ON THE PONTRYAGIN MAXIMUM PRINCIPLE FOR SDEs WITH A POISSON-TYPE DRIVING NOISE

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The paper contains results on Pontryagin maximum principle for systems linear in phase variables and disturbed by Poisson-type noise. The cases of linear and quadratic cost are studied.

Key words: Pontryagin maximum principle, stochastic systems, Poisson-type process, backward stochastic differential equation.

1 Introduction

The important role of the Pontryagin maximum principle in the optimal control of deterministic systems is well-known. It is a powerful tool, in many cases more convenient and efficient than the dynamic programming based on the Bellman equation. The maximum principle is formulated as necessary conditions of optimality. In an initial value problem, it asserts that for the optimal control the corresponding solutions of the (forward) equation describing the dynamics of the controlled object and the conjugate (backward) equation are related through a maximization problem for a certain function called the Hamiltonian. An analysis of these conditions, sometimes rather involved, usually starts with inspection of the solution of the conjugate equation in backward direction since the boundary ("transversality") conditions for the latter are given at the right-hand extremity of the time interval, where, quite often, a structure of the optimal control is clear. For deterministic models the time direction is not essential and the conjugate equation is a usual ordinary differential equation.

The situation with stochastic systems is radically different: for a fixed increasing family of σ-algebras a boundary condition for stochastic differential equations is always formulated as an initial value problem with data given at the left-hand extremity of the interval. During a relatively long period it was not clear at all what is a natural analog of the conjugate equation which is so important object in the deterministic optimal control. Only in 1973 J.-M. Bismut suggested a construction which he called a backward stochastic differential equation (BSDE). With the use of backward stochastic equations the maximum principle for certain models resembles strongly its deterministic prototype. However, for models where the amplitude of the disturbances depends on the control, the structure of the Hamiltonian may be surprisingly different.

Here we present two simple results that are variants of the Pontryagin maximum principle for linear equations where the "stochastic" term is an integral with respect to a centered counting process of the Poisson type. In our first problem, the cost functional is linear in the phase variable. Due to this specificity there is no need in variational methods and the maximum principle can be deduced by straightforward calculations revealing the nature of BSDEs and their relation with the predictable representation theorem. In the second problem, with a quadratic cost function, we use variations by a single step function.

It necessary to note that more general variation schemes of the modern optimal control lead to analogues of the Pontryagin maximum principle for nonlinear SDEs of rather general type. However, one can observe that, in spite of the considerable efforts, the available results are far from the state of art of the deterministic theory and the examples of successful applications of the necessary conditions in stochastic setting are rare. Up to now, the stochastic maximum principle can not compete with Bellman's method. One of the possible reason of such a discouraging situation are "algorithmic" difficulties encountered with determining a solution of a backward SDEs constrained to be an element of a certain specific space. The recent investigations extended the concept of the BSDE to a nonlinear formulation and revealed that they have important applications and worth to be studied as objects of independent interest in more general setting.

We end the introduction by some historical comments. The backward stochastic differential equation was introduced by Bismut's [4] in the context of the duality approach to to stochastic optimal control and was exploited in a number of his subsequent publications, see, e.g., [4] for LQ-problem with Poisson disturbances. The first attempt to apply backward SDEs to finance is also due to Bismut, [3]; in particular, he considered Merton's portfolio problem and provided an economic interpretation for dual variables. It worth to notice that Bismut considered filtrations more general than natural and his backward equations are related to Kunita–Watanabe decomposition rather than to the simple predictable representation theorem. In 70-s Bismut's idea to use BSDE in a formulation of stochastic maximum principle was discussed intensively by participants of the seminar at Central Economics and Mathematics Institute;

the result was a number of publications but, unfortunately, the only paper [1] is available in English. The recent boom in mathematical finance activates an interest to the stochastic maximal principle and there is a new wave of papers contributing to this subject, see, e.g., [12], [5]. Nonlinear backward stochastic equations was introduced by Pardoux and Peng in [11], an important work [7] contains a good exposition of the theory with applications to mathematical finance as well as a long list of references.

The optimality criteria for the linear model (1), (2), but driven by a Wiener process, was proved in [9], the Poisson case was considered in author's thesis [10]; lemmas 3.1 – 3.3 are adapted from [4]. Theorem 5.1 for the linear-quadratic model and Example 2 is an extension of Saksonov's approach [13] to the Poisson-type disturbances. In spite of a difference in corresponding Ito calculus, the resulting maximum principle has a similar structure. At last, the books [8] and [6] can be recommended as references in point processes and measurable selection.

2 A linear model

Let us consider the following optimal control problem:

$$J(u) := E\int_{[0,T]}((b_t, y_t) + f_0(u_t))dt \to \max, \tag{1}$$

$$\begin{aligned} dy_t &= (Ay_{t-} + f_1(u_t))dt + (By_{t-} + f_2(u_t))(dN_t - \lambda_t dt), \\ y_0 &= \eta \end{aligned} \tag{2}$$

(in the sequel the subscript t will be usually omitted).

We assume that a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ is given with the filtration $\mathbf{F}$ generated by a counting process N and the P-null sets of $\mathcal{F}$. The intensity λ of N is a bounded predictable process, the initial condition $\eta \in \mathbf{R}^n$ and the time horizon $T \in \mathbf{R}_+$ are fixed.

The set $\mathcal{U}$ of admissible controls consists of all predictable processes with values in $U \subseteq \mathbf{R}^m$.

The coefficients are assumed to satisfy the following conditions:

1) The functions $f_i = f_i(\omega, t, u)$ are defined on $\Omega \times \mathbf{R}_+ \times U$, f_1 and f_2 take values in $\mathbf{R}^n$ and f_0 is a scalar function. For any $u \in U$ the process $f_i(., u)$ is predictable and for any ω, t the function $f(\omega, t, \cdot)$ is continuous.

2) The process $b = (b_t)$ with values in $\mathbf{R}^n$ is predictable.

3) The components of $n \times n$ matrices $A = (A_t)$ and $B = (B_t)$ are predictable processes.

4) All coefficients are bounded by a certain constant c.

3 Backward stochastic equations

Put $\|X\|_t := \sup_{s\le t}|X_s|$. Let LC_T^m be the space of regular (=càdlàg) adapted processes $X=(X_t)$ with $E\|X_t\|_T^m<\infty$ and $LC_T^\infty := \cap_{m\ge 1} LC_T^m$. We shall use also the spaces L_T^{2m} which contains all predictable processes h such that

$$E\Big(\int_{[0,T]}|h_s|^2 ds\Big)^m < \infty$$

and $L_T^{2\infty} := \cap_{m\ge 1} LC_T^{2m}$.

For a matrix or a vector process X a notation like $X\in LC_T^m$ means that all components belong to this space.

Lemma 3.1 *Let Φ be the solution of the linear matrix equation*

$$\begin{aligned} d\Phi &= A\Phi_- dt + B\Phi_-(dN-\lambda dt),\\ \Phi_0 &= I, \end{aligned} \tag{3}$$

where I is the $n\times n$ identity matrix. Then

(a) $\Phi\in LC_T^\infty$;

(b) if $\det(I+B_t(\omega))\neq 0$ for all ω,t then the matrix $\Phi_t(\omega)$ is always invertible for all ω,t and the process $\Psi:=\Phi^{-1}$ is the solution of the following equation:

$$\begin{aligned} d\Psi &= -\Psi_- A dt + \Psi_-((I+B)^{-1}-I)B\lambda dt && (4)\\ &+ \Psi_-((I+B)^{-1}-I)(dN-\lambda dt),\\ \Psi_0 &= I; && (5) \end{aligned}$$

(c) if, moreover $\|(I+B)^{-1}\|\le K$ where K is a constant then $\Psi\in LC_T^\infty$.

Proof. (a) Let $g_t := \sup_{s\le t}|\Phi_s|^m$, $m\ge 1$. Using the boundedness of the coefficients and the Jensen inequality we get from (5.3) that

$$g_t \le K_m\Big(1+\int_{[0,t]} g_{s-}\lambda_s ds + \int_{[0,t]} g_{s-} dN_s\Big) \tag{6}$$

where K_m is a constant depending on T, m, and c.

Let $\sigma_n := \inf\{t:\ g_t\ge n\}$, $G_t^{(n)} := E g_t^{\sigma_n}$. It follows from (6) that

$$G_t^{(n)} \le K_m\Big(1+2c\int_{[0,t]} G_s^{(n)} ds\Big).$$

By the Gronwall–Bellman lemma the function $G_t^{(n)}$ is bounded by a constant not depending on n, and the same constant gives a bound for $E\sup_{t\le T}|\Phi_t|^m$.

(b) Put $V := \Phi\Psi$. Then V solves the equation

$$
\begin{aligned}
dV &= [(A-B\lambda)V_- - V_-(A-B\lambda)]dt + [(I+B)V_-(I+B)^{-1} - V_-]dN,\\
V_0 &= I.
\end{aligned}
$$

Since the process identically equal to I is also a solution of the above equation, we have by uniqueness that $\Phi\Psi = I$. Similarly, $\Psi\Phi = I$.

(c) The assumption ensures the boundedness of the coefficients of the linear equation (4) and the assertion follows from (a). □

From now on we shall assume the $|(I+B)^{-1}|$ is uniformly bounded.

The following assertion is nothing but a version of the Cauchy formula.

Lemma 3.2 *The solution of the problem (2) (when $u \in \mathcal{U}$ is fixed) admits the following representation:*

$$
\begin{aligned}
y_t &= \Phi_t\Big[\eta + \int_{[0,t]} \Phi_{s-}^{-1}[f_1(u_s) - B_s(I+B_s)^{-1}f_2(u_s)\lambda_s]ds + \\
&+ \int_{[0,t]} \Phi_{s-}^{-1}(I+B_s)^{-1}f_2(u_s)(dN_s - \lambda_s ds)\Big]. \qquad (7)
\end{aligned}
$$

Proof. For y defined by (7), evidently, $y_0 = \eta$; by the product formula $d(XY) = X_-dY + dXY_- + d[X,Y]$ we have

$$
\begin{aligned}
dy &= Ay_-dt + By_-(dN - \lambda dt) + [f_1 - B(I+B)^{-1}f_2\lambda]dt + \\
&+ (I+B)^{-1}f_2(dN - \lambda dt) + B(I+B)^{-1}f_2 dN = \\
&= (Ay_- + f_1)dt + (By_- + f_2)(dN - \lambda dt)
\end{aligned}
$$

where $f_i := f(u_i)$. □

Lemma 3.3 *(a) There are processes $p \in LC_T^{\infty}$ and $h \in L_T^{2\infty}$ such that*

$$
\begin{aligned}
dp &= -(A^*p_- + B^*h\lambda + b)dt + h(dN - \lambda dt), \qquad (8)\\
p_T &= 0; \qquad (9)
\end{aligned}
$$

(b) the pair $(p,h) \in LC_T^{\infty} \times L_T^{2\infty}$ satisfying (8) and (9) is uniquely defined;

(c) this pair (p,h) has the following structure:

$$
\begin{aligned}
p &= \Phi^{*-1}(EG_T + M - G), \qquad (10)\\
h &= ((I+B^*)^{-1} - I)p_- + (I+B^*)^{-1}\Phi^{*-1}\phi, \qquad (11)
\end{aligned}
$$

where Φ is the solution of (3),

$$G_t := \int_{[0,t]} \Phi^*_{s-} ds, \tag{12}$$

$$M_t := \int_{[0,t]} \phi_s (dN_s - \lambda_s ds), \tag{13}$$

and the process $\phi \in L^{2\infty}_T$ is uniquely defined by the predictable representation

$$G_T = EG_T + \int_{[0,T]} \phi_s (dN_s - \lambda_s ds). \tag{14}$$

Proof. At first, let us check that the formulae (10) – (14) define a pair $(p,h) \in LC^\infty_T \times LC^\infty_T$. Since we assume that $|(I+B)^{-1}| \leq K$, Lemma 3.1 implies that $\Phi^{-1} \in LC^\infty_T$ and hence, by the boundedness of b, we have $G \in LC^\infty_T$. In particular, $E|G_T|^2 < \infty$. By the predictable representation theorem there exists the process $\phi \in L^{22}_T$ such that (14) holds and the components of M are square integrable martingales on $[0,T]$. Because the components of M_T have finite moments of any order, the Doob inequality implies that $M \in LC^\infty_T$. It follows easily from the above properties that the process p given by (7) is in LC^∞_T.

Let M^i and ϕ^i be the components of M and ϕ, $i = 1, \dots, n$. Using the Jensen inequality and the boundedness of λ we have:

$$\begin{aligned} E\Big(\int_{[0,T]} |\phi^i_s|^2 \lambda_s ds\Big)^m &\leq c^{m-1} T^m E \int_{[0,T]} |\phi^i_s|^{2m} \lambda_s ds = \\ = \quad c^{m-1} T^m E \int_{[0,T]} |\phi^i_s|^{2m} dN_s &\leq c^{m-1} T^m E\Big(\int_{[0,T]} |\phi^i_s|^2 dN_s\Big)^m. \end{aligned}$$

The Burkholder–Gundi–Davis inequality asserts that

$$E\Big(\int_{[0,T]} |\phi^i_s|^2 dN_s\Big)^m = E[M^i, M^i]^m_T \leq c_m E\|M^i\|^{2m}_T.$$

As $M \in LC^\infty_T$, the above bounds imply that $\phi \in L^{2\infty}_T$. From the properties of Φ^{-1}, p, and ϕ, proven under the assumption that $\|I+B)^{-1}\|$ is bounded, we infer easily that the formula (11) defines the process $h \in L^{2\infty}_T$.

Now we check that the relations (10) – (14) define the processes p and h related through the backward equation (8), (9). To this aim we apply the product formula (10) and make use the equation (4) for $\Psi = \Phi^{-1}$ and (12) –

(14). As a result, we get:

$$\begin{aligned}
dp &= -[A^*p_- + B^*((I+B^*)^{-1} - I)\lambda p_-]dt \\
&+ ((I+B^*)^{-1} - I)p_-(dN - \lambda dt) + \\
&+ \Phi_-^{*-1}\phi(dN - \lambda dt) - bdt + ((I+B^*)^{-1} - I)\Phi_-^{*-1}\phi dN = \\
&= -\{A^*p_- + B^*[((I+B^*)^{-1} - I)p_- + (I+B^*)^{-1}\Phi_-^{*-1}\phi]\lambda + b\}dt + \\
&+ [((I+B^*)^{-1} - I)p_- + (I+B^*)^{-1}\Phi_-^{*-1}\phi](dN - \lambda dt) = \\
&= -(A^*p + B^*h\lambda + b)dt + h(dN - \lambda dt).
\end{aligned}$$

The condition (9) holds obviously.

So, (a) and (c) are proved. The assertion (b) follows from the uniqueness of the trivial solution in the class $LC_T^\infty \times L_T^{2\infty}$ in the case $b \equiv 0$.

By (3) and (8) we have with $b = 0$ that

$$\begin{aligned}
d(\Phi^*p) &= \Phi_-^*A^*p_-dt + \Phi_-^*B^*p_-(dN - \lambda dt) - \Phi_-^*A^*p_-dt - \Phi_-^*B^*h\lambda dt + \\
&+ \Phi_-^*h(dN - \lambda dt) + \Phi_-^*hdN = \Phi_-^*(B^*p_- + B^*h + h)(dN - \lambda dt) = \\
&= \psi(dN - \lambda dt)
\end{aligned}$$

where $\psi := \Phi_-^*(B^*p_- + B^*h + h) \in L_T^{2\infty}$. We deduce from here, making use of the boundary condition (9) that

$$\int_{[0,T]} \psi_s(dN_s - \lambda_s ds) = 0.$$

By the uniqueness of the predictable representation in the class L_T^{22} for the square integrable random variable, in our case, identically equal to zero, we get $\psi = 0$. The matrix Φ_t^* is invertible and thus $p = 0$. Since the matrix $I + B_t^*$ is also invertible, $h = 0$ as well. □

The problem (8), (9) to find $p \in LC_T^\infty$ and $h \in L_T^{22}$ will referred to as the backward stochastic equation.

Remark 1. Evidently, the relation (10) can be written as $p = \Phi^{*-1}\pi(Y)$ where

$$Y_t := \int_{[t,T]} \Phi_{s-}^{*-1} b_s ds$$

and $\pi(Y)$ is the optional projection of the process Y, i.e.

$$p_t = E\Big(\int_{[t,T]} \Phi_{s-}^{*-1} b_s ds \Big| \mathcal{F}_t\Big).$$

4 The optimality criterion for the linear model

Now we formulate the analog of the Pontryagin maximum principle for the control problem (1), (2).

Let us consider the Hamiltonian function

$$H(y,u,p,h) := (p, Ay + f_1(u)) + (h, By + f_2(u))\lambda + (b,y) + f_0(u)$$

(arguments ω and t are omitted).

Theorem 4.1 *(a) Let $u^\circ \in \mathcal{U}$ be the optimal control for the problem (1), (2) and $y^\circ = (y^\circ_t)$ the corresponding process.*

Then there exist n-dimensional processes $p \in LC^\infty_T$ and $h \in L^{2\infty}_T$ satisfying the backward stochastic equation

$$\begin{aligned} dp_t &= -\frac{\partial H}{\partial y}(y^\circ_{t-}, u^\circ_t, p_{t-}, h_t)dt + h_t(dN - \lambda_t dt), && (15)\\ p_T &= 0, && (16) \end{aligned}$$

and such that

$$\sup_{u\in U} H(y^\circ_{t-}, u, p_{t-}, h_t) = H(y^\circ_{t-}, u^\circ_t, p_{t-}, h_t) \qquad dP \times dt\text{-a.e.} \tag{17}$$

(b) If the processes $y^\circ \in LC^\infty_T$, $u^\circ \in \mathcal{U}$, $p \in LC^\infty_T$, and $h \in L^{2\infty}_T$ are such that the equations (2), (15) – (17) hold then u° is the optimal control and y° is the optimal process.

Proof. (a) First of all we notice that the backward equation (15), (16) coincides with (8), (9) and the existence of its solution is guaranteed by Lemma 3.3.

Let us substitute the expression (7) for y into the definition (1) of the cost functional. We have:

$$\begin{aligned} J(u) &= E\int_{[0,T]} (\Phi^*_t b_t, \eta)dt + \\ &+ E\int_{[0,T]} \Big(\Phi^*_t b_t, \int_{[0,t]} \Phi^{-1}_s [f_1(u_s) - B(I+B)^{-1} f_2(u_s)\lambda_s]ds\Big)dt + \\ &+ E\int_{[0,T]} \Big(\Phi^*_t b_t, \int_{[0,t]} \Phi^{-1}_{s-}(I+B)^{-1} f_2(u_s)(dN_s - \lambda_s ds)\Big)dt + \\ &+ E\int_{[0,T]} f_0(u_t)dt. \end{aligned} \tag{18}$$

Integration by parts yields the equality

$$\begin{aligned} &\int_{[0,T]} \Big(\Phi_t^* b_t, \int_{[0,t]} \Phi_s^{-1}[f_1(u_s) - B(I+B)^{-1} f_2(u_s)\lambda_s] ds\Big) dt = \\ = \ &\int_{[0,T]} \Big(\Phi \int_{[t,T]} \Phi_s^* b_s ds, f_1(u_t) - B(I+B)^{-1} f_2(u_t)\lambda_t\Big) dt. \end{aligned} \tag{19}$$

Further, using the martingale property of the integral with respect to $dN - \lambda_t dt$ and the formula for covariance of such integrals we obtain that

$$\begin{aligned} &E\int_{[0,T]} \Big(\Phi_t^* b_t, \int_{[0,t]} \Phi_{s-}^{-1}(I+B)^{-1} f_2(u_s)(dN_s - \lambda_s ds)\Big) dt = \\ = \ &E\int_{[0,T]} \Big(\Phi_t^* b_t dt, \int_{[0,T]} \Phi_{t-}^{-1}(I+B)^{-1} f_2(u_t)(dN_t - \lambda_t dt)\Big) = \\ = \ &E\Big(\int_{[0,T]} \phi_t(dN_t - \lambda_t dt), \int_{[0,T]} \Phi_{t-}^{-1}(I+B)^{-1} f_2(u_t)(dN_t - \lambda_t dt)\Big) = \\ = \ &E\int_{[0,T]} ((I+B^*)^{-1}\Phi_{t-}^{*-1}\phi, f_2(u_t)\lambda_t dt). \end{aligned} \tag{20}$$

Notice that the expectation of the right-hand side of (19) is equal to

$$\begin{aligned} &E\int_{[0,T]} \Big(\Phi_t^{*-1}\pi_t\Big(\int_{[.,T]} \Phi_s^* b_s ds\Big), f_1(u_t) - B(I+B)^{-1} f_2(u_t)\lambda_t\Big) dt = \\ = \ &E\int_{[0,T]} (\Phi_t^{*-1}(EG_T - M_t - G_t), f_1(u_t) - B(I+B)^{-1} f_2(u_t)\lambda_t) dt. \end{aligned} \tag{21}$$

From (18) – (21) and (10), (11) we get the following expression for the cost functional:

$$J(u) = E\int_{[0,T]} (\Phi_t^* b_t, \eta) dt + E\int_{[0,T]} [(p_t, f_1(u_t)) + (h_t, f_2(u_t))\lambda_t + f_0(u_t)] dt. \tag{22}$$

It follows from the definitions that the control u° satisfying the maximum principle brings the pointwise maximum to the expression under the sign of the integral in the second term of the above representation, hence, $J(u^\circ) \geq J(u)$ for all $u \in \mathcal{U}$.

(b) Reciprocally, let u° be the optimal control. This means that $J(u^\circ) \geq J(u)$ for all $u \in \mathcal{U}$. Assume that u° does not satisfy the maximum principle. Since in the considered case maximization of the Hamiltonian H is equivalent to maximize of the function

$$\chi(u) := (p_-, f_1(u)) + \lambda(h, f_2(u)) + f_0(u),$$

for a certain $\varepsilon > 0$ the set

$$\{(\omega, t) : \sup_{u \in U} \chi(\omega, t, u) - \varepsilon > \chi(\omega, t, u^\circ)\}$$

has positive $dP \times dt$-measure. Let us consider the set-valued mapping Γ from $(\Omega \times \mathbf{R}_+, P)$ into U defined by

$$\Gamma(\omega, t) := \{u \in U : \chi(\omega, t, u) \geq \sup_{u \in U} \chi(\omega, t, u) - \varepsilon\}.$$

Clearly, Γ has the $\mathcal{P} \otimes \mathcal{U}$-measurable graph and the set $\Delta = \{(\omega, t) : u_t^\circ(\omega) \in \Gamma(\omega, t)\}$ is $\mathcal{P}$-measurable. Let $\tilde{u}$ be a $\mathcal{P}$-measurable selector of Γ. For the control $\tilde{u}^\circ := \tilde{u} I_\Delta + u^\circ I$ we have $J(u^\circ) < J(\tilde{u}^\circ)$. Thus, the assumption that u° does not satisfy (17) contradicts to optimality of u°. $\square$

Remark 2. If the set U is a compact then the control $u^\circ \in \mathcal{U}$ always exists. Indeed, the maximum principle is satisfied by any $\mathcal{P}$-measurable selector of the set-valued mapping Σ from $(\Omega \times \mathbf{R}_+, P)$ into U defined by

$$\Sigma(\omega, t) := \{u \in U : \chi(\omega, t, u) = \sup_{u \in U} \chi(\omega, t, u)\}.$$

Example 1. As an illustration we consider the following optimal control problem

$$J(u) := E \int_{[0,T]} I_{[0,\tau]}(t)(y_t^1 + \beta u_t^2) dt \to \max, \tag{23}$$

$$\begin{aligned} dy_t^1 &= (y_t^2 + u_t^1)dt + C^1(dN_t - dt), \qquad y_0^1 = \eta^1, \\ dy_t^2 &= (y_t^1 + u_t^2)dt + C^2(dN_t - dt), \qquad y_0^2 = \eta^2, \end{aligned} \tag{24}$$

where $u^1, u^2 \geq 0$ and $u^1 + u^2 = 1$, C^1 and C^2 are some real numbers, may be, equal to zero, the constant β is such that $e^{-T} < 1 - \beta < 1$, $T \in \mathbf{R}_+$, τ is a stopping time and N is the unit rate Poisson process.

The transition matrix Φ for the system (24) has the form

$$\Phi_t = \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{sh} t \end{bmatrix}.$$

In this case the Hamiltonian is

$$H(y, u, p, h) = p^1(y^2 + u^1) + p^2(y^1 + u^2) + h^1 C^1 + h^2 C^2 + I_{[0,\tau]}(y^1 + \beta u^2),$$

and to obtain u° from the maximum principle one needs to know only p^1 and p^2.

According to Remark 1, $p = \Phi^{*-1}\pi(Y)$ where

$$Y_t := \int_{[t,T]} I_{[0,\tau]}(s)\Phi_s^* e_1 ds, \quad e_1 := (1,0)^{\mathrm{T}}.$$

Hence,

$$p_t^1 = E[\mathrm{sh}\,(T\wedge\tau - t\wedge\tau) \mid \mathcal{F}_t],$$
$$p_t^2 = E[\mathrm{ch}\,(T\wedge\tau - t\wedge\tau) \mid \mathcal{F}_t].$$

Let us consider the set

$$\Gamma := \{p^1 > p^2 + \beta\} = \{(\omega, t) : \; E(e^{T\wedge\tau - t\wedge\tau} \mid \mathcal{F}_t) > 1 - \beta\}.$$

It follows from the maximum principle that that for $t \le \tau$ the components of the optimal control u° has the form $u^{\circ 1} = I_U$, $u^{\circ 2} = I$. Evidently, for $t > \tau$ the values of u° can be chosen arbitrary.

In particular, let $\tau := T$. Then $u^{\circ 1} = I_{[0,t_0]}$, $u^{\circ 2} = I_{]t_0,T]}$, $t_0 = T + \ln(1-\beta)$.

Another particular case: $\tau := \tau_{-1}$ where $\tau_\alpha := \inf\{t : \; Y_t \le \alpha\}$. Here the optimal control is given by the formula $u_t^{\circ 1} = I_A(t, M_{t\wedge\tau})$ where $M_t := N_t - t$ and

$$A := \Big\{(t,z) \in [0,T] \times \mathbf{R} : \; e^{t-T}P(\tau_{-z-1} \ge T-t) + \int_{[0,T-t]} e^{-s} dF_s < 1 - \beta\Big\}$$

where F is the distribution function of τ_{-z-1}.

5 A linear-quadratic problem

Let us consider the problem of optimal control of the same linear equation (2) but with the quadratic cost functional

$$J(u) := E\int_{[0,T]} \Big(\frac{1}{2}a_t|y_t|^2 + (b_t, y_t) + f_0(u_t)\Big)dt \to \max, \tag{25}$$

where $a = (a_t)$ is a bounded predictable process and all other parameters satisfy the hypotheses assumed above including the boundedness of the process $|(I+B)^{-1}|$.

It happens that this more general criteria leads to some new surprising features with respect to the corresponding deterministic result.

One could expect, by analogy, that for the problem (2), (25) the assertion (a) of Theorem 4.1 should hold with the Hamiltonian

$$H(y,u,p,h) := (p, Ay + f_1(u)) + (h, By + f_2(u))\lambda + \frac{1}{2}|y|^2 + (b,y) + f_0(u). \quad (26)$$

However, the example below shows that such a straightforward generalization fails.

Example 2. Let consider the following particular case of the linear-quadratic model:

$$J(u) := -\frac{1}{2}E\int_{[0,T]} y_t^2\,dt \to \max, \quad (27)$$

$$dy_t = u_t(dN_t - dt), \quad y_0 = 0, \quad (28)$$

where N is the unit rate Poisson process and the phase space of controls U consists only from two points: -1 and 1.

Evidently, the cost functional J does not depend at the choice of a control and is equal identically to

$$-\frac{1}{2}\int_{[0,T]} Ey_t^2\,dt = -\frac{1}{2}\int_{[0,T]} E(N_t - t)^2\,dt = -\frac{1}{4}T^2.$$

According to the formula (26) the Hamiltonian $H(y,u,p,h) = hu - (1/2)y^2$. In particular, the control $u^\circ \equiv 1$ is (as all the others) an optimal one; it generates the dynamics $y_t^\circ = N_t - t$. The backward SDE

$$dp_t = -\frac{\partial H}{\partial y}(y_t^\circ, u_t^\circ, p_t, h)dt + h_t(dN_t - dt), \quad p_T = 0,$$

reduces to

$$dp_t = y_t^\circ dt + h_t(dN_t - dt), \quad p_T = 0,$$

It is easy to check, by applying the formula for a product to the processes y° and h with $h_t = (t-T)$, that (y°, h) is the solution of this backward SDE. But $h_t < 0$ for $t < T$ and hence the maximum of the Hamiltonian H cannot be attained on the control $u^\circ \equiv 1$.

Nevertheless, the maximum principle holds for our LQ problem (even in the case on non-convex U) but with a modified Hamiltonian which contains one extra term, namely, with

$$\begin{aligned} H(y,u,p,h) \quad &:= \quad (p, Ay + f_1(u)) + (h, By + f_2(u))\lambda + \\ &+ \quad \frac{1}{2}a|y|^2 + (b,y) + f_0(u) - \frac{1}{2}(M_N(Y|\mathcal{P})\widehat{\psi}(u), \widehat{\psi}(u))\lambda, \end{aligned} \quad (29)$$

where

$$Y_t := \int_{[t,T]} \Phi_s^* \Phi_s ds, \tag{30}$$

$$\widehat{\psi}_t(u) := \Phi_{t-}^{-1}(I+B)^{-1} f_2(u_t^\circ) - \Phi_{t-}^{-1}(I+B)^{-1} f_2(u), \tag{31}$$

and $M_N(.|\mathcal{P})$ denotes conditioning with respect to the predictable σ-algebra on the measure space $(\Omega \times \mathbf{R}_+, \mathcal{F} \otimes \mathcal{B}_+, M_N)$ with $M_N(d\omega, dt) := P(d\omega)dN_t(\omega)$.

Notice that the extra term depends on the control u° to be tested on optimality.

Theorem 5.1 *Let $u^\circ \in \mathcal{U}$ be the optimal control for the problem (25), (2) and y° be the corresponding process.*

Then there is a pair $(p, h) \in LC_T^\infty \times L_T^{2\infty}$ which is the solution of the (n-dimensional) backward SDE

$$dp_t = -\frac{\partial H}{\partial y}(y_{t-}^\circ, u_t^\circ, p_{t-}, h_t)dt + h_t(dN - \lambda_t dt), \tag{32}$$

$$p_T = 0, \tag{33}$$

such that

$$\sup_{u \in U} H(y_{t-}^\circ, u, p_{t-}, h_t) = H(y_{t-}^\circ, u_t^\circ, p_{t-}, h_t) \qquad dP \times dt\text{-a.e.} \tag{34}$$

where H is given by (29).

Proof. The relation (32) is nothing but

$$dp_t = -(A^* p_{t-} + B^* h_t \lambda_t + a y_t^\circ + b)dt + h_t(dN_t - \lambda_t dt). \tag{35}$$

Since $y^\circ \in LC_T^\infty$ and the coefficient a is bounded, in accordance with Remark 1 there exists the unique solution $(p, h) \in LC_T^\infty \times L_T^{2\infty}$. Let $u \in \mathcal{U}$ be an arbitrary control with its corresponding process y solving (2); put $\widehat{y} := y^\circ - y$. Then

$$d\widehat{y} = (A\widehat{y}_- + f_1(u^\circ) - f_1(u))dt + (B\widehat{y}_- + f_2(u^\circ) - f_2(u)(dN - \lambda dt) \tag{36}$$

where $\widehat{y}_0 = 0$. Let us calculate the increment of the cost functional J when u° is substituted by u. We have:

$$J(u^\circ) - J(u) = E\int_{[0,T]} ((a_t y_t^\circ + b_t, \widehat{y}_t) + f_0(u_t^\circ) - f_0(u_t))dt - \frac{1}{2}E\int_{[0,T]} a_t |\widehat{y}_t|^2 dt. \tag{37}$$

Let us introduce the following notations:

$$\begin{aligned}
\chi_t(u_t) &:= (p_{t-}, f_1(u_t)) + (h_t, f_2(u_t)) + f_0(u_t), \\
\phi_t(u_t) &:= \Phi_{t-}^{-1}[f_1(u_t) - B(I+B)^{-1} f_2(u_t)\lambda], \\
\psi_t(u_t) &:= \Phi_{t-}^{-1}(I+B)^{-1} f_2(u_t), \\
\widehat{\chi}_t &:= \chi_t(u_t^\circ) - \chi_t(u_t), \\
\widehat{\phi}_t &:= \phi_t(u_t^\circ) - \phi_t(u_t), \\
\widehat{\psi}_t &:= \psi_t(u_t^\circ) - \psi_t(u_t), \\
m_t &:= N_t - \int_{[0,t]} \lambda_s ds.
\end{aligned}$$

We rewrite the first term in the right-hand side of the equality (37) involving $\widehat{y}$ in a linear way using (22). In the second term we substitute the expression for $\widehat{y}$ given by the formula Cauchy (7). As a result we get

$$J(u^\circ) - J(u) = E\int_{[0,T]} \widehat{\chi}_t dt - \frac{1}{2} E\int_{[0,T]} a_t \left|\Phi_t \int_{[0,t]} \widehat{\phi}_s ds\right|^2 dt -$$

$$-E\int_{[0,T]} a_t \Big(\Phi_t \int_{[0,t]} \widehat{\phi}_s ds, \Phi_t \int_{[0,t]} \widehat{\psi}_s dm_s\Big) dt - \frac{1}{2} E\int_{[0,T]} a_t \left|\Phi_t \int_{[0,t]} \widehat{\psi}_s dm_s\right|^2 dt. \tag{38}$$

For any $r \in]0,T]$ and $\varepsilon \in]0,r]$ we can define the control

$$u^{\varepsilon,r} := u I^{\varepsilon,r} + u^\circ(1 - I^{\varepsilon,r})$$

where $I^{\varepsilon,r}$ is the indicator function of $]r-\varepsilon, r]$, i.e. $u^{\varepsilon,r}$ coincides with the optimal control u° everywhere on $[0,T]$ except the interval $]r-\varepsilon, r]$ where $u^\varepsilon(r)$ is equal to u.

It follows from (38) and the definition of $u^{\varepsilon,r}$ that

$$J(u^\circ) - J(u^{\varepsilon,r}) = G_1^{\varepsilon,r} - (1/2)G_2^{\varepsilon,r} - G_3^{\varepsilon,r} - 1/2)G_4^{\varepsilon,r}$$

with

$$\begin{aligned}
G_1^{\varepsilon,r} &:= E\int_{[0,T]} \widehat{\chi}_t I_t^{\varepsilon,r} dt = \int_{]r,r-\varepsilon]} E\widehat{\chi}_t dt, \\
G_2^{\varepsilon,r} &:= E\int_{[0,T]} a_t \left|\Phi_t \int_{[0,t]} I_s^{\varepsilon,r} \widehat{\phi}_s ds\right|^2 dt,
\end{aligned}$$

$$
\begin{aligned}
G_3^{\varepsilon,r} &:= E\int_{[0,T]} a_t\Big(\Phi_t\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\phi}_s ds, \Phi_t\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\psi}_s dm_s\Big)dt,\\
G_4^{\varepsilon,r} &:= E\int_{[0,T]} a_t\left|\Phi_t\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\psi}_s dm_s\right|^2 dt.
\end{aligned}
$$

We show that

$$
\delta J^r := \lim_{\varepsilon\to 0}\varepsilon^{-1}(J(u^\circ)-J(u^{\varepsilon,r}))
$$

exists for almost all $r\in]0,T]$ and calculate the explicit expression for this limit. Since the functions f_i are bounded and $(p,h)\in LC_T^\infty\times L_T^{2\infty}$, the function $\int_{[0,.]}E\widehat{\chi}_t dt$ is absolute continuous on $[0,T]$. Hence for almost all $r\in]0,T]$

$$
\lim_{\varepsilon\to 0}\varepsilon^{-1}G_1^{\varepsilon,r} = E\widehat{\chi}_r. \tag{39}
$$

Notice that

$$
\varepsilon^{-1/2}\left|\Phi_t\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\phi}_s ds\right| \le \varepsilon^{-1/2}|\Phi_t|\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\phi}_s|ds \le \varepsilon^{1/2}|\Phi_t|\|\widehat{\phi}\|_T.
$$

The processes Φ and Φ^{-1} are in LC_T^∞, the functions a, f_i, λ , B, and $(I+B)^{-1}$ are bounded. Therefore, by the bounded convergence theorem we have for all r that

$$
\lim_{\varepsilon\to 0}\varepsilon^{-1}G_2^{\varepsilon,r} = 0.
$$

Further,

$$
\varepsilon^{-1}\left|\Big(\Phi_t\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\phi}_s ds, \Phi_t\int_{[0,t]} I_s^{\varepsilon,r}\widehat{\psi}_s dm_s)\Big)\right| \le |\Phi_t||\widehat{\phi}_s|\|\widehat{\psi}|_T(N_r-N_{r-\varepsilon}+c\varepsilon),
$$

hence, for all r

$$
\lim_{\varepsilon\to 0}\varepsilon^{-1}G_3^{\varepsilon,r} = 0.
$$

At last, we check that for almost all r

$$
\lim_{\varepsilon\to 0}\varepsilon^{-1}G_4^{\varepsilon,r} = E(M_N(Y|\mathcal{P})_r\widehat{\psi}_r,\widehat{\psi}_r) \tag{40}
$$

where Y is given by (30). Put

$$
\xi_t^{\varepsilon,r} := \int_{[0,t]} I_s^{\varepsilon,r}\widehat{\psi}_s dm_s.
$$

It follows from the product formula that

$$G_4^{\varepsilon,r} = G_{4,1}^{\varepsilon,r} + G_{4,2}^{\varepsilon,r}$$

where

$$\begin{aligned} G_{4,1}^{\varepsilon,r} &:= 2E\int_{[0,T]} a_t\Big(\int_{[0,t]} \Phi_t^*\Phi_t\xi_s^{\varepsilon,r}, d\xi_s^{\varepsilon,r}\Big)dt, \\ G_{4,2}^{\varepsilon,r} &:= E\int_{[0,T]} a_t\int_{[0,t]} (\Phi_t^*\Phi_t\widehat{\psi}_s, \widehat{\psi}_s)I_s^{\varepsilon,r}dN_s dt. \end{aligned}$$

The total variation of any component of $\xi^{\varepsilon,r}$ is bounded by $\eta(N_r - N_{r-\varepsilon} + c\varepsilon)$ where the random variable η has moments of any order. It follows easily that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}G_{4,1}^{\varepsilon,r} = 0.$$

The expression for $G_{4,1}^{\varepsilon,r}$ can be easily transformed in the following way:

$$G_{4,2}^{\varepsilon,r} = E\int_{[0,T]} I_s^{\varepsilon,r}\int_{[s,t]}(\Phi_t^*\Phi_t\widehat{\psi}_s, \widehat{\psi}_s)dtdN_s =$$

$$= E\int_{[0,T]} I_s^{\varepsilon,r}(Y_s\widehat{\psi}_s, \widehat{\psi}_s)dN_s = E\int_{[0,T]} I_s^{\varepsilon,r}(M_N(Y|\mathcal{P})_s\widehat{\psi}_s, \widehat{\psi}_s)dN_s =$$

$$= E\int_{[0,T]} I_s^{\varepsilon,r}(M_N(Y|\mathcal{P})_s\widehat{\psi}_s, \widehat{\psi}_s)\lambda_s ds.$$

We infer from the obtained expression, in the same way as for (39), that the relation (40) holds for almost all $r \in]0,T]$.

Thus, we have shown that for almost all r

$$\delta J^r = E[\widehat{\chi}_r - (1/2)(M_N(Y \mid \mathcal{P})_r\widehat{\psi}_r, \widehat{\psi}_r)\lambda_r]. \tag{41}$$

But u° is an optimal control, i.e. $J(u^\circ) - J(u^\varepsilon(r)) \geq 0$. Therefore, for almost all r

$$E[\widehat{\chi}_r - (1/2)(M_N(Y|\mathcal{P})_r\widehat{\psi}_r, \widehat{\psi}_r)\lambda_r] \geq 0.$$

Integrating this inequality with respect to r we get the bound

$$E\int_{[0,T]}[\widehat{\chi}_r - (1/2)(M_N(Y|\mathcal{P})_r\widehat{\psi}_r, \widehat{\psi}_r)\lambda_r]dr \geq 0. \tag{42}$$

Since the control $u \in \mathcal{U}$ is arbitrary, the relation (42) implies that

$$\sup_{u\in U}[\widehat{\chi}_r(u) - (1/2)(M_N(Y|\mathcal{P})_r\widehat{\psi}_r(u), \widehat{\psi}_r(u))\lambda_r] = \widehat{\chi}_r(u^\circ) \tag{43}$$

$dP \times dt$-a.e. Indeed, if it would not be the case and (43) were violated on the set of positive measure, the standard use of the measurable selection theorem (see the proof of the assertion (b) of Theorem 4.1) would allow to construct a control $u \in \mathcal{U}$ violating (42).

To accomplish the proof we notice that (34) and (43) are equivalent. $\square$

6 Final remarks

Our hypotheses on the boundedness of the functions $f_i(u)$ make possible to avoid some technicalities in the proof of the maximum principle. However, the results obtained above are not applicable to the generalization of the classic LQ-problem where the phase variables have jumps at stopping times which form a Poisson type point process, e.g., to the problem:

$$E \int_{[0,T]} ((1/2)|y_t|^2 + |u_t|^2) dt \to \min, \tag{44}$$

$$dy_t = (Ay_{t-} + Cu_t)dt + (By_{t-} + Du_t)(dN_t - \lambda dt), \quad y_0 = \eta, \tag{45}$$

where $a \geq 0$, $U = \mathbf{R}^n$, and the class of admissible controls consists of all predictable processes $u = (u_t)$ with values in $\mathbf{R}^n$ and satisfying a certain integrability condition. In the problem (44), (45) it is naturally to assume that $\mathcal{U}$ coincides with the Hilbert space L^{22}_T. The strictly concave functional J on this space attains its minimum and only at a single point. In this case the stochastic maximum principle holds as well but it is more natural to consider the processes involved in its formulation as the elements of other functional spaces. The analysis of necessary conditions lead to the definitive solution of the LQ-problem similar to that as was done by J.-M. Bismut for the case of the Poisson disturbances.

The discussion of this result as well as of nonlinear problems and problems with incomplete information is beyond of the scope of this work.

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