ON THE FTAP OF KREPS–DELBAEN–SCHACHERMAYER

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Abstract

We analyze the proof of the Fundamental Theorem on Asset Pricing and show that the closedness result does not require any assumption on the price process S.

1 Main Results

In 1981 Kreps [13] established a theorem relating the existence of an equivalent "separating" measure with a certain no-arbitrage property: No Free Lunch (NFL). Delbaen and Schachermayer [4] observed that in a model driven by a semimartingale price process NFL coincides with another no-arbitrage property of a clear financial meaning: No Free Lunch with Vanishing Risk (NFLVR). Following closely the line of their proof we study here a more general setting with value processes as the primary objects, covering the case of bond market models and allowing some types of constraints. Our main message is that the closedness result holds without any additional hypothesis.

Let $\mathcal{S}$ be the space of semimartingales X defined on a finite interval $[0,T]$ and starting from zero; $\mathcal{S}$ is a Frechet space [7] with the quasinorm

$$\mathbf{D}(X) := \sup\{E1 \wedge |h \cdot X_T| : \ h \text{ is predictable}, \ \ |h| \leq 1\}.$$

We fix in $\mathcal{S}$ a closed convex subset $\mathcal{X}^1$ of processes $X \geq -1$ which contains 0 and satisfies the following condition: if $X, Y \in \mathcal{X}^1$, $H, G \geq 0$ are bounded predictable processes, $HG = 0$, and $Z := H \cdot X + G \cdot Y \geq -1$ then $Z \in \mathcal{X}^1$.

Put $\mathcal{X} := \cup_{\lambda > 0} \lambda \mathcal{X}^1$.

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Define the convex sets $K_0^1 := \{X_T : \ X \in \mathcal{X}^1\}$, $K_0 := \{X_T : \ X \in \mathcal{X}\}$, $C_0 := K_0 - L_+^0$, $C := C_0 \cap L^\infty$. We denote by $\bar{C}$, $\tilde{C}^*$, and $\bar{C}^*$ the norm closure, the union of weak* closures of denumerable subsets, and weak* closure of C in L^∞; $C_+ := C \cap L_+^\infty$ etc.

The properties NA, NFLVR, NFLBR, and NFL mean that $C_+ = \{0\}$, $\bar{C}_+ = \{0\}$, $\tilde{C}_+^* = \{0\}$, and $\bar{C}_+^* = \{0\}$, respectively. Consecutive inclusions induce the hierarchy of these properties:

$$\begin{array}{ccccccc} C & \subseteq & \bar{C} & \subseteq & \tilde{C}^* & \subseteq & \bar{C}^* \\ \text{NA} & \Leftarrow & \text{NFLVR} & \Leftarrow & \text{NFLBR} & \Leftarrow & \text{NFL.} \end{array}$$

Define the ESM property as the existence of $\tilde{P} \sim P$ such that $\tilde{E} X_T \le 0$ for all $X \in \mathcal{X}$. We introduce also the BK property: K_0^1 is bounded in L^0.

The following result, hidden in a more abstract context of [13], is referred sometimes, especially, in the case of the example below, as the Fundamental Theorem of Asset Pricing (FTAP).

Theorem 1.1 *NFL $\Leftrightarrow$ ESM.*

Proof. ($\Leftarrow$) Let $f \in \bar{C}^* \cap L_+^\infty$. Since $d\tilde{P}/dP \in L^1$, there are $f_n \in C$ with $\tilde{E} f_n \to \tilde{E} f$. By definition, $f_n \le X_T^n$ where $X^n \in \mathcal{X}$. Thus, $\tilde{E} f_n \le 0$ implying that $\tilde{E} f \le 0$ and $f = 0$.

($\Rightarrow$) Since $\bar{C}^* \cap L_+^\infty = \{0\}$, the Kreps–Yan separation theorem (Lemma F) provides $\tilde{P} \sim P$ such that $\tilde{E} f \le 0$ for all $f \in C$, hence, for all $f \in K_0$. $\Box$

Surprisingly, we have also NFLVR $\Leftrightarrow$ NFLBR $\Leftrightarrow$ NFL due to following

Theorem 1.2 *Under NFLVR $C = \bar{C}^*$.*

Example. Let $\mathcal{X}^1$ be the set of all integrals $H \cdot S \ge -1$ with respect to a fixed semimartingale S (it is closed by Mémin's theorem [14]). By definition, S has EMM (resp. ELMM) property if there is $P' \sim P$ such that $S \in \mathcal{M}(P')$ (resp., $S \in \mathcal{M}_{loc}(P')$. If S is bounded, ESM coincides with EMM. Indeed, being bounded from below, the stochastic integral $H \cdot S \in \mathcal{M}(P')$ (see [1]) and, by the Fatou lemma, $E' H \cdot S_T \le 0$. On the other hand, for each stopping time τ the processes $I_{[0,\tau]} \cdot S$ and $-I_{[0,\tau]} \cdot S$ belong to $\mathcal{X}$, $\tilde{E} S_\tau = 0$ and, hence, $S \in \mathcal{M}(\tilde{P})$. Of course, if S is locally bounded, ESM coincides with ELMM.

Delbaen and Schachermayer [4] studied this very model and proved the corresponding version of Theorem 1.2, a remarkable result important in its own right not to less extent than the reformulations of the FTAP, based on it. While they considered the case of a locally bounded semimartingale S, their method works for a more general situation.

We give the proof of Theorem 1.2 in Section 3 after some preparatory work in Section 2. Auxiliary results (Lemmas A, B etc.) are listed in Appendix.

For a sequence $x = (x_k)$ we put $\mathcal{T}_n(x) := \operatorname{conv}\{x_k, \ k \ge n\}$. A special semimartingales X is indicated by its canonical decomposition always of the form $X = M + A$ (with various attributes) where $M \in \mathcal{M}_{loc}$ and A is a predictable process of bounded variation; $|A|_t := \operatorname{Var} A$; the density $dA/d|A|$ is taken predictable with values in $\{-1, 1\}$. We write $H \cdot M^*$ instead of $(H \cdot M^*)$.

2 NFLVR and BK

2.1 Elementary properties

Lemma 2.1 *Let $X \in \mathcal{X}$. If NA holds then $X \in \lambda\mathcal{X}^1$ with $\lambda = \|X_T^-\|_\infty$.*

Proof. If $P(X_s < -\lambda) > 0$ then $I_{\{X_s<-\lambda\}} I_{]s,T]} \cdot X_T$ violates NA. □

Lemma 2.2 *The following conditions are equivalent:*
(a) NFLVR;
(b) P-$\lim g_n = 0$ for every sequence $g_n \in K_0$ with $\lim \|g_n^-\|_\infty = 0$;
(c) NA & BK.

Proof. $(a) \Rightarrow (b)$ Let $g_n \in K_0$, $\lim \|g_n^-\|_\infty = 0$ but $\lim P(g_n \geq \alpha) \geq \alpha > 0$. Clearly, $f_n := g_n \wedge 1 \in C$. Lemma A provides a sequence $\tilde{f}_n \in \mathcal{T}_n(f) \subseteq C$ converging a.s. to $\tilde{f} \geq 0$ with $P(\tilde{f} > 0) = 2\beta > 0$. By Egorov's theorem there is Γ such that $P(\Gamma) > 1 - \beta$ and $\lim \|\tilde{f}_n I_\Gamma - \tilde{f} I_\Gamma\|_\infty = 0$ and NFLVR fails since $C \ni \tilde{f}_n I_\Gamma - \tilde{f}_n^- I_{\Gamma^c} \to \tilde{f} I_\Gamma$ in L^∞ where $P(\tilde{f} I_\Gamma > 0) \geq \beta$.

$(b) \Rightarrow (c)$ NA follows trivially. If BK fails we can find $X^n \in \mathcal{X}^1$ with $\lim P(X_T^n \geq n) > 0$ and get a contradiction with $g_n := n^{-1} X_T^n$.

$(c) \Rightarrow (a)$ If NFLVR fails, there are a sequence $f_n \in C$ and $f \geq 0$ with $P(f > 0) > 0$ such that $\|f_n - f\|_\infty \leq n^{-1}$. By definition, $f_n \leq h_n = X_T^n$ where $X^n \in \mathcal{X}$. Obviously, $\|h_n^-\|_\infty \leq n^{-1}$ and, by NA, $nX^n \in \mathcal{X}^1$. By Lemma A we may assume that $h_n \to h$ a.s. Since $P(h > 0) > 0$, the sequence $nX_T^n \in K_0^1$, tending to infinity with positive probability, violates BK. □

Lemma 2.3 *Let $X, X^1, X^2 \in \mathcal{X}^1$ and $\alpha > 0$.*
(a) Let θ be stopping time. Then $I_{[0,\theta]} \cdot X \in \mathcal{X}^1$;
(b) Let $\tau := \inf\{t : \ X_t^1 \geq X_t^2 + \alpha\}$. Then $I_{[0,\tau]} \cdot X^1 + I_{]\tau,\infty[} \cdot X^2 \in \mathcal{X}^1$.
(c) Let $X^i = M^i + A^i$, B be a predictable increasing process dominating A^i, $r^i := dA^i/dB$, and $\sigma := \inf\{t : \ I_\Gamma \cdot M_t^1 + I_{\Gamma^c} \cdot M_t^2 < M_t^1 \vee M_t^2 - \alpha\}$ where $\Gamma = \{r^1 \geq r^2\}$. Then $\tilde{X} := I_{[0,\sigma]\cap\Gamma} \cdot X^1 + I_{[0,\sigma]\cap\Gamma^c} \cdot X^2 \in (1+\alpha)\mathcal{X}^1$.

Proof. The assertion (a) is obvious; (b) holds since

$$I_{[0,\tau]} \cdot X^1 + I_{]\tau,\infty[} \cdot X^2 \geq X^1 I_{[0,\tau]} + (\alpha + X^2) I_{]\tau,\infty[} \geq -1 + \alpha.$$

To show (c) we notice that $I_\Gamma \cdot A^1 + I_{\Gamma^c} \cdot A^2 \geq A^1 \vee A^2$. Thus, on $[0, \sigma[$

$$\tilde{X} \geq A^1 \vee A^2 + M^1 \vee M^2 - \alpha \geq (M^1 + A^1) \vee (M^2 + A^2) - \alpha = X^1 \vee X^2 - \alpha$$

and $\tilde{X}_\sigma \geq -1 - \alpha$ because $\Delta\tilde{X}_\sigma = I_\Gamma \Delta X_\sigma^1 + I_{\Gamma^c} \Delta X_\sigma^2$. □

Lemma 2.4 *Let $Y^n = I_{\Gamma^n} \cdot Z^n$ with predictable Γ^n and $Z^n \in \mathcal{X}^1$ be such that*
1) $\limsup P(Y^n_T \geq \beta) \geq \beta > 0$,
2) $\beta_n := \|((\Delta Y^n)^-)^*_T\|_\infty \to 0$, $n \to 0$,
3) $\lim P(((Y^n)^-)^*_T \geq \gamma_n) = 0$ *where* $\gamma_n \downarrow 0$.
Then BK (hence, NFLVR) fails.

Proof. Take $(\beta_n + \gamma_n)^{-1} I_{[0,\tau_n]} \cdot Y_n \in \mathcal{X}^1$ where $\tau_n := \inf\{t : \ (Y^n_t)^- > \gamma_n\}$. □

2.2 Absence of Compensation

Let $X^n = M^n + A^n \in \mathcal{X}^1$. One can imagine the situation where M^n_T and A^n_T may diverge though the sequence X^{n*}_T is bounded in L^2. Fortunately, BK excludes this compensation phenomenon.

Lemma 2.5 *Assume BK. Let the sequence $X^n = M^n + A^n$ in $\mathcal{X}^1$ be such that $\|X^{n*}_T\|_2 \leq \lambda$. Then the sequence M^{n*}_T is bounded in L^0.*

Proof. Suppose that the claim fails. We may assume, taking a subsequence, that $P(M^{n*}_T > n^3) \geq 8\alpha > 0$. By the Chebyshev inequality, $P(X^{n*}_T > n) \leq \alpha$ for sufficiently large n (we skip further these words to avoid repetitions).

Let $\tau_n := \inf\{t : \ M^{n*}_t > n^3 \text{ or } X^{n*}_t > n\}$ and $\tilde{X}^n := n^{-3} I_{[0,\tau_n]} \cdot X^n$. Clearly, $P(\tilde{M}^{n*}_T \geq 1) \geq 7\alpha$. By Lemma C $\|(\Delta \tilde{M}^n)^*_T\|_2 \leq 6n^{-3}\lambda \leq n^{-1}$. Let (T^n_i) be the n^{-1}-chain for $\tilde{M}^n$. By Lemma D

$$P(\tilde{M}^n_{T^n_i} - \tilde{M}^n_{T^n_{i-1}} \leq -\alpha n^{-1}) \geq \alpha^2 \qquad \forall i \leq k_n := [n\alpha/4].$$

For arbitrary stopping times σ and τ the Chebyshev inequality yields

$$P(|\tilde{X}^n_\tau - \tilde{X}^n_\sigma| > \alpha n^{-1}/2) \leq 8\lambda^2 n^{-4}/\alpha^2 \leq \alpha^2/2.$$

It follows that

$$P(\tilde{A}^n_{T^n_i} - \tilde{A}^n_{T^n_{i-1}} \geq \alpha n^{-1}/2) \geq \alpha^2/2 \qquad \forall i \leq k_n.$$

Let H^n be the indicator function of $\{r^n = 1\} \cap [0, T^n_{k_n}]$ where $r^n := d\tilde{A}^n/|\tilde{A}^n|$. Put $Y^n := H^n \cdot \tilde{X}^n$. Then $H^n \cdot \tilde{A}^n$ and $H^n \cdot \tilde{A}^n - \tilde{A}^n$ are increasing on $[0, T^n_{k_n}]$, the last bound holds for $H^n \cdot \tilde{A}^n$, and by Lemma B

$$P(H^n \cdot \tilde{A}^n_{T^n_{k_n}} \geq k_n n^{-1}\alpha^3/8) \geq \alpha^2/4,$$

i.e., $P(H^n \cdot \tilde{A}^n_T \geq 2\beta) \geq 2\beta$ for some $\beta > 0$.

Since X^n on $[0, \tau_n[$ is in $[-1, n]$ we have $\Delta \tilde{X}^n \geq -(n+1)$ on $[0, \tau_n]$. Thus,

$$(\Delta Y^n)^- \leq (\Delta \tilde{X}^n)^- \leq (n+1)n^{-3}.$$

Clearly, $\|H^n \cdot \tilde{M}^n_{T_{k_n}}\|_2^2 \le \|\tilde{M}^n_{T_{k_n}}\|_2^2 \le 4n^{-2}k_n$ and, by the Doob inequality,

$$P(H^n \cdot \tilde{M}^{n*}_T \ge \gamma_n) \le \gamma_n^{-2}\|H^n \cdot \tilde{M}^{n*}_T\|_2^2 \le 16\gamma_n^{-2}n^{-2}k_n \to 0$$

if $\gamma_n^{-1} = o(n^{1/2})$. Since $Y^n \ge H^n \cdot \tilde{M}^n$ we get in this case also that

$$P(((Y^n)^-)^*_T \ge \gamma_n) \le P(H^n \cdot \tilde{M}^{n*}_T \ge \gamma_n) \to 0,$$

$P(Y^n_T \ge \beta) \ge \beta$ and the result follows from Lemma 2.4. □

2.3 Residual Processes

Notations: $\tau^n_c := \inf\{t: \ M^{n*}_t > c\}$, $X^n_c := I_{[\tau^n_c,\infty[} \cdot X^n$, and $\xi := \sup_n X^{n*}_T$.

Lemma 2.6 *Assume BK. Let $X^n = M^n + A^n \in \mathcal{X}^1$ and $\xi \in L^2$. Then for any $\varepsilon > 0$ there is c_0 such that for all $\tilde{X} \in \cup_{c \ge c_0}\mathcal{T}_1(X_c)$ we have $P(\tilde{M}^*_T > \varepsilon) \le \varepsilon$.*

Proof. Suppose that the claim fails with $\varepsilon = \alpha > 0$. Take $\delta \in]0, \alpha/4[$. By Lemma 2.5 (with $\lambda = \|\xi\|_2$) there is c_0 such that for all $c \ge c_0$ and all n

$$P(\tau^n_c < \infty) = P(M^{n*}_T > c) \le \delta^2.$$

By our hypothesis there are $c \ge c_0$ and $\tilde{X} = \sum \lambda_i X^i_c$ (λ_i are convex weights) with $P(\tilde{M}^*_T > \alpha) > \alpha$. So, for $\rho := \inf\{t: \ \tilde{M}^*_t > \alpha\}$ we have $P(\rho < \infty) > \alpha$.

Let $\vartheta := \inf\{t: \ F_t > \delta\}$ where $F := \sum \lambda_i I_{]\tau^i_c,\infty[}$; F is an increasing left-continuous process and, therefore, $F_\vartheta \le \delta$. By the Chebyshev inequality

$$P(\vartheta < \infty) = P(F_\infty > \delta) \le \delta^{-1}\sum \lambda_i P(\tau^i_c < \infty) \le \delta < \alpha/4.$$

Fix $N \ge 2$ such that $P(\xi > N-1) < \alpha/4$. Then $P(\tau < \infty) < \alpha/4$ where $\tau := \inf\{t: \ \sup_n X^{n*}_t > N-1\}$. For $t \le \vartheta \wedge \tau$ we have

$$2\delta\xi \ge 2F_t\xi \ge \sum \lambda_i I_{\{t > \tau^i_c\}}(X^i_t - X^i_{\tau^i_c}) \ge -NF_t \ge -N\delta.$$

Thus, $X' := (N\delta)^{-1}I_{[0,\tau\wedge\vartheta\wedge\rho]} \cdot \tilde{X} \in \mathcal{X}^1$, $\|X'^*_T\|_2 \le \|1 \vee \xi\|_2$, and

$$P(M'^*_T \ge \alpha(N\delta)^{-1}) \ge P(\rho < \infty, \ \tau = \infty, \ \vartheta = \infty) \ge \alpha/2.$$

Letting $\delta \downarrow 0$ we come to a contradiction with Lemma 2.5. □

Lemma 2.7 *Assume BK. Let $X^n = M^n + A^n \in \mathcal{X}^1$ and $\xi \in L^2$. Then for any $\delta > 0$ there is c_0 such that $\mathbf{D}(\tilde{M}) \le \delta$ for all $\tilde{X} \in \cup_{c \ge c_0}\mathcal{T}_1(X_c)$.*

Proof. Let $\varepsilon > 0$. Take c_0 as in Lemma 2.6, i.e. such that $P(\tilde{M}^*_T > \varepsilon) \le \varepsilon$ for all $c \ge c_0$ and $\tilde{X} \in \mathcal{T}_1(X_c)$. By Lemma 2.5 $\sup_n P(\tau^n_c < \infty) \to 0$ and, hence, $\|(X^n_c)^*_T\|_2 \le 2\|\xi I_{\tau^n_c < \infty}\| \le \varepsilon/6$ for all n if c is large enough. Enlarging eventually c_0, we get by Lemma C that $\sup_n \|(\Delta\tilde{M}^n)^*_T\|_2 \le \varepsilon$ if $c \ge c_0$. Take predictable h with $|h| \le 1$, $\tilde{X} \in \mathcal{T}_1(X_c)$, and put $\rho := \inf\{t:\ \tilde{M}^*_t > \varepsilon\}$. As

$$\|h \cdot \tilde{M}_\rho\|_2 = \|h^2 \cdot [\tilde{M}, \tilde{M}]_\rho\|_1^{1/2} \le \|[\tilde{M}, \tilde{M}]_\rho\|_1^{1/2} = \|\tilde{M}_\rho\|_2 \le 2\varepsilon,$$

we have, in virtue of the Chebyshev and Doob inequalities, that

$$P(h \cdot \tilde{M}^*_T \ge \sqrt{\varepsilon}) \le P(h \cdot \tilde{M}^*_\rho \ge \sqrt{\varepsilon}) + P(\rho < \infty) \le 17\varepsilon.$$

Thus, $D(\tilde{M}) = \sqrt{\varepsilon} + 17\varepsilon$ and the result follows. □

2.4 Convergence in the Semimartingale Topology

Lemma 2.8 *Assume BK. Let $X^n = M^n + A^n \in \mathcal{X}^1$ and $\xi \in L^2$. Then there exist $\tilde{X}^n \in \mathcal{T}_n(X)$ with $\tilde{M}^n$ converging in $\mathcal{S}$.*

Proof. By Lemma 2.7 there is c_k such that $\mathbf{D}(\tilde{M}) \le k^{-1}$ for all $\tilde{X} \in \mathcal{T}_1(X_{c_k})$. We consider the martingales $x^n_k := I_{[0,\tau^n_{c_k}]} \cdot M^n$ as elements of the Hilbert space $\mathcal{M}^2$. Since $\|I_{[0,\tau^n_{c_k}]} \cdot M^n_T\|_2 \le c_k + 6\|\xi\|_2$, by Lemma E there are convex weights $\Lambda^n := (\lambda^n_j)_{j \le N_n}$ such that for every k the sequence

$$Y^n_k := \sum_{j=0}^{N_n} \lambda^n_j I_{[0,\tau^{n+j}_{c^k}]} \cdot M^{n+j}$$

converges in $\mathcal{M}^2$, hence, in $\mathcal{S}$. This implies that

$$Z^n := \sum_{j=0}^{N_n} \lambda^n_j M^{n+j} = Y^n_k + \sum_{j=0}^{N_n} \lambda^n_j M^{n+j}_{c_k} = Y^n_k + \tilde{M}^n_k$$

is a Cauchy sequence (hence, convergent) in $\mathcal{S}$. Indeed,

$$\mathbf{D}(Z^n - Z^m) \le \mathbf{D}(Y^n_k - Y^m_k) + \mathbf{D}(\tilde{M}^n_k) + \mathbf{D}(\tilde{M}^m_k) \le \mathbf{D}(Y^n_k - Y^m_k) + 2k^{-1}$$

and it remains to take at first the limit as $m, n \to \infty$ and then as $k \to \infty$. □

3 Closedness

A convex bounded set C in L^∞ is closed in L^∞ iff every bounded and weakly* convergent sequence of its elements has the limit in C ([9], Ch. 5-3, Ex. 1).

It is easy to see, using, e.g., Lemma A, that in our case, where $C = C_0 \cap L^\infty$, the second condition holds if C_0 is Fatou closed, i.e. for any sequence $h_n \in C_0$ uniformly bounded from below and such that $h_n \to h$ a.s. we have $h \in C_0$.

Thus, Theorem 1.2 follows from

Theorem 3.1 *Assume NFLVR. Then C_0 is Fatou closed.*

Before the proof we recall the following fact: a non empty closed bounded subset of L^0 has a maximal element. Indeed, each linearly ordered subset $\{f_\alpha\}$ has as a majorant $\operatorname{ess\,sup}_\alpha f_\alpha < \infty$ and the claim holds by Zorn's lemma.

Notations: $\lfloor f, \infty \lfloor := \{g \in L^0 : \ g \geq f\}$, $\mathcal{D}_f := \widehat{K}_0^1 \cap \lfloor f, \infty \lfloor$ where $\widehat{K}_0^1$ in a closure of K_0^1 in L^0.

Proof. Let $f_n \geq -1$ be a sequence in C_0 converging to f a.s. The claim follows if the set $K_0 \cap \lfloor f, \infty \lfloor$ is non empty. We show that it contains the set of maximal elements of $\mathcal{D}_f$. By Lemma 2.2 $\widehat{K}_0^1$, and hence $\mathcal{D}_f$, is bounded in L^0. We have $-1 \leq f_n \leq h_n = X_T^n$ where $X^n \in \mathcal{X}$; NA yields that $X^n \in \mathcal{X}^1$. By Lemma A there are $\tilde{h}_n \in \mathcal{T}_n(h) \subset K_0^1$ converging a.s. to some finite $\tilde{h}_0$ which is, clearly, in $\mathcal{D}_f$. Thus, $\mathcal{D}_f \neq \emptyset$ and a maximal element h_0 in $\mathcal{D}_f$ does exist. It remains to check that $h_0 = \tilde{X}_T$ for some $\tilde{X} \in \mathcal{X}$.

Since $h_0 \in \widehat{K}_0^1$, there is a sequence $X^n \in \mathcal{X}^1$ such that $X_T^n \to h_0$ a.s.

Lemma 3.2 *We have P-$\lim_{m,n}(X^m - X^n)_T^* = 0$.*

Proof. If the claim fails then $P(((X^{i_k} - X^{j_k})^+)_T^* > \alpha) \geq \alpha > 0$ with some $i_k, j_k \to \infty$. For $T_k := \inf\{t : \ X_t^{i_k} - X_t^{j_k} > \alpha\}$ we have $P(T_k < \infty) \geq \alpha$. In virtue of Lemma 2.3 $\tilde{X}^k := I_{[0,T_k]} \cdot X^{i_k} + I_{]T_k,T]} \cdot X^{j_k} \in \mathcal{X}^1$. Notice that

$$\tilde{X}_T^k = X_T^{i_k} I_{\{T_k = \infty\}} + X_T^{j_k} I_{\{T_k < \infty\}} + \xi_k$$

where $\xi_k := (X_{T_k}^{i_k} - X_{T_k}^{j_k}) I_{\{T_k < \infty\}} \geq 0$ and $P(\xi_k \geq \alpha) \geq \alpha$. Applying Lemma A to (ξ_k) we infer that $\mathcal{D}_f$ contains an element $h_0 + \eta$ with $\eta \geq 0$, $\eta \neq 0$, in a contradiction with the maximality of h_0. $\square$

In particular, $\xi := \sup_n X_T^{n*} < \infty$. Take $Q \sim P$ such that $\xi \in L^2(Q)$; X^n are special semimartingales under Q. Working with the properties invariant under an equivalent change of measure, we may assume $Q = P$. Lemma 2.8 provides $\tilde{X}^n \in \mathcal{T}_n(X)$ with the martingale components $\tilde{M}^n$ converging in $\mathcal{S}$. By Lemma 3.3 below $\tilde{A}^n$ also converge in $\mathcal{S}$. So, $\tilde{X}^n = \tilde{M}^n + \tilde{A}^n$ converge in $\mathcal{S}$ to $\tilde{X} \in \mathcal{X}^1$ since $\mathcal{X}^1$ is convex and closed. Thus, $h_0 = \lim \tilde{X}_T^n = \tilde{X}_T$. $\square$

Lemma 3.3 *Let $X^n \in \mathcal{X}^1$ be such that $X_T^n \to h_0$ a.s. and M^n converge in $\mathcal{S}$. Then A^n converge in $\mathcal{S}$.*

Proof. Let $r^n := dA^n/dB$, B is a predictable increasing process dominating all A^n. If the claim fails, A^n is not a Cauchy sequence in $\mathcal{S}$ and there are $i_k, j_k \to \infty$ such that $P(|r^{i_k} - r^{j_k}| \cdot B_T > 2\gamma) \geq 2\gamma > 0$. Let $\Gamma_k := \{r^{i_k} \geq r^{j_k}\}$. We may assume $i_k \wedge j_k > i_{k-1} \vee j_{k-1}$ and, interchanging eventually i_k and j_k,

$$P((r^{i_k} - r^{j_k}) I_{\Gamma_k} \cdot B_T > \gamma) \geq \gamma.$$

Take $\alpha_k \downarrow 0$ and define $\bar{X}^k := I_{\Gamma_k} \cdot X^{i_k} + I_{\Gamma_k^c} \cdot X^{j_k}$, $\tilde{X}^k := I_{[0,\sigma_k]} \cdot \bar{X}^k$ where

$$\sigma_k := \inf\{t: \ I_{\Gamma_k} \cdot M_t^{i_k} + I_{\Gamma_k^c} \cdot M_t^{j_k} < M_t^{i_k} \vee M_t^{j_k} - \alpha_k\}.$$

By Lemma 2.3 $\tilde{X}^k \in (1+\alpha_k)\mathcal{X}^1$. Note that $\bar{M}^k - M^{i_k} = I_{\Gamma_k^c} \cdot (M^{j_k} - M^{i_k})$ tends to zero in $\mathcal{S}$. Hence, $(\bar{M}^k - M^{i_k})^*_T$ tends to zero in probability and the same holds for $(\bar{M}^k - M^{j_k})^*_T$. We may take i_k and j_k growing fast enough to ensure $P(\sigma_k < \infty) \to 0$. From the representation

$$\tilde{X}^k_T = X^{j_k}_{T\wedge\sigma_k} + I_{\Gamma_k \cap [0,\sigma_k]} \cdot (M^{i_k} - M^{j_k})_T + \xi_k,$$

applying Lemma A to $\xi_k := (r^{i_k} - r^{j_k}) I_{\Gamma_k} \cdot B_{T\wedge\sigma_k}$, we get easily that $\mathcal{D}_f$ contains an element $h_0 + \eta$ with $\eta \geq 0$, $\eta \neq 0$, in a contradiction with the maximality of h_0. $\Box$

Appendix

Facts from Probability

Lemma A. *Let (ξ_n) be a sequence of nonnegative r.v. Then there exist a sequence $\eta_n \in \mathcal{T}_n(\xi)$ and a r.v. η with values in $[0,\infty]$ such that $\eta_n \to \eta$ a.s.*

If $\mathcal{T}_1(\xi)$ is bounded in L^0 then $\eta < \infty$.

If $P(\xi_n \geq \alpha) \geq \alpha > 0$ for all n then $P(\eta > 0) > 0$.

Proof. Clearly, $J_n := \inf_{\eta \in \mathcal{T}_n(\xi)} Ee^{-\eta}$ increase to some $J \leq 1$. Take $\eta_n \in \mathcal{T}_n(\xi)$ with $Ee^{-\eta_n} \leq J_n + 1/n$. For any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$e^{-(x+y)/2} \leq (e^{-x} + e^{-y})/2 - \delta I_{B_\varepsilon}(x,y)$$

where $B_\varepsilon := \{(x,y) \in \mathbf{R}^2_+ : \ |x-y| \geq \varepsilon, \ x \wedge y \leq 1/\varepsilon\}$. Therefore,

$$J_{n\wedge m} \leq Ee^{-(\eta_n + \eta_m/2} \leq (Ee^{-\eta_n} + Ee^{-\eta_m})/2 - \delta P((\eta_n, \eta_m) \in B_\varepsilon).$$

It follows that $\lim_{m,n\to\infty} P((\eta_n,\eta_m) \in B_\varepsilon) = 0$. We infer from the inequality

$$E|e^{-\eta_n} - e^{-\eta_m}| \leq \varepsilon + 2e^{-1/\varepsilon} + P((\eta_n,\eta_m) \in B_\varepsilon)$$

that $e^{-\eta_n}$ is a Cauchy sequence in L^1. It remains to recall that a sequence convergent in L^1 (hence, in L^0) contains a subsequence convergent a.s.

The first property of limits is obvious.

To prove the second one we observe that if $P(\xi_n \geq \alpha) \geq \alpha > 0$ for all n then $Ee^{-\zeta} \leq 1 - \alpha + \alpha e^{-\alpha} < 1$ for any $\zeta \in \mathcal{T}_1(\xi)$. $\Box$

Lemma B. *Let $\eta_n := \xi_1 + \ldots \xi_n$ where all $\xi_i \geq 0$ and $P(\xi_i \geq a) \geq 2b$ for some $a, b > 0$. Then $P(\eta_n \geq nab) > b$.*

Proof. Let $A := \{\eta_n \geq nab\}$. We have:

$$nab(1 - P(A)) \geq E\eta_n I_{A^c} \geq \sum_{i=1}^{n} E\xi_i I_{\cap\{\xi_i \geq a\}\cap A^c} \geq a \sum_{i=1}^{n} P(\{\xi_i \geq a\} \cap A^c) \geq$$

$$\geq a \sum_{i=1}^{n} (P(\xi_i \geq a) - P(A)) \geq na(2b - P(A)).$$

Thus, $P(A) \geq b/(1-b)$ and the result follows. □

Facts from Stochastic Calculus

Lemma C. *Let $X = M + A$ be a special semimartingale. Then*

$$\|(\Delta A)^*_T\|_2 \leq 2\|(\Delta X)^*_T\|_2, \qquad \|(\Delta M)^*_T\|_2 \leq 3\|(\Delta X)^*_T\|_2.$$

Proof. Obviously, for the predictable process A we have $(\Delta A)^* \leq Y$ where $Y_t = E((\Delta X)^*_T|\mathcal{F}_t)$. The bound for A follows by the Doob inequality. □

Lemma D. *Let $N \in \mathcal{M}^2_0$ and (T_i) be the ε-chain of N defined as follows:*

$$T_0 := 0, \qquad T_{i+1} := \inf\{t \geq T_i : \ |N_t - N_{T_i}| > \varepsilon\}, \quad n \geq 1.$$

Assume that $\|(\Delta N)^*_T\|_2 \leq \varepsilon$ and $P(N^*_T \geq 1) \geq 7\alpha > 0$. Then

$$P(N_{T_i} - N_{T_{i-1}} > -\alpha\varepsilon) \geq \alpha^2 \quad \forall\ i = 1, 2, \ldots, k := [\varepsilon^{-1}\alpha/4].$$

Proof. Let $f_i := N_{T_i} - N_{T_{i-1}}$, $\Gamma := \{T_k < \infty\}$, and $B_i := \{f_i^- > \varepsilon\alpha\}$ (the set of interest). By the Doob inequality $\|I_{]T_{i-1},T_i]} \cdot N^*_T\|_2 \leq 2\|f_i\|_2 \leq 4\varepsilon$. Hence,

$$\|N^*_T I_{\Gamma^c}\|_2 = \left\| \left(\sum_{i=1}^{k} I_{]T_{i-1},T_i]} \right) \cdot N^*_T I_{\Gamma^c} \right\|_2 \leq \sum_{i=1}^{k} \|I_{]T_{i-1},T_i]} \cdot N^*_T\|_2 \leq 4\varepsilon k \leq \alpha$$

and, by the Chebyshev inequality, $P(N^*_T I_{\Gamma^c} \geq 1) \leq \alpha^2$. It follows that

$$P(\Gamma) \geq P(N^*_T \geq 1) - P(\{N^*_T \geq 1\} \cap \Gamma^c) \geq 7\alpha - \alpha^2 \geq 6\alpha.$$

For any $i \leq k$ we have $\{|f_i| > \varepsilon\} \supseteq \Gamma$ and, due to the martingale property,

$$Ef_i^- = Ef_i^+ = E|f_i|/2 \geq \varepsilon P(|f_i| > \varepsilon)/2 \geq 3\varepsilon\alpha.$$

Using the Cauchy–Schwarz inequality and the above estimate we have

$$\|f_i^-\|_2 \sqrt{P(B^i)} \geq Ef_i^- I_{B_i} = Ef_i^- - Ef_i^- I_{B_i^c} \geq 3\varepsilon\alpha - \varepsilon\alpha = 2\varepsilon\alpha$$

and the result follows since $\|f_i^-\|_2 \leq \|f_i\|_2 \leq 2\varepsilon$. □

Facts from Functional Analysis

Lemma E. *Let $(x_k^n)_{k,n\in\mathbf{N}}$ be a double sequence in a Hilbert space $\mathcal{H}$ such that $C_k := \sup_n \|x_k^n\| < \infty$ for every k. Then there is a sequence $\Lambda^n = (\lambda_j^n)_{0\le j\le N_n}$ of convex weights such that for every k the sequence $(y_k^n)_{n\in\mathbf{N}}$ with*

$$y_k^n := \sum_{j=0}^{N_n} \lambda_j^n x_k^{j+n}$$

converges in $\mathcal{H}$.

Proof. Let $\mathcal{K} := \oplus\sum_1^\infty \mathcal{H}$ be the Hilbert space with elements $x = (x_k)$, $x_k \in H$, and $\|x\|_{\mathcal{K}}^2 := \sum \|x_k\|_{\mathcal{H}}^2 < \infty$. As $\tilde{x}^n = (\tilde{x}_k^n)_{k\in\mathbf{N}}$ with $\tilde{x}_k^n := x_k^n/(2^k C_k)$ is a bounded sequence in $\mathcal{K}$, by the Mazur theorem [16] there are $y^n \in \mathcal{T}_n(\tilde{x})$ convergent in $\mathcal{K}$, thus, componentwise in $\mathcal{H}$. The corresponding convex weights meet the requirement. □

Lemma F. [13], [15],[2] *Let $\mathcal{C} \supseteq -L_+^\infty$ be a convex weakly* closed cone in L^∞ such that $\mathcal{C} \cap L_+^\infty = \{0\}$. Then there is $\tilde{P} \sim P$ such that $\tilde{E}f \le 0$ for all $f \in \mathcal{C}$.*

Proof. By the Hahn–Banach theorem any nonzero $x \in L_+^\infty$ can be separated from $\mathcal{C}$: there is $z_x \in L^1$ such that $Ez_x x > 0$ and $Ez_x f \le 0$ for all $f \in \mathcal{C}$. Since $\mathcal{C} \supseteq -L_+^\infty$, the latter property yields that $z_x \ge 0$; we may assume $Ez_x = 1$. By the Halmos–Savage lemma the dominated family $\{P_x = z_x P : \ x \in L_+^\infty,\ x \ne 0\}$ contains a countable equivalent family $\{P_{x_i}\}$. But then $z := \sum 2^{-i} z_{x_i} > 0$ and we can take $\tilde{P} := zP$. □

Addendum

When this note was completed, F. Delbaen communicated the summary [5] which contains, for the case $\mathcal{X}^1 = \{H\cdot S : \ H\cdot S \ge -1\}$ without restrictions on S, formulations of the closedness result and an approximation property of separating measures implying a new equivalence announced at Ascona meeting in September 1996 (see Theorems 2 and 1 below). The detailed exposition [6] is now available but for the sake of completeness we give our proof of Theorem 2 using the measurable selection technology of [8] which goes back to [3].

A semimartingale S is a σ-martingale (notation: $S \in \Sigma_m$) if $G\cdot S \in \mathcal{M}_{loc}$ for some G with values in $]0,1]$. The property EσMM means that there is $Q \sim P$ such that $S \in \Sigma_m(Q)$.

Theorem 1 *Let $\mathcal{X}^1$ be the set of stochastic integrals $H\cdot S \ge -1$. Then*

$$NFLVR \Leftrightarrow NFLBR \Leftrightarrow NFL \Leftrightarrow ESM \Leftrightarrow E\sigma MM.$$

The only remaining nontrivial implication ESM $\Rightarrow$ EσMM follows from

Theorem 2 *Let P be a separating measure. Then for any $\varepsilon > 0$ there is $Q \sim P$ with* $\mathrm{Var}\,(P-Q) \le \varepsilon$ *such that S is a σ-martingale under Q.*

In contrast with the intriguing implication above, this formulation contains an instruction[1] how to proceed. Our arguments use intensively notations and results of [10], cited directly.

Let (B, C, ν) be the characteristics (relative to the truncation function $h(x) := xI_{\{|x|\le 1\}}$) of the semimartingale S written in the canonical form

$$S = S^c + h * (\mu - \nu) + \bar{h} * \mu + B,$$

$\bar{h} := x - h$. We choose a "good" version of the triplet, i.e. such that $B = b \cdot A$, $\nu(\omega, dt, dx) = dA_t(\omega)K_{\omega,t}(dx)$ where A is a predictable process in $\mathcal{A}^+$, b is predictable, $K_{\omega,t}(dx)$ is a transition kernel from $(\Omega \times \mathbf{R}_+, \tilde{\mathcal{P}})$ into $(\mathbf{R}^d, \mathcal{B}^d)$ with $\int (|x|^2 \wedge 1)K_{\omega,t}(dx) < \infty$; if $\Delta A_t(\omega) > 0$ then $\Delta A_t(\omega)K_{\omega,t}(\mathbf{R}^d) \le 1$ and $b_t(\omega) = \int h(x)K_{\omega,t}(dx)$, II.2.9. We may assume that $A = \alpha \cdot A^o$ where $\alpha > 0$ is predictable and $A^o_T \le 1$. Let $\bar{\mathcal{P}}$ be the completion of $\mathcal{P}$ with respect to the measure $m(d\omega, dt) := P(d\omega)dA_t(\omega)$. As usual, $a_t := \nu(\{t\}, \mathbf{R}^d)$. We write $K_{\omega,t}(Y)$ instead of $\int Y(x)K_{\omega,t}(dx)$ and omit often ω, t. Let $\theta := K(|x|^2 \wedge |x|)$.

The following assertion is an obvious corollary of II.2.29.

Lemma 3 $S \in \Sigma_m$ *(with $1/G := 1 + \theta$) $\Leftrightarrow$ $\theta < \infty$ and $b + K(\bar{h}) = 0$ m-a.e.*

Proof of Theorem 2. Let $\mathbf{Y}$ be the set of functions $Y > 0$, $Y \in C(\bar{\mathbf{R}}^d)$; $\mathbf{Y}$ with its Borel σ-algebra $\mathcal{Y}$ is a Lusin space. Let $\delta > 0$ be a predictable process. For every (ω, t) we consider in $\mathbf{Y}$ the convex subsets

$$\begin{aligned}
\Gamma^1_{\omega,t} &:= \{Y:\ K_{\omega,t}((\sqrt{Y}-1)^2) \le \delta_t(\omega)\},\\
\Gamma^2_{\omega,t} &:= \{Y:\ K_{\omega,t}((|x|^2 \wedge |x|)Y) < \infty\},\\
\Gamma^3_{\omega,t} &:= \{Y:\ I_{\{a_t(\omega)>0\}}K_{\omega,t}(Y) = I_{\{a_t(\omega)>0\}}K_{\omega,t}(\mathbf{R}^d)\},\\
\Gamma^0_{\omega,t} &:= \{Y:\ K_{\omega,t}(|xY - h|) < \infty,\ K_{\omega,t}(xY - h) = -b_t(\omega)\}.
\end{aligned}$$

Put $\Gamma_{\omega,t} := \Gamma^1_{\omega,t} \cap \Gamma^2_{\omega,t} \cap \Gamma^3_{\omega,t}$. Clearly, $\{(\omega, t, Y):\ Y \in \Gamma^i_{\omega,t}\} \in \mathcal{P} \otimes \mathcal{Y}$.

Lemma 4 *Let P be a separating measure. Then $\Gamma_{\omega,t} \cap \Gamma^0_{\omega,t} \ne \emptyset$ m-a.e.*

With this lemma we get the result immediately. Indeed, let $\delta := \varepsilon^2/(16\alpha)$. Applying the measurable selection theorem with $(\Omega \times \mathbf{R}_+, \bar{\mathcal{P}}, m)$ as the x-axis and $\mathbf{Y}$ as the y-axis we find a $\mathcal{P}$-measurable mapping $(\omega, t) \mapsto Y(\omega, t, .)$ into $\mathbf{Y}$ such that $Y(\omega, t, .) \in \Gamma_{\omega,t} \cap \Gamma^0_{\omega,t}$ m-a.e. The function $(\omega, t, x) \mapsto Y(\omega, t, x)$ is $\tilde{\mathcal{P}}$-measurable. Put $Z := \mathcal{E}((Y-1) * (\mu - \nu))$. Since $(\sqrt{Y}-1)^2 * \nu_T \le \varepsilon^2/16$ we have by Th. 12 in [11] that $Z \in \mathcal{M}$ and $Z_T > 0$. Let $Q := Z_T P$ and let $h(P, Q)$ be the Hellinger process. In virtue of IV.3.39, $h_T(P, Q) \le (\sqrt{Y}-1)^2 * \nu_T$, Thus, by V.4.22 (see also [12])

$$\mathrm{Var}\,(P-Q) \le 4\sqrt{Eh_T(P,Q)} \le \varepsilon.$$

[1] At least, for colleagues of Robert Liptser due to [11], [12].

The conditions of Lemma 3 are fulfilled for characteristics of S under Q, see Girsanov's theorem III.3.24. $\square$

Proof of Lemma 4. We start with the case $d=1$, omitting, as usual, ω, t. Let $r := \sup\{x \le 0 : \ K(]-\infty, x[) = 0\}$, $R := \inf\{x \ge 0 : \ K(]x, \infty[) = 0\}$. Define the predictable processes $j^n := I_{\{r > -n\}}$ and $J^n := I_{\{R < n\}}$. Notice that $\Delta(j^n \cdot S) \ge -n$. Hence, $j^n \bar{h}^- * \mu \in \mathcal{A}^+$. From the separation property of P we infer easily that $j^n \bar{h}^+ * \mu \in \mathcal{A}^+$ and $j^n \bar{h} * \nu + j^n \cdot B \in -\mathcal{A}^+$. The conclusion for J^n is symmetric. It follows that, outside of a m-negligible set,

$$\begin{array}{lll} \text{if } r > -\infty & \text{then} & K(|\bar{h}|) < \infty \text{ and } -b \in [K(\bar{h}), \infty[, \\ \text{if } R < \infty & \text{then} & K(|\bar{h}|) < \infty \text{ and } -b \in]-\infty, K(\bar{h})]. \end{array}$$

In particular, $-b = K(\bar{h})$ if $r < \infty$ and $R > -\infty$.

Put $\Psi(Y) := K(xY - h)$. Obviously, $\Gamma \cap \Gamma_0 \ne \emptyset$ iff $-b \in \Psi(\Gamma)$. The image of the convex set Γ under the affine mapping Ψ is an interval. By above $\Psi(\Gamma)$ always contains the point $\Psi(Y) = K(\bar{h})$ and we may conclude in the scalar case using

Lemma 5 *If $R = \infty$ then the interval $\Psi(\Gamma)$ is unbounded from above.*

Proof. Let $K_n(dx) := I_{\{x > n\}}(x) K(dx)$. For $\gamma > 0$ we define the set $\mathcal{W}_{n,\gamma}$ of $W \in \mathbf{Y}$ such that $W(n) = 1$, $xW(x) \to 0$ as $x \to \infty$, and $K_n(W) = \gamma$. Then the interval $K_n(x\mathcal{W}_{n,\gamma})$ is unbounded. Indeed, "deforming" a continuous function $V > 0$ such that $V(n) = 1$, $\gamma < K_n(V) < \infty$ but $K_n(xV) = \infty$, it is easy to construct $W \in \mathcal{W}_{n,\gamma}$ with arbitrary large $K_n(xW)$.

Fix $n \ge 1$ such that $K(\mathbf{R} \setminus [-n, n]) \le \delta/2$. Take $U \in \mathbf{Y}$ such that $U \le 1$, $U(-n) = 1$, $xU(x) \to 0$ as $x \to -\infty$, and $K(U) < K(\mathbf{R} \setminus [-n, n])$. Take $W_N \in \mathcal{W}_{n,\gamma}$ with $\gamma = K(\mathbf{R} \setminus [-n, n]) - K(U)$ and $K_n(xW_N) \ge N$. Then

$$Y_N(x) := U(x) I_{\{x < -n\}} + I_{\{|x| \le n\}} + W_N(x) I_{\{x > n\}} \in \Gamma$$

and $\Psi(Y_N) \to \infty$ as $N \to \infty$. $\square$

The vector case is easily reduced to the just considered. Indeed, the sets $\Xi_{\omega,t} := \Psi_{\omega,t}(\Gamma_{\omega,t}) + b_t(\omega) \subseteq \mathbf{R}^d$ are convex and $\{(\omega, t, x) : \ x \in \Xi_{\omega,t}\} \in \mathcal{P} \otimes \mathcal{B}^d$. By the measurable version of the separation theorem, there is a predictable process l with values in $(\mathbf{R}^d)^* = \mathbf{R}^d$ such that, outside of a m-negligible set, $|l_{\omega,t}| = 1$ and $l_{\omega,t} x < 0$ for every $x \in \Xi_{\omega,t}$ if $0 \notin \Xi_{\omega,t}$, and $l_{\omega,t} = 0$, otherwise. Let us consider the scalar semimartingale $S^l := l \cdot S$. We use the superscript l to denote objects related to S^l and subscripts for dimensions of the truncation functions. Obviously, $\nu^l(\omega, dt, dx) = K^l_{\omega,t}(dx) dA_t(\omega)$ with $K^l_{\omega,t}(dx) = (K_{\omega,t} l^{-1}_{\omega,t})(dx)$ and $B^l = lb \cdot A + K(lh_d(x) - h_1(lx)) \cdot A$, see IX.5.3; P is a separating measure for S^l. We have proved that for every fixed (ω, t) outside of a m-negligible set the equation $\Psi^l_{\omega,t}(Y) = -b^l_t(\omega)$ has a solution $Y \in \Gamma^l_{\omega,t}$. Due to the above relations, the function $Y(l_{\omega,t} x)$ belongs to $\Gamma_{\omega,t}$ and solves the equation $\Psi_{\omega,t}(Y(l_{\omega,t} x)) = -b_t(\omega)$. Thus, $l = 0$ m-a.e. $\square$

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