are no asymptotic arbitrage opportunities for any subsequence of market models $\{M^{n'}\}$.

A simple but useful remark: the NAA property remains the same if we replace (a) in the definition of AAO by the weaker property $\limsup_n E^n \xi^n > 0$ ("if one can become rich, one can become infinitely rich").

Let ρ_n be the L^2-distance of R_T^{0n} from the unit, i.e.

$$\rho_n := \inf_{\xi \in R_T^{0n}} E^n (\xi - 1)^2,$$

Proposition 5.1 NAA $\Leftrightarrow \liminf_n \rho_n > 0$.

Proof. ($\Rightarrow$) Assume that $\liminf_n \rho_n = 0$. This means (modulo passage to a subsequence) that there are $\xi^n \in R_T^{0n}$ such that $E^n(\xi^n - 1)^2 \to 0$. It follows from the identity

$$E^n(\xi^n - 1)^2 = D^n \xi^n + (E^n \xi^n - 1)^2$$

that $D^n \xi^n \to 0$ and $E^n \xi^n \to 1$, violating NAA.

($\Leftarrow$) Assume that NAA fails. This means (modulo passage to a subsequence) that there are $\xi^n \in R_T^{0n}$, $\xi_n \neq 0$, satisfying (a) and (b). It follows that

$$E^n(\xi^n)^2 = D^n \xi^n + (E^n \xi^n)^2 \to \infty.$$

Put $\tilde{\xi}^n := \xi^n / \sqrt{E^n(\xi^n)^2}$. Then $\tilde{\xi}^n \in R_T^{0n}$,

$$D^n \tilde{\xi}^n = (1/E^n(\xi^n)^2) D^n \xi^n \to 0$$

and

$$(E^n \tilde{\xi}^n)^2 = E^n(\tilde{\xi}^n)^2 - D^n \tilde{\xi}^n = 1 - D^n \tilde{\xi}^n \to 1.$$

Thus,

$$E^n(\tilde{\xi}^n - 1)^2 = D^n \tilde{\xi}^n + (E^n \tilde{\xi}^n - 1)^2 \to 0$$

and we get a contradiction. $\square$

Suppose now that in the n-th model we are given a d-dimensional square integrable price process (S_t^n) where $t \in \{0, T\}$. In general, $d = d(n)$. Suppose that $S_0^{in} = 1$ (this is just a choice of scales).

The crucial hypothesis of the k-factor APM is that there are k common sources of randomness affecting the prices of all securities and there are also individual sources of randomness related to each security. Specifically, we suppose that

$$\Delta S_T^{in} = \mu^{in} + \sum_{j=1}^{k} \zeta_j^n b_j^{in} + \eta^{in}, \qquad i \leq d,$$

or, in vector notation,

$$\Delta S_T^n = \mu^n + \sum_{j=1}^{k} \zeta_j^n b_j^n + \eta^n.$$

Here μ^n, $b_j^n \in \mathbf{R}^d$, the scalar random variables ζ_j^n with zero means are square integrable and the d-dimensional random vector η^n with zero mean has uncorrelated components (representing randomness proper to each asset).

Assume that $D\eta^{in} \leq C$ for all $i \leq d$ and $n \in \mathbf{N}$ for a certain constant C.

A (self-financing) portfolio strategy H^n is a vector in $\mathbf{R}^d$ such that

$$H^n \mathbf{1}_d := \sum_{i=1}^d H^{in} = 0.$$

At the final date the corresponding portfolio value is

$$V_T^n = H^n \Delta S_T^n = \sum_{i=1}^d H^{i,n} \Delta S_T^{in}$$

and these random variables form the set R_T^{0n}.

Lemma 5.2 *Let $\mathcal{L}_n$ be the linear subspace in $\mathbf{R}^d$ spanned by the set $\{\mathbf{1}_d,\ b_j^n,\ j \leq k\}$ and let c^n be the projection of μ^n onto $\mathcal{L}_n^\perp$. Then*

$$\text{NAA} \quad \Rightarrow \quad \sup_n |c^n| < \infty.$$

Proof. Let a_n be a real number. The vector $H^n := a_n c^n$ (being orthogonal to $\mathbf{1}_d$) is a self-financing strategy with the corresponding terminal value

$$V_T^n = a_n |c^n|^2 + a_n c^n \eta^n.$$

It follows that

$$E^n V_T^n = a_n |c^n|^2,$$

$$D^n V_T^n = a_n^2 E(c^n \eta^n)^2 = a_n^2 \sum_{i=1}^d (c^{in})^2 D^n \eta^{in} \leq C a_n^2 |c^n|^2.$$

In particular, for $a_n = |c^n|^{-3/2}$ we have an asymptotic arbitrage opportunity for any subsequence along which $|c^n|$ converges to infinity. $\square$

As is easily seen from the proof, the conditions of the lemma are equivalent if $D^n \eta^{in} \geq \varepsilon > 0$ for all i and n.

Proposition 5.3 *Assume that NAA holds. Then there exist a constant A and real-valued sequences $\{r^n\}$, $\{g_j^n\}$, $j \leq k$, such that*

$$\Big|\mu^n - r^n \mathbf{1}_d - \sum_{j=1}^k g_j^n b_j^n\Big|^2 := \sum_{i=1}^d \Big(\mu^{in} - r^n - \sum_{j=1}^k g_j^n b_j^{in}\Big)^2 \leq A.$$

The assertion is an obvious corollary of the above lemma: the vector c^n is a difference of μ^n and the projection of μ^n onto $\mathcal{L}_n$; the latter is a linear combination of the generating vectors $\mathbf{1}_d$, b_1^n,..., b_k^n. Of course, if the generators are not linearly independent, the coefficients r^n, g_1^n,...,g_k^n are not uniquely defined.

The most interesting case of the APM is the "stationary" one where all random variables "live" on the same probability space and do not depend on n. All model parameters also do not depend on n except the dimension $d=n$. In other words, we are given infinite-dimensional vectors $\mu=(\mu^1,\mu^2,...)$, $\eta=(\eta^1,\eta^2,...)$, etc., and the ingredients of the n-th model, μ^n, η^n, etc., are composed of the first n coordinates of these vectors. One can think that the "real-world" market has an infinite number of securities, enumerated somehow, and the agent uses the first n of them in his portfolios. That is, the increment of the n-dimensional price process in the n-th model is

$$\Delta S_T^i = \mu^i + \sum_{j=1}^k \zeta_j b_j^i + \eta^i, \qquad i \le n.$$

Theorem 5.4 *Assume that NAA holds. Then there are constants r and g_j, $j \le k$, such that*

$$\sum_{i=1}^{\infty} \Big(\mu^i - r - \sum_{j=1}^k g_j b_j^i\Big)^2 < \infty.$$

Proof. Let us consider the vector space spanned by the infinite-dimensional vectors $\mathbf{1}_\infty=(1,1,...)$, $b_j=(b_j^1,b_j^2,...)$, $j\le k$. Without loss of generality we may assume that $\mathbf{1}_\infty$, b_j, $j\le l$, is a basis in this space. There is n_0 such that for every $n\ge n_0$ the vectors formed by the first n components of the latter are linearly independent. For every $n\ge n_0$ we define the set

$$K^n := \Big\{(r,g_1,...,g_l,0,...,0)\in\mathbf{R}^{k+1}:\ \sum_{i=1}^n \Big(\mu^i - r - \sum_{j=1}^k g_j b_j^i\Big)^2 \le A\Big\}$$

where choosing A as in Proposition 5.3 ensures that K^n is non-empty. Clearly, K^n is closed and $K^{n+1}\subseteq K^n$. It is easily seen that K^n is bounded (otherwise we could construct a linear relation between the vectors assumed to be linearly independent). Thus, the sets K^n are compact, $\cap_{n\ge n_0} K^n \ne \emptyset$, and the result follows. □

In the case where the numéraire is a traded security, say, the first one (i.e., $\Delta S_T^{1n}=0$) we can take $r^n=0$ for all n in Proposition 5.3 and $r=0$ in Theorem 5.4. To see this, we repeat the arguments above with "truncated" price vectors and strategies, the first component being excluded. In this specification an admissible strategy is just a vector from $\mathbf{R}^{d-1}$ and the projection onto the vector with unit coordinates is not needed.

To make the relation between CAPM and APM clear, let us consider the one-factor stationary model where the numéraire is a traded security and the increments of the risky asset (enumerating from zero) are of the following structure:

$$\begin{aligned}\Delta S_T^0 &= \mu_0 + b_0\zeta, \\ \Delta S_T^i &= \mu_i + b_i\zeta + \eta_i, \quad i \geq 1.\end{aligned}$$

where all random variables ζ and η_i are uncorrelated and have zero means. Assume that $D\eta_i \leq C$. The 0-th asset plays a particular role: all other price movements are conditionally uncorrelated given ΔS_T^0. It can be viewed as a kind of "market portfolio" or "market index".

If there is no asymptotic arbitrage, then there exists a constant g such that

$$\sum_{i=0}^{\infty}(\mu_i - gb_i)^2 < \infty$$

i.e. $\mu_i = gb_i + u_i$ where $u_i \to 0$. If the residual u_0 is small, then $\mu_0 \approx gb_0$. We can use the latter relation to specify g and conclude that $\mu_i \approx \mu_0\beta_i$ (at least, for sufficiently large i) with $\beta_i := b_i/b_0$. Of course, this reasoning is far from being rigorous: the empirical data, even being in accordance with APM, may or may not follow the conclusion of CAPM.

Note that the approach of APT is based on the assumption that the agents have certain risk-preferences and in the asymptotic setting they may accept the possibility of large losses with small probabilities; the variance is taken as an appropriate measure of risk.

A specific feature of the classical APT is that it does not deal with the problem of existence of equivalent martingale measures which is the key point of the Fundamental Theorem of Asset Pricing. For a long time these two arbitrage theories were considered as unrelated. In [35] an approach was suggested which puts together basic ideas of both of them and allows us to solve the long-standing problem of extension of APT to the continuous-time setting. A brief account of its further development is given in the next subsections.

5.2 Asymptotic arbitrage and contiguity

The theory of large financial markets contains four principal ingredients: basic concepts, functional-analytic methods, probabilistic results, and analysis of specific models. The fundamentals of this theory were established in [35] where the definitions of asymptotic arbitrage of the first and the second kind were suggested. Assuming the uniqueness of equivalent martingale measures (i.e. the completeness) for each market model, the authors proved necessary and sufficient conditions for NAA1 and NAA2 in terms of contiguity of sequences of equivalent martingale measures and objective ("historical") probabilities. A particular model of a "large Black–Scholes market" (where the price processes are correlated geometric Brownian motions) was investigated. It was shown that the boundedness condition similar to that of Ross–Huberman can be obtained as a direct application of the Liptser–Shiryaev criteria of contiguity in terms of the Hellinger processes. The restricting uniqueness hypothesis was removed by Klein and Schachermayer (see [45], [46], and [44]). They discovered the importance of duality methods of geometric functional analysis in the context of large financial markets and found non-trivial extensions of NAA1 and NAA2 criteria for the case of incomplete market models. These criteria were complemented in [37] by new ones. In particular, it was shown that the strong asymptotic arbitrage is equivalent to the complete asymptotic separability of the historic probabilities and equivalent martingale measures. Our presentation follows the latter paper where also several modifications of classical models were analyzed and necessary and sufficient conditions for absence of asymptotic arbitrage were obtained in terms of model specifications.

In the terminology of [37], a *large financial market* is a sequence of ordinary semimartingale models of a frictionless market $\{(\mathbf{B}^n, S^n, T^n)\}$, where $\mathbf{B}^n$ is a stochastic basis with the trivial initial σ-algebra. A semimartingale price process S^n takes values in $\mathbf{R}^d$ for some $d = d(n)$. To simplify notation we shall often omit the superscript for the time horizon.

We denote by $\mathcal{Q}^n$ the set of all probability measures Q^n equivalent to P^n such that S^n is a local martingale with respect to Q^n. It is assumed that each set $\mathcal{Q}^n$ of equivalent local martingale measures is non-empty.

We define a trading strategy on $(\mathbf{B}^n, S^n, T^n)$ as a predictable process H^n with values in $\mathbf{R}^d$ such that the stochastic integral with respect to the semimartingale S^n $H^n \cdot S^n$ is well-defined on $[0, T]$.

For a trading strategy H^n and an initial endowment x^n the value process

$$V^n = V(n, x^n, H^n) := x^n + H^n \cdot S^n.$$

A sequence V^n *realizes asymptotic arbitrage of the first kind* (AA1) if

(1a) $V_t^n \geq 0$ for all $t \leq T$;

(1b) $\lim_n V_0^n = 0$ (i.e. $\lim_n x^n = 0$);

(1c) $\lim_n P^n(V_T^n \geq 1) > 0$.

A sequence V^n *realizes asymptotic arbitrage of the second kind* (AA2) if

(2a) $V_t^n \leq 1$ for all $t \leq T$;

(2b) $\lim_n V_0^n > 0$;

(2c) $\lim_n P^n(V_T^n \geq \varepsilon) = 0$ for any $\varepsilon > 0$.

A sequence V^n *realizes strong asymptotic arbitrage of the first kind* (SAA1) if

(3a) $V_t^n \geq 0$ for all $t \leq T$;

(3b) $\lim_n V_0^n = 0$ (i.e. $\lim_n x^n = 0$);

(3c) $\lim_n P^n(V_T^n \geq 1) = 1$.

One can continue and give also the definition SAA2. It is easy to understand that the existence of SAA1 implies the existence of SAA2 and vice versa (provided that there are no specific constraints). So existence criteria are the same in both cases.

A large security market $\{(\mathbf{B}^n, S^n, T^n)\}$ has *no asymptotic arbitrage of the first kind* (respectively, of the *second kind*) if for any subsequence (m) there are no value processes V^m realizing asymptotic arbitrage of the first kind (respectively, of the second kind) for $\{(\mathbf{B}^m, S^m, T^m)\}$.

To formulate the results we need to extend some notions from measure theory.

Let $\mathcal{Q} = \{Q\}$ be a family of probabilities on a measurable space $(\Omega, \mathcal{F})$. Define the upper and lower envelopes of measures from $\mathcal{Q}$ as the set functions with

$$\overline{\mathbf{Q}}(A) := \sup_{Q \in \mathcal{Q}} Q(A), \quad \underline{\mathbf{Q}}(A) := \inf_{Q \in \mathcal{Q}} Q(A), \qquad A \in \mathcal{F}.$$

We say that $\mathcal{Q}$ is *dominated* if any element of $\mathcal{Q}$ is absolutely continuous with respect to some fixed probability measure.

In our setting, where for every n a family $\mathcal{Q}^n$ of equivalent local martingale measures is given, we use the obvious notations $\overline{\mathbf{Q}}^n$ and $\underline{\mathbf{Q}}^n$.

Generalizing in a straightforward way the well-known notion of contiguity to set functions other than measures, we introduce the following definitions:

The sequence (P^n) is *contiguous* with respect to $(\overline{\mathbf{Q}}^n)$ (notation: $(P^n) \lhd (\overline{\mathbf{Q}}^n)$) when the implication

$$\lim_{n\to\infty} \overline{\mathbf{Q}}^n(A^n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} P^n(A^n) = 0$$

holds for any sequence $A^n \in \mathcal{F}^n$, $n \geq 1$.

Obviously, $(P^n) \lhd (\overline{\mathbf{Q}}^n)$ if and only if the implication

$$\lim_{n\to\infty} \sup_{Q \in \mathcal{Q}^n} E_Q g^n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} E_{P^n} g^n = 0$$

holds for any uniformly bounded sequence g^n of positive $\mathcal{F}^n$-measurable random variables.

A sequence (P^n) is *asymptotically separable* from $(\overline{\mathbf{Q}}^n)$ (notation: $(P^n) \triangle (\overline{\mathbf{Q}}^n)$) if there exists a subsequence (m) with sets $A^m \in \mathcal{F}^m$ such that

$$\lim_{m\to\infty} \overline{\mathbf{Q}}^m(A^m) = 0, \quad \lim_{m\to\infty} P^m(A^m) = 1.$$

Proposition 5.5 *The following conditions are equivalent:*

(a) there is no asymptotic arbitrage of the first kind (NAA1);

(b) $(P^n) \lhd (\overline{\mathbf{Q}}^n)$*;*

(c) there exists a sequence $R^n \in \mathcal{Q}^n$ *such that* $(P^n) \lhd (R^n)$.

Proof. $(b) \Rightarrow (a)$ Let (V^n) be a sequence of value processes realizing asymptotic arbitrage of the first kind. For any $Q \in \mathcal{Q}^n$ the process V^n is a non-negative local Q-martingale, hence a Q-supermartingale, and

$$\sup_{Q \in \mathcal{Q}^n} E_Q V_T^n \leq \sup_{Q \in \mathcal{Q}^n} E_Q V_0^n = x^n \to 0$$

by (1b). Thus,

$$\overline{\mathbf{Q}}^n(V_T^n \geq 1) := \sup_{Q \in \mathcal{Q}^n} Q(V_T^n \geq 1) \to 0$$

and, by contiguity $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$, we have $P^n(V_T^n \geq 1) \to 0$ in contradiction to (1c).

$(a) \Rightarrow (b)$ Assume that (P^n) is not contiguous with respect to $(\overline{\mathbf{Q}}^n)$. Taking, if necessary, a subsequence we can find sets $\Gamma^n \in \mathcal{F}^n$ such that $\overline{\mathbf{Q}}^n(\Gamma^n) \to 0$, $P^n(\Gamma^n) \to \gamma$ as $n \to \infty$ where $\gamma > 0$. According to Proposition 4.7 the process

$$X_t^n = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^n} E_Q(I_{\Gamma^n} | \mathcal{F}_t^n)$$

is a supermartingale with respect to any $Q \in \mathcal{Q}^n$. By Theorem 4.6 it admits a decomposition $X^n = X_0^n + H^n \cdot S^n - A^n$ where A^n is an increasing process. Let us show that $V^n := X_0^n + H^n \cdot S^n$ are value processes realizing AA1. Indeed, $V^n = X^n + A^n \geq 0$,

$$V_0^n = \sup_{Q \in \mathcal{Q}^n} E_Q I_{\Gamma^n} = \overline{\mathbf{Q}}^n(\Gamma^n) \to 0,$$

and

$$\lim_n P^n(V_T^n \geq 1) \geq \lim_n P^n(X_T^n \geq 1) = \lim_n P^n(X_T^n = 1) = \lim_n P^n(\Gamma^n) = \gamma > 0.$$

$(b) \Leftrightarrow (c)$ This relation follows from the convexity of $\mathcal{Q}^n$ and a general result given below. $\square$

Proposition 5.6 *Assume that for any $n \geq 1$ we are given a probability space $(\Omega^n, \mathcal{F}^n, P^n)$ with a dominated family $\mathcal{Q}^n$ of probability measures. Then the following conditions are equivalent:*

(a) $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$;

(b) there is a sequence $R^n \in \operatorname{conv} \mathcal{Q}^n$ such that $(P^n) \triangleleft (R^n)$;

(c) the following equality holds:

$$\lim_{\alpha \downarrow 0} \liminf_{n \to \infty} \sup_{Q \in \operatorname{conv} \mathcal{Q}^n} H(\alpha, Q, P^n) = 1,$$

where $H(\alpha, Q, P) = \int (dQ)^\alpha (dP)^{1-\alpha}$ is the Hellinger integral of order $\alpha \in]0,1[$.

The sequence of sets of probability measures $(\mathcal{Q}^n)$ is said to be *weakly contiguous with respect to* (P^n) (notation: $(\mathcal{Q}^n) \triangleleft_w (P^n)$) if for any $\varepsilon > 0$ there are $\delta > 0$ and a sequence of measures $Q^n \in \mathcal{Q}^n$ such that for any sequence $A^n \in \mathcal{F}^n$ with the property $\limsup_n P^n(A^n) < \delta$ we have $\limsup_n Q^n(A^n) < \varepsilon$.

For the case where the sets $\mathcal{Q}^n$ are singletons containing only the measure Q^n, the relation $(\mathcal{Q}^n) \triangleleft_w (P^n)$ means simply that $(Q^n) \triangleleft (P^n)$.

Obviously, the property $(\mathcal{Q}^n) \triangleleft_w (P^n)$ can be formulated in terms of random variables:

for any $\varepsilon > 0$ there are $\delta > 0$ and a sequence of measures $Q^n \in \mathcal{Q}^n$ such that for any sequence of $\mathcal{F}^n$-measurable random variables g^n taking values in the interval $[0,1]$ *with the property* $\limsup_n E_{P^n} g^n < \delta$, *we have* $\limsup_n E_{Q^n} g^n < \varepsilon$.

Proposition 5.7 *The following conditions are equivalent:*

(a) there is no asymptotic arbitrage of the second kind (NAA2);

(b) $(\underline{\mathbf{Q}}^n) \lhd (P^n)$;

(c) $(\mathcal{Q}^n) \lhd_w (P^n)$.

The proof of Proposition 5.7 is similar to that of Proposition 5.5. Notice that the conditions (b) in both statements look rather symmetric in contrast to the conditions (c). In general, the condition (b) of Proposition 5.7 may hold though a sequence $Q^n \in \mathcal{Q}^n$ such that $(Q^n) \lhd (P^n)$ does not exist (see an example in [45]). The reason is that the set functions $\overline{\mathbf{Q}}$ and $\underline{\mathbf{Q}}$ are of a radically different nature.

The following assertion gives criteria of existence of strong asymptotic arbitrage.

Proposition 5.8 *The following conditions are equivalent:*

(a) there is SAA1;

(b) $(P^n) \triangle (\overline{\mathbf{Q}}^n)$;

(c) $(\underline{\mathbf{Q}}^n) \triangle (P^n)$;

(d) $(P^n) \triangle (Q^n)$ *for any sequence* $Q^n \in \mathcal{Q}^n$.

Let P and $\tilde{P}$ be two equivalent probability measures on a stochastic basis $\mathbf{B}$ and let $R := (P + \tilde{P})/2$. Let us denote by z and $\tilde{z}$ the density processes of P and $\tilde{P}$ with respect to R. For arbitrary $\alpha \in]0,1[$ the process $Y = Y(\alpha) := z^\alpha \tilde{z}^{1-\alpha}$ is an R-supermartingale admitting the multiplicative decomposition $Y = M\mathcal{E}(-h)$ where $M = M(\alpha)$ is a local Q-martingale, $\mathcal{E}$ is the Doléan-Dade exponential, and $h = h(\alpha, P, \tilde{P})$ is an increasing predictable process, $h_0 = 0$, called the Hellinger process of order α. These Hellinger processes play an important role in criteria of absolute continuity and, more generally, contiguity of probability measures, see [28] for details.

In the abstract setting of Proposition 5.6 when the probability spaces are equipped with filtrations (i.e. they are stochastic bases) we have the following results which are helpful in analysis of particular models arising in mathematical finance.

Theorem 5.9 *The following conditions are equivalent:*

(a) $(P^n) \lhd (\overline{\mathbf{Q}}^n)$;

(b) for all $\varepsilon > 0$

$$\lim_{\alpha \downarrow 0} \limsup_{n\to\infty} \inf_{Q \in \operatorname{conv} \mathcal{Q}^n} P^n(h_\infty(\alpha, Q, P^n) \geq \varepsilon) = 0.$$

Theorem 5.10 *Assume that the family* $\mathcal{Q}^n$ *is convex and dominated for any* n. *Then the following conditions are equivalent:*

(a) $(\underline{\mathbf{Q}}^n) \lhd (P^n)$;

(b) for all $\varepsilon > 0$

$$\lim_{\alpha \downarrow 0} \limsup_{n\to\infty} \inf_{Q \in \mathcal{Q}^n} Q(h_\infty(\alpha, P^n, Q) \geq \varepsilon) = 0.$$

The concept of contiguity is useful in relation with an important question whether the option prices calculated in "approximating" models converge to the "true" option price, see [24] and [58].

5.3 A large BS-market

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$ be a stochastic basis with a countable set of independent one-dimensional Wiener processes w^i, $i \in \mathbf{Z}_+$, $\mathbf{w}^n = (w^0, \dots, w^n)$, and let $\mathbf{F}^n = (\mathcal{F}^n_t)$ be a filtration generated by $\mathbf{w}^n$. For simplicity, assume that T is fixed.

The behavior of the stock prices is described by the following stochastic differential equations:

$$dX^0_t = \mu_0 X^0_t dt + \sigma_0 X^0_t dw^0_t,$$
$$dX^i_t = \mu_i X^i_t dt + \sigma_i X^i_t (\gamma_i dw^0_t + \bar{\gamma}_i dw^i_t), \quad i \in \mathbf{N},$$

with (deterministic strictly positive) initial points X^i_0. Here γ_i is a function taking values in $[0,1[$ and $\gamma_i^2 + \bar{\gamma}_i^2 = 1$, We assume that $\mu_i, \sigma_i \in L^2[0,T]$ and $\sigma_i > 0$.

Notice that the process ξ^i with

$$d\xi^i_t = \gamma_i dw^0_t + \bar{\gamma}_i dw^i_t, \quad \xi^i_0 = 0,$$

is a Wiener process. Thus, in the case of constant coefficients price processes are geometric Brownian motions as in the classical case of Black and Scholes. The model is designed to reflect the fact that in the market there are two different types of randomness: the first type is proper to each stock while the second one originates from some common source and it is accumulated in a "stock index" (or "market portfolio") whose evolution is described by the first equation. Set

$$\beta_i := \frac{\gamma_i \sigma_i}{\sigma_0} = \frac{\gamma_i \sigma_i \sigma_0}{\sigma_0^2}.$$

In the case of deterministic coefficients, β_i is a well-known measure of risk which is the covariance between the return on the asset with number i and the return on the index, divided by the variance of the return on the index.

Let $\mathbf{b}_n(t) := (b_0(t), b_1(t), ..., b_n(t))$ where

$$b_0 := -\frac{\mu_0}{\sigma_0}, \quad b_i := \frac{\beta_i \mu_0 - \mu_i}{\sigma_i \bar{\gamma}_i}.$$

Assume that for every n

$$\int_0^T |\mathbf{b}_n(t)|^2 dt < \infty.$$

We consider the stochastic basis $\mathbf{B}^n = (\Omega, \mathcal{F}, \mathbf{F}^n = (\mathcal{F}^n_t)_{t \le T}, P^n)$ with the $(n+1)$-dimensional semimartingale $S^n := (X^0_t, X^1_t, \dots, X^n_t)$ and $P^n := P|\mathcal{F}^n_T$. The sequence $\{(\mathbf{B}^n, S^n, T)\}$ is a large security market. In our case each $(\mathbf{B}^n, S^n, T)$ is a model of a complete market and the set $\mathcal{Q}^n$ is a singleton which consists of the measure $Q^n = Z_T(\mathbf{b}_n)P^n$ where

$$Z_T(\mathbf{b}_n) := \exp\left\{\int_0^T (\mathbf{b}_n(t), d\mathbf{w}_t^n) - \frac{1}{2}\int_0^T |\mathbf{b}_n(t)|^2 dt\right\}.$$

The Hellinger process has an explicit expression

$$h(\alpha, Q^n, P^n) = \frac{\alpha(1-\alpha)}{2}\int_0^T \left[\left(\frac{\mu_0}{\sigma_0}\right)^2 + \sum_{i=1}^n \left(\frac{\mu_i - \beta_i\mu_0}{\sigma_i\bar{\gamma}_i}\right)^2\right] ds.$$

As a corollary of Theorem 5.9 we have

Proposition 5.11 *The condition NAA1 holds if and only if*

$$\int_0^T \left[\left(\frac{\mu_0}{\sigma_0}\right)^2 + \sum_{i=1}^\infty \left(\frac{\mu_i - \beta_i\mu_0}{\sigma_i\bar{\gamma}_i}\right)^2\right] ds < \infty.$$

In fact, in this model both conditions NAA1 and NAA2 hold simultaneously.

In the particular case of constant coefficients, finite T, and $0 < c \le \sigma_i\bar{\gamma}_i \le C$ we get that the property NAA1 holds if and only if

$$\sum_{i=1}^\infty (\mu_i - \beta_i\mu_0)^2 < \infty,$$

i.e. the Huberman–Ross boundedness is fulfilled.

5.4 One-factor APM revisited

We consider the "stationary" one-factor model of the following specific structure (cf. with the model given at the end of Subsection 5.1). Let $(\epsilon_i)_{i\ge 0}$ be independent random variables given on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a finite interval $[-N, N]$, $E\epsilon_i = 0$, $E\epsilon_i^2 = 1$. At time zero all asset prices $S_0^i = 1$ and

$$\begin{aligned}\Delta S_T^0 &= 1 + \mu_0 + \sigma_0\epsilon_0,\\ \Delta S_T^i &= 1 + \mu_i + \sigma_i(\gamma_i\epsilon_0 + \bar{\gamma}_i\epsilon_i), \quad i \ge 1.\end{aligned}$$

The coefficients here are deterministic, $\sigma_i > 0$, $\bar{\gamma}_i > 0$ and $\gamma_i^2 + \bar{\gamma}_i^2 = 1$. The asset with number zero is interpreted as a market portfolio, γ_i is the correlation coefficient between the rate of return for the market portfolio and the rate of return for the asset with number i.

For $n \ge 0$ we consider the stochastic basis $\mathbf{B}^n = (\Omega, \mathcal{F}^n, \mathbf{F}^n = (\mathcal{F}_t^n)_{t\in\{0,1\}}, P^n)$ with the $(n+1)$-dimensional random process $\mathbf{S}^n := (S_t^0, S_t^1, \ldots, S_t^n)_{t\in\{0,1\}}$ where $\mathcal{F}_0^n$ is the trivial σ-algebra, $\mathcal{F}_1^n = \mathcal{F}^n := \sigma\{\epsilon_0, ..., \epsilon_n\}$, and $P^n = P|\mathcal{F}^n$. According to our definition, the sequence $\mathbf{M} = \{(\mathbf{B}^n, \mathbf{S}^n, 1)\}$ is a large security market.

Let $\beta_i := \gamma_i \sigma_i / \sigma_0$,

$$b_0 := -\frac{\mu_0}{\sigma_0}, \qquad b_i := \frac{\mu_0 \beta_i - \mu_i}{\sigma_i \bar{\gamma}_i}, \quad i \geq 1.$$

It is convenient to rewrite the price increments as follows:

$$\begin{aligned}
\Delta S_T^0 &= 1 + \sigma_0(\epsilon_0 - b_0), \\
\Delta S_T^i &= 1 + \sigma_i \gamma_i (\epsilon_0 - b_0) + \sigma_i \bar{\gamma}_i (\epsilon_i - b_i)), \quad i \geq 1.
\end{aligned}$$

The set $\mathcal{Q}^n$ of equivalent martingale measures for $\mathbf{S}^n$ has a very simple description: $Q \in \mathcal{Q}^n$ iff $Q \sim P^n$ and

$$E_Q(\epsilon_i - b_i) = 0, \quad 0 \leq i \leq n,$$

i.e. the b_i are mean values of ϵ_i under Q. Obviously, $\mathcal{Q}^n \neq \emptyset$ iff $P(\epsilon_i > b_i) > 0$ and $P(\epsilon_i < b_i) > 0$ for all $i \leq n$.

As usual, we assume that $\mathcal{Q}^n \neq \emptyset$ for all n; this implies, in particular, that $|b_i| < N$.

Let F_i be the distribution function of ϵ_i. Put

$$\underline{s}_i := \inf\{t : \ F_i(t) > 0\}, \qquad \bar{s}_i := \inf\{t : \ F_i(t) = 1\},$$

$\underline{d}_i := b_i - \underline{s}_i$, $\bar{d}_i := \bar{s}_i - b_i$, and $d_i := \underline{d}_i \wedge \bar{d}_i$. In other words, d_i is the distance from b_i to the end points of the interval $[\underline{s}_i, \bar{s}_i]$.

Proposition 5.12 *The following assertions hold:*

(a) $\inf_i d_i = 0 \ \Leftrightarrow \ SAA \ \Leftrightarrow \ (P^n) \triangle (\overline{\mathbf{Q}}^n)$,

(b) $\inf_i d_i > 0 \ \Leftrightarrow \ NAA1 \ \Leftrightarrow \ (P^n) \triangleleft (\overline{\mathbf{Q}}^n)$,

(c) $\limsup_i |b_i| = 0 \ \Leftrightarrow \ NAA2 \ \Leftrightarrow \ (\underline{\mathbf{Q}}^n) \triangleleft (P^n)$.

The hypothesis that the distributions of ϵ_i have finite support is important: it excludes the case where the value of every non-trivial portfolio is negative with positive probability. For the proof of this result, we send the reader to the original paper [37].

A Facts from convex analysis

1. By definition, a subset K in $\mathbf{R}^n$ (or in a linear space X) is a *cone* if it is convex and stable under multiplication by the non-negative constants. It defines the partial ordering:

$$x \geq_K y \quad \Leftrightarrow \quad x - y \in K;$$

in particular, $x \geq_K 0$ means that $x \in K$.

A closed cone K is *proper* if the linear space $F := K \cap (-K) = \{0\}$, i.e. if the relations $x \geq_K$ and $x \leq_K = 0$ imply that $x = 0$.

Let K be a closed cone and let $\pi : \mathbf{R}^n \to \mathbf{R}^n/F$ be the canonical mapping onto the quotient space. Then πK is a proper closed cone.

For a set C we denote by $\operatorname{cone} C$ the set of all conic combinations of elements of C. If C is convex then $\operatorname{cone} C = \cup_{\lambda \geq 0} \lambda C$.

Let K be a cone. Its *dual positive cone*

$$K^* := \{z \in \mathbf{R}^n : \ zx \geq 0 \ \forall x \in K\}$$

is closed (the *dual cone* K° is defined using the opposite inequality, i.e. $K^\circ = -K^*$); K is closed if and only if $K = K^{**}$.

We use the notations $\operatorname{int} K$ for the interior of K and $\operatorname{ri} K$ for the relative interior (i.e., the interior in $K - K$, the linear subspace generated by K).

A closed cone K in the Euclidean space $\mathbf{R}^n$ is proper if and only if there exists a compact convex set C such that $0 \notin C$ and $K = \operatorname{cone} C$. One can take as C the convex hull of the intersection of K with the unit sphere $\{x \in \mathbf{R}^n : \ |x| = 1\}$.

A closed cone K is proper if and only if $\operatorname{int} K^* \neq \emptyset$.

We have

$$\operatorname{ri} K^* = \{w : \ wx > 0 \ \forall x \in K, \ x \neq F\};$$

in particular, if K is proper then

$$\operatorname{int} K^* = \{w : \ wx > 0 \ \forall x \in K, \ x \neq 0\}.$$

By definition, the cone K is *polyhedral* if it is the intersection of a finite number of half-spaces $\{x : \ p_i x \geq 0\}$, $p_i \in \mathbf{R}^n$, $i = 1, ..., N$.

The Farkas–Minkowski–Weyl theorem:

a cone is polyhedral if and only if it is finitely generated.

The following result is a direct generalization of the Stiemke lemma.

Lemma A.1 *Let K and R be closed cones in $\mathbf{R}^n$. Assume that K is proper. Then*

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

Proof. ($\Leftarrow$) The existence of w such that $wx \leq 0$ for all $x \in R$ and $wy > 0$ for all y in $K \setminus \{0\}$ obviously implies that R and $K \setminus \{0\}$ are disjoint.

($\Rightarrow$) Let C be a convex compact set such that $0 \notin C$ and $K = \operatorname{cone} C$. By the separation theorem (for the case where one set is closed and another is compact) there is a non-zero $z \in \mathbf{R}^n$ such that

$$\sup_{x \in R} zx < \inf_{y \in C} zy.$$

Since R is a cone, the left-hand side of this inequality is zero, hence $z \in -R^*$ and, also, $zy > 0$ for all $y \in C$. The latter property implies that $zy > 0$ for $z \in K$, $z \neq 0$, and we have $z \in \operatorname{int} K$. $\square$

In the classical Stiemke lemma $K = \mathbf{R}^n$ and $R = \{y \in \mathbf{R}^n : \ y = Bx, \ x \in \mathbf{R}^d\}$ where B is a linear mapping. Usually, it is formulated as the alternative:

either there is $x \in \mathbf{R}^d$ such that $Bx \geq_K 0$ and $Bx \neq 0$ or there is $y \in \mathbf{R}^n$ with strictly positive components such that $B^*y = 0$.

Lemma A.1 can be slightly generalized.

Let π be the natural projection of $\mathbf{R}^n$ onto $\mathbf{R}^n/F$.

Theorem A.2 *Let K and R be closed cones in $\mathbf{R}^n$. Assume that the cone πR is closed. Then*

$$R \cap K \subseteq F \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{ri} K^* \neq \emptyset.$$

Proof. It is easy to see that $\pi(R \cap K) = \pi R \cap \pi K$ and, hence,

$$R \cap K \subseteq F \quad \Leftrightarrow \quad \pi R \cap \pi K = \{0\}.$$

By Lemma A.1

$$\pi R \cap \pi K = \{0\} \quad \Leftrightarrow \quad (-\pi R)^* \cap \operatorname{int} (\pi K)^* \neq \emptyset.$$

Since $(\pi R)^* = \pi^{*-1} R^*$ and $\operatorname{int} (\pi K)^* = \pi^{*-1}(\operatorname{ri} K^*)$, the condition in the right-hand side can be written as

$$\pi^{*-1}((-R^*) \cap \operatorname{ri} K^*) \neq \emptyset$$

or, equivalently,

$$(-R^*) \cap \operatorname{ri} K^* \cap \operatorname{Im} \pi^* \neq \emptyset.$$

But $\operatorname{Im} \pi^* = (K \cap (-K))^* = K^* - K^* \supseteq \operatorname{ri} K^*$ and we get the result. $\square$

Notice that if R is polyhedral then πR is also polyhedral, hence closed.

2. The following result is referred to as the Kreps–Yan theorem, see [48], [63], [5]. It holds for arbitrary $p \in [1, \infty]$, $p^{-1} + q^{-1} = 1$, but the cases $p = 1$ and $p = \infty$ are the most important.

Theorem A.3 *Let $\mathcal{C}$ be a convex cone in L^p closed in $\sigma\{L^p, L^q\}$, containing $-L^p_+$ and such that $\mathcal{C} \cap L^p_+ = \{0\}$. Then there is $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^q$ such that $\tilde{E}\xi \leq 0$ for all $\xi \in \mathcal{C}$.*

Proof. By the Hahn–Banach theorem any non-zero $x \in L^p_+ := L^p(\mathbf{R}_+, \mathcal{F})$ can be separated from $\mathcal{C}$: there is a $z_x \in L^q$ such that $Ez_x x > 0$ and $Ez_x \xi \leq 0$ for all $\xi \in \mathcal{C}$. Since $\mathcal{C} \supseteq -L^p_+$, the latter property yields that $z_x \geq 0$; we may assume $||z_x||_q = 1$. By the Halmos–Savage lemma the dominated family $\{P_x = z_x P : \ x \in L^p_+, \ x \neq 0\}$

contains a countable equivalent family $\{P_{x_i}\}$. But then $z := \sum 2^{-i} z_{x_i} > 0$ and we can take $\tilde{P} := zP$. $\square$

Recall that the Halmos–Savage lemma, though important, is, in fact, very simple. It suffices to prove its claim for the case of a convex family (in our situation we even have this property). A family $\{P_{x_i}\}$ such that the sequence $I_{\{z_{x_i}>0\}}$ increases to $\operatorname{ess\,sup} I_{\{z_x>0\}}$ (existing because of convexity) meets the requirement.

The above theorem has the following "purely geometric" version, [5].

Theorem A.4 *Suppose J and K are non-empty convex cones in a separable Banach space X such that $J \cap \overline{K - J} = \{0\}$. Then there is a continuous linear functional z such that $zx > 0 \ \forall\, x \in J$ and $zx \le 0 \ \forall\, x \in K$.*

The first step of the proof is the same as of the previous theorem: the separation of single points allows us to construct the set of $\{z_x \in X',\ x \in K\}$ with unit norms. The second step is to select a countable weak* dense subset. This can be done because the separability of X implies that the weak*-topologie on the unit ball of X' (always weak* compact) is metrisable. For the Lebesgue spaces the separability means that the σ-algebra is countably generated. Specific properties of these spaces allow us, by means of the Halmos–Savage lemma, to avoid such an unpleasant assumption on the σ-algebra.

References

[1] Ansel J.-P., Stricker C. Couverture des actifs contingents. *Ann. Inst. Henri Poincaré*, **30** (1994), 2, 303-315.

[2] Bouchard-Denize B., Touzi N. Explicit solution of the multivariate super-replication problem under transaction costs. *The Annals of Applied Probability*, **10** (2000), 685-708.

[3] Chamberlain G. Funds, factors, and diversification in arbitrage pricing models. *Econometrica*, **51** (1983), 5, 1305–1323.

[4] Chamberlain G., Rothschild M. Arbitrage, factor structure, and mean-variance analysis on large asset markets. *Econometrica*, **51** (1983), 5, 1281–1304.

[5] Clark S.A. The valuation problem in arbitrage price theory. *J. Math. Economics*, **22** (1992), 463-478.

[6] Cvitanić J., Karatzas I. Hedging and portfolio optimization under transaction costs: a martingale approach. *Mathematical Finance*, **6** (1996), 2, 133-165.

[7] Cvitanić J., Pham H., Touzi N. A closed form solution to the problem of super-replication under transaction costs. *Finance and Stochastics*, **3** (1999), 1, 35–54.

[8] Dalang R.C., Morton A., Willinger W. Equivalent martingale measures and no-arbitrage in stochastic securities market model. *Stochastics and Stochastic Reports*, **29** (1990), 185–201.

[9] Davis M.H.A., Clark J.M.C. A note on super-replicating strategies. *Philos. Trans. Roy. Soc. London*, A **347** (1994), 485–494.

[10] Delbaen F. Representing martingale measures when asset prices are continuous and bounded. *Mathematical Finance*, **2** (1992), 107–130.

[11] Delbaen F., Kabanov Yu.M, Valkeila S. Hedging under transaction costs in currency markets: a discrete-time model. *Mathematical Finance*. To appear.

[12] Delbaen F., Schachermayer W. A general version of the fundamental theorem of asset pricing. *Math. Annalen*, **300** (1994), 463–520.

[13] Delbaen F., Schachermayer W. A compactness principle for bounded sequence of martingales with applications. Proceedings of the Seminar of Stochastic Analysis, Random Fields and Applications, 1999.

[14] Delbaen F., Schachermayer W. The Fundamental Theorem of Asset Pricing for unbounded stochastic processes. *Math. Annalen*, 312 (1998), 215 - 250.

[15] Dellacherie C., Meyer P.-A. *Probabilités et Potenciel.* Hermann, Paris, 1980.

[16] El Karoui N., Quenez M.-C. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization*, **33** (1995), 1, 27–66.

[17] Émery M. Une topologie sur l'espace de semimartingales. Séminaire de Probabilités XIII. Lect. Notes Math., 721 (1979), 260-280.

[18] Föllmer H., Kabanov Yu.M. Optional decomposition and Lagrange multipliers. *Finance and Stochastics*, **2** (1998), 1, 69–81.

[19] Föllmer H., Kabanov Yu.M. Optional decomposition theorems in discrete time. *Atti del convegno in onore di Oliviero Lessi, Padova, 25-26 marzo 1996*, 47–68.

[20] Föllmer H., Kramkov D.O. Optional decomposition theorem under constraints. *Probability Theory and Related Fields*, **109** (1997), 1, 1-25.

[21] Gordan P. Über di Auflösung linearer Gleichungen mit reelen Koefficienten. *Math. Annalen*, 6 (1873), 23–28.

[22] Hall P., Heyde C.C. *Martingale Limit Theory and Its Applications.* Academic Press, NY, 1980.

[23] Harrison M., Pliska S. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, **11** (1981), 215–260.

[24] Hubalek F., Schachermayer W. When does convergence of asset price processes imply convergence of option prices? *Mathematical Finance* **8** (1998), 4, 215–233.

[25] Huberman G. A simple approach to Arbitrage Pricing Theory. *Journal of Economic Theory*, **28** (1982), 1, 183–191.

[26] Ingersoll J. E., Jr. Some results in the theory of arbitrage pricing. *Journal of Finance*, **39** (1984), 1021–1039.

[27] Ingersoll J. E., Jr. *Theory of Financial Decision Making.* Rowman and Littlefield, 1989.

[28] Jacod J., Shiryaev A.N. *Limit Theorems for Stochastic Processes.* Springer, Berlin–Heidelberg–New York, 1987.

[29] Jacod J., Shiryaev A.N. Local martingales and the fundamental asset pricing theorem in the discrete-time case. *Finance and Stochastics*, **2** (1998), 3, 259–273.

[30] Jouini E., Kallal H. Martingales and arbitrage in securities markets with transaction costs. *J. Economic Theory*, **66** (1995), 178–197.

[31] Jouini E., Kallal H. Arbitrage in securities markets with short sale constraints. *Mathematical Finance*, **5** (1995), 3, 197–232.

[32] Jouini E., Kallal H. Viability and equilibrium in securities markets with frictions. *Mathematical Finance*, **9** (1999), 3, 275–292.

[33] Kabanov Yu.M. On the FTAP of Kreps–Delbaen–Schachermayer. *Statistics and Control of Random Processes. The Liptser Festschrift. Proceedings of Steklov Mathematical Institute Seminar*, World Scientific, 1997, 191–203.

[34] Kabanov Yu.M. Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics*, **3** (1999), 2, 237–248.

[35] Kabanov Yu. M., Kramkov D.O. Large financial markets: asymptotic arbitrage and contiguity. *Probability Theory and its Applications*, **39**, (1994), 1, 222–229.

[36] Kabanov Yu. M., Kramkov D.O. No-arbitrage and equivalent martingale measure: an elementary proof of the Harrison–Pliska theorem. *Probability Theory and Its Applications*, **39** (1994), 3, 523–527.

[37] Kabanov Yu. M., Kramkov D.O. Asymptotic arbitrage in large financial markets. *Finance and Stochastics*, **2** (1998), 2, 143–172.

[38] Kabanov Yu.M., Last G. Hedging in a model with transaction costs. Preprint.

[39] Kabanov Yu.M., Last G. Hedging under transaction costs in currency markets: a continuous-time model. *Mathematical Finance*. To appear.

[40] Kabanov Yu.M., Liptser R.Sh., Shiryayev A.N. On the variation distance for probability measures defined on a filtered space. *Probability Theory and Related Fields*, **71** (1981), 19-36.

[41] Kabanov Yu.M., Stricker Ch. The Harrison–Pliska arbitrage pricing theorem under transaction costs. *J. Math. Econ.* To appear.

[42] Kabanov Yu.M., Stricker Ch. A teachers' note on no-arbitrage criteria. Séminaire de Probabilités. To appear.

[43] Kabanov Yu.M., Stricker Ch. On equivalent martingale measures with bounded densities. Séminaire de Probabilités. To appear.

[44] Klein I. A Fundamental Theorem of Asset Pricing for large financial markets. Preprint.

[45] Klein I., Schachermayer W. Asymptotic arbitrage in non-complete large financial markets. *Probability Theory and its Applications*, **41** (1996), 4, 927–934.

[46] Klein I., Schachermayer W. A quantitative and a dual version of the Halmos–Savage theorem with applications to mathematical finance. *Annals of Probability*, **24** (1996), 2, 867–881.

[47] Kramkov D.O. Optional decomposition of supermartingales and hedging in incomplete security markets. *Probability Theory and Related Fields*, **105** (1996), 4, 459–479.

[48] Kreps D.M. Arbitrage and equilibrium in economies with infinitely many commodities. *J. Math. Economics*, **8** (1981), 15-35.

[49] Levental S., Skorohod, A.V. On the possibility of hedging options in the presence of transaction costs. *The Annals of Applied Probability*, **7** (1997), 410–443.

[50] Mémin J. Espace de semimartingales et changement de probabilité. *Zeitschrift für Wahrscheinlichkeitstheorie und Verw. Geb.*, **52** (1980), 9-39.

[51] Pshenychnyi B.N. *Convex Analysis and Extremal Problems.* Nauka, Moscow, 1980 (in Russian).

[52] Rockafellar R.T. *Convex Analysis.* Princeton University Press, Princeton, 1970.

[53] Rogers L.C.G. Equivalent martingale measures and no-arbitrage. *Stochastic and Stochastics Reports*, **51** (1994), 41-51.

[54] Ross S. A. The arbitrage theory of asset pricing. *Journal of Economic Theory*, **13** (1976), 1, 341–360.

[55] Sin C.A. Strictly local martingales and hedge ratios on stochastic volatility models. PhD–dissertation, Cornell University, 1996.

[56] Schachermayer W. A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time. *Insurance: Mathematics and Economics*, **11** (1992), 249–257.

[57] Shiryaev A.N. *Probability.* Springer, Berlin–Heidelberg–New York, 1984.

[58] Shiryaev A.N. *Essentials of Stochastic Finance.* World Scientific, Singapore, 1999.

[59] Soner H.M., Shreve S.E., Cvitanić J. There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability*, **5** (1995), 327–355.

[60] Stricker Ch. Arbitrage et lois de martingale. *Annales de l'Institut Henri Poincaré. Probabilité et Statistiques*, **26** (1990), 3, 451-460.

[61] Schrijver A. *Theory of Linear and Integer Programming.* Wiley, 1986.

[62] Stiemke E. Über positive Lösungen homogener linearer Gleichungen. *Math. Annalen*, 76 (1915), 340–342.

[63] Yan J.A. Caractérisation d'une classe d'ensembles convexes de L^1 et H^1. Séminaire de Probabilités XIV. Lect. Notes Math., 784 (1980), 220-222.