Dimensional Analysis, Leverage Neutrality, and Market Microstructure Invariance

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Big Questions

We think of trading a stock as playing a trading game: Long-term traders buy and sell shares to implement "bets," and intermediaries with short-term strategies–market makers, high frequency traders, and other arbitragers–clear markets.

- Can we derive quantitative predictions about microstructure variables? Is there a simple empirical measure of liquidity and how can theoretical liquidity parameters, like market impact coefficient λ, be implemented empirically?
- Trading games look different across assets that different in terms of their trading activity: dollar trading volume, volatility etc. Are there any fundamental (universal) laws in financial markets as in physics?

Answers: Use General Principles

- Trading games look different across assets only at first sight! They are similar if one looks through invariance lenses.
- Invariance implies a simple measure of liquidity as function of volume $V \cdot P$ and returns volatility σ^2:

$$L = \left(\frac{m^2 \cdot P \cdot V}{C \cdot \sigma^2} \right)^{1/3}$$

where constants $m^2 = 0.25$ and $C = 2000$.

- Liquidity L can be mapped to permanent price impact λ, temporary price impact κ, funding and trading liquidity etc.
- Theoretical dynamic models can link liquidity L to resiliency of prices ρ and error variance of prices Σ.

A General Picture

The scaling laws can be derived using different approaches:

- “Market Microstructure Invariance: Empirical Hypotheses” (Ecma, 2016): Empirical conjectures and tests.
- “Market Microstructure Invariance: A Dynamic Equilibrium Model”: Dynamic equilibrium model of speculative trading in which liquidity constrained investors seek to profit from trading on signals with invariant cost.
- “Adverse Selection and Liquidity: From Theory to Practice”: A meta-model.
- This paper: Physicists’ approach, apply dimensional analysis (consistency of units, Buckingham π-theorem)

APPROACH I: DIMENSIONAL ANALYSIS AND LEVERAGE NEUTRALITY

Derivation of Invariance for Physicists

Overview

This paper combines dimensional analysis, leverage neutrality, and a principle of invariance to derive scaling laws.

- Scaling laws relate transaction costs functions, bid-ask spreads, bet sizes, number of bets, and other financial variables in terms of dollar trading volume and volatility.
- These laws are tested using a data set of trades in the Russian and U.S. stock markets and find a strong support in the data.
- These scaling laws provide useful metrics for risk managers and traders; scientific benchmarks for evaluating issues related to high frequency trading, market crashes, and liquidity measurement; and guidelines for designing policies.

Dimensional Analysis

Physics researchers obtain powerful results by using dimensional analysis to reduce the dimensionality of problems (the size and number of molecules in a mole of gas, the size of the explosive energy, turbulence).

- **Physics:** fundamental units of mass, distance, and time & conservation laws based on laws of physics.
- **Finance:** fundamental units of time, currency, and shares & conservation laws based on no-arbitrage restrictions.

Oscillation of a Pendulum?

Suppose the time of oscillation T of a pendulum $T = f(M, L, g)$.

$$[T] = \text{s} \qquad \text{and} \qquad \begin{aligned}[M] &= \text{kg} \\ [L] &= \text{m} \\ [g] &= \text{m/s}^2.\end{aligned}$$

Buckingham π theorem: Rescaled T is a function of $N-3$ rescaled dimensionless variables

$$T = \sqrt{\frac{L}{g}} \cdot f(\text{dimensionless variables}) = \sqrt{\frac{L}{g}} \cdot f(\cdot) = \sqrt{\frac{L}{g}} \cdot \text{const}.$$

The law of conservation of energy implies that const $= 2\pi$.

If $T = f(M, L, g, x_1, x_2)$, then $T = \sqrt{\frac{L}{g}} \cdot f(\text{scaled x1, scaled x2})$.

How Big was the Bomb?

The first atomic blast, the Trinity Test in New Mexico in 1945, had an explosive yield of about 20 kilotons, but this value was secret.

Based on photographs of the Trinity Test released by the US Army in 1947 and dimensional analysis, Taylor guessed the size E from

$$R = \left(\frac{E \cdot t^2}{\rho}\right)^{1/5},$$

where R is radius, E is energy, t is time, ρ is density of air.

$$[R] = \text{m} \qquad \text{and} \qquad \begin{aligned} [E] &= \text{kg} \cdot \text{m}^2/\text{s}^2 \\ [t] &= \text{s} \\ [\rho] &= \text{kg}/\text{m}^3. \end{aligned}$$

Dimensional Analysis and Finance

In financial markets, institutional investors trade by implementing speculative "bets" which move prices. A bet is a decision to buy or sell a quantity of institutional size.

Trading is costly; bets tend to move market prices.

Dimensional analysis can be used to find formulas for the number of bets, their average size, market depth, transaction costs, etc.

Market Microstructure Variables

Easy-to-observe quantities include price, volume, and volatility:

$$\text{Price} = P_{jt} = 40.00 \text{ dollars/share}$$

$$\text{Trading Volume} = V_{jt} = 1.00 \text{ million shares/day}$$

$$\text{Returns Volatility} = \sigma_{jt} = 0.02/\text{day}^{1/2}$$

Hard-to-measure quantities that vary greatly across assets and time include bet size, number of bets, and the price impact coefficient:

$$\text{Size of Bet} = Q_{jt} = 10\,000 \text{ shares}$$

$$\text{Number of Bets} = \gamma_{jt} = 100/\text{day}$$

$$\text{Execution Horizon} = H_{jt} = 1 \text{ day}$$

$$\text{Price Change per Bet} = \Delta P_{jt} = 0.04 = \text{ dollars/share}$$

$$\text{Price Impact Coefficient} = \lambda_{jt} = 5 \times 10^{-5} \text{dollars/share}^2$$

$$\text{Price Error} = \Sigma_{jt}^{1/2} = \text{var}^{1/2}\{\log\left(\frac{F_{jt}}{P_{jt}}\right)\} = \log(2)(\text{dimensionless})$$

$$\text{Price Resiliency} = \rho_{jt} = 0.0040/\text{day}$$

Transaction Costs

Transaction costs are usually also hard to measure.

$$\text{Price Impact} = \Delta P_{jt} = 0.04 \text{ dollars/share}$$

Price impact cost G_{jt} as fraction of value traded:

$$G_{jt} = \frac{\Delta P_{jt}}{P_{jt}} = \frac{\Delta P_{jt} \cdot Q_{jt}}{P_{jt} \cdot Q_{jt}} = 10 \text{ basis points}$$

Price impact cost in dollars:

$$\text{Dollar Price Impact Cost} = \Delta P_{jt} \cdot Q_{jt} = 400 \text{ dollars}$$

Avg Dollar Cost per Bet C:

$$C = E_Q\{\Delta P_{jt} \cdot Q_{jt}\}.$$

Dimensional Analysis and Finance

Basic idea: Use dimensional analysis like a physicist (units consistency, Buckingham π Theorem):

- "Guess" correct functional form, i.e, correct list of explanatory variables.
  - Warning: Incorrect guess may lead to nonsense.
- Reduce dimensionality of problem by factoring out units, making remaining parameters dimensionless.
- Add restriction of "leverage neutrality" (Modigliani–Miller Theorem) to reduce dimensionality further. The cost of exchanging cash is zero. Dollar market impact cost of exchanging a risky bundle of assets is the same for any positive or negative amount of cash-equivalent assets included with the bundle.
- Make empirically motivated invariance assumption.

Dimensional Analysis Approach: Example of γ

I. Assume that the variable of interest—the number of bets γ_{jt}—is determined by some unknown functions f_γ, which take share volume V_{jt}, share price P_{jt}, returns volatility σ_{jt}, and expected dollar costs C as their arguments:

$$\gamma_{jt} = f_\gamma(V_{jt}, P_{jt}, \sigma^2_{jt}, C),$$

II. Reduce the dimensionality by applying dimensional analysis:

$$[\gamma_{jt}] = 1/\text{day} \qquad \text{and} \qquad \begin{aligned} [\sigma^2_{jt}] &= 1/\text{day} \\ [V_{jt}] &= \text{shares/day} \\ [P_{jt}] &= \text{dollars/share} \\ [C] &= \text{dollars.} \end{aligned}$$

$$\gamma_{jt} = \sigma^2_{jt} \cdot g_\gamma\Big(C \cdot \frac{\sigma^2_{jt}}{V_{jt} \cdot P_{jt}}\Big) \sim \sigma^2_{jt} \cdot \Big(\frac{C \cdot \sigma^2_{jt}}{V_{jt} \cdot P_{jt}}\Big)^{\alpha_\gamma}.$$

Example Cont'd: Leverage Neutrality

III. Impose a leverage neutrality restriction:

Exchanging cash-equivalent assets incurs zero cost. Exchanging risky securities is costly. The economic cost of trading bundles of risky securities and cash-equivalent assets is the same for any positive or negative amount of cash-equivalent assets included into a bundle.

$$\begin{aligned} \gamma_{jt} &\to \gamma_{jt}, & P_{jt} &\to P_{jt} \cdot A, \\ V_{jt} &\to V_{jt}, & \sigma_{jt} &\to \sigma_{jt} \cdot A^{-1}, \\ C &\to C, & P_{jt} \cdot \sigma &\to P_{jt} \cdot \sigma_{jt}. \end{aligned}$$

This restriction implies that $\alpha_\gamma = -2/3$.

$$\gamma_{jt} \sim \sigma_{jt}^2 \cdot \left(\frac{C \cdot \sigma_{jt}^2}{V_{jt} \cdot P_{jt}} \right)^{\alpha_\gamma} \Rightarrow \gamma_{jt} \sim \sigma_{jt}^2 \cdot \left(\frac{C \cdot \sigma_{jt}^2}{V_{jt} \cdot P_{jt}} \right)^{-2/3}.$$

Example Cont'd: Invariance

IV. Impose invariance restriction:

Average dollar cost C is hard to observe. Suppose C is approximately constant across assets and time, perhaps due to equilibrium in allocating resources and skills across markets. Then,

$$\gamma_{jt} \sim \sigma_{jt}^2 \cdot \left(\frac{C \cdot \sigma_{jt}^2}{V_{jt} \cdot P_{jt}} \right)^{-2/3} \quad \Rightarrow \quad \gamma_{jt} \sim \Big(\sigma_{jt} \cdot V_{jt} \cdot P_{jt} \Big)^{2/3}.$$

In terms of liquidity measure $L_{jt} = \left(\dfrac{m^2 \cdot P_{jt} \cdot V_{jt}}{C \cdot \sigma_{jt}^2} \right)^{1/3}$, we get

$$\gamma_{jt} \sim \sigma_{jt}^2 \cdot L_{jt}^2.$$

Russian Data

- One-minute data from the Moscow Exchange for January–December 2015 provided by Interfax Ltd.
- 50 Russian stocks in the RTS index as of June 15, 2015.
- The Russian stock market is centralized with all trading implemented in a consolidated limit-order book.
- Small tick and lot sizes.

U.S. Data

- One-minute data from the Trades and Quotes (TAQ) dataset for January–December 2015.
- 500 U.S. stocks in the S&P 500 index as of June 15, 2015.
- The U.S. stock market is fragmented, and securities are traded simultaneously at dozens of exchanges.
- Tick size of one cent, and lot sizes of 100 shares.

Tests for Number of Trades

Let N_{jt} denote the number of trades. Suppose

$$N_{jt} \sim \gamma_{jt}$$

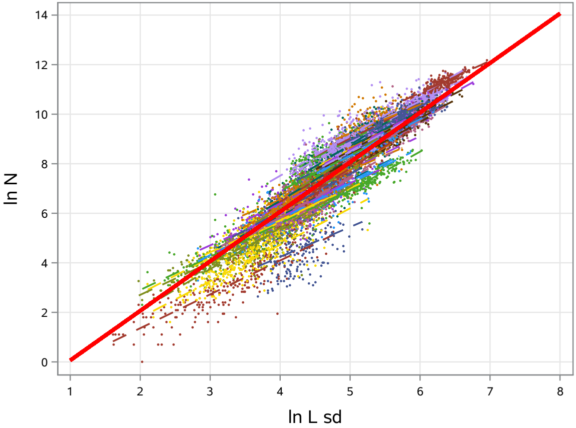
Then, from prediction

$$\gamma_{jt} \sim \sigma_{jt}^2 \cdot L_{jt}^2,$$

we get

$$\log\left(N_{jt}\right) = \mathrm{const} + \mathbf{2} \cdot \log(\sigma_{jt} L_{jt}).$$

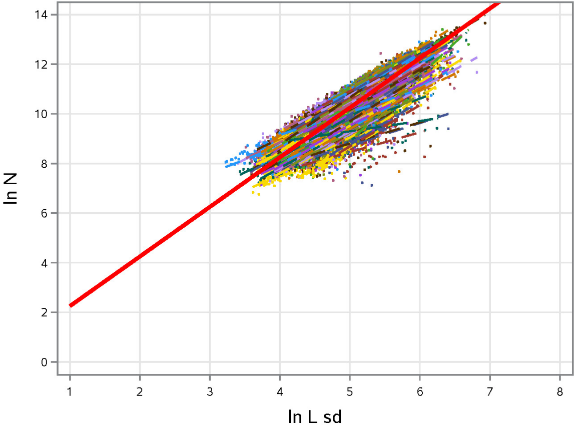
Number of Trades: Results for Russian Data



In aggregate sample, the slope is close to 2! R-square is 0.882.

$$\log(N_{jt}) = -3.085 + \mathbf{2.239} \cdot \log(\sigma_{jt} L_{jt})$$

Number of Trades: Results for U.S. Data



In aggregate sample, the slope is close to 2! R-square is 0.702.

$$\log(N_{jt}) = 1.005 + \mathbf{1.842} \cdot \log(\sigma_{jt} L_{jt})$$

Dimensional Analysis: Transaction Costs

Let G_{jt} denote the price impact cost as a fraction of the value traded $Q_{jt} \cdot P_{jt}$. The price impact G_{jt} is dimensionless, e.g. in basis points, and it is a function of

$$G_{jt} := \frac{\Delta P_{jt}(Q_{jt})}{P_{jt}} = g(Q_{jt}; P_{jt}, V_{jt}, \sigma_{jt}^2, C).$$

- bet size Q_{jt} in units of shares,
- stock price P_{jt} in units of dollars per share,
- share volume V_{jt} in units of shares-per-day,
- volatility σ_{jt}^2 in units of per-day,
- bet cost C in units of dollars.

Dimensional Analysis

Since the value of $G_{jt} := g(Q_{jt}, P_{jt}, V_{jt}, \sigma^2_{jt}, C)$ is dimensionless, consistency of units implies that it cannot depend on the dimensional quantities P_{jt}, Q_{jt}, and σ^2_{jt}.

Thus, *dimensional analysis* implies that the function $g()$ can be further simplified by writing it as function of two dimensionless variables.

$$\begin{aligned} G_{jt} &= P^0_{jt} \cdot Q^0_{jt} \cdot (\sigma^2_{jt})^0 \cdot f(\text{two dimensionless variables}) \\ &= f(\text{two dimensionless variables}). \end{aligned}$$

Dimensional Analysis

There are three sets of distinct units and five dimensional quantities—Q_{jt}, P_{jt}, V_{jt}, σ_{jt}^2, C.

Form two independent dimensionless quantities:

$$L_{jt} := \left(\frac{m^2 \cdot P_{jt} \cdot V_{jt}}{\sigma_{jt}^2 \cdot C} \right)^{1/3}, \qquad Z_{jt} := \frac{P_{jt} \cdot Q_{jt}}{L_{jt} \cdot C},$$

where m^2 is a dimensionless scaling constant.

Thus, *dimensional analysis* implies that the function g can be further simplified by writing it as $g(L_{jt}, Z_{jt})$.

$$G_{jt} := g(L_{jt}, Z_{jt}).$$

Leverage Neutrality

The cost of exchanging cash is zero. Adding cash/riskless debt to a risky asset or changing margin requirements must not affect economic costs and trading.

If $(A-1)P$ dollars of cash or debt is added to P_{jt}, then

$$P_{jt} \to P_{jt} \cdot A \qquad L_{jt} \to L_{jt} \cdot A$$

$$\sigma^2_{jt} \to \sigma^2_{jt} \cdot A^{-2} \qquad Z_{jt} \to Z_{jt}$$

$$Q_{jt} \to Q_{jt} \qquad C \to C$$

$$V_{jt} \to V_{jt} \qquad G_{jt} \to G_{jt} \cdot A^{-1}$$

The dollar costs $G_{jt} \cdot Q_{jt} \cdot P_{jt}$ are the same, but dollar bet size $Q_{jt} \cdot P_{jt}$ changes. $1/L_{jt}$ has the same leverage scaling as G_{jt}.

Leverage Neutrality

Percentage cost G_{jt} of executing a bet of Q_{jt} shares changes by a factor A^{-1}, since dollar cost did not change but dollar value changed. Leverage neutrality implies that

$$g(A \cdot L_{jt}, Z_{jt}) = A^{-1} \cdot g(L_{jt}, Z_{jt}).$$

If $A = L_{jt}^{-1}$, then $g(L_{jt}, Z_{jt}) = L_{jt}^{-1} \cdot g(1, Z_{jt})$.

Define $f(Z_{jt}) := g(1, Z_{jt})$ and get a very **important formula**:

$$G_{jt} = g(L_{jt}, Z_{jt}) = \frac{1}{L_{jt}} \cdot f(Z_{jt}).$$

Transaction Costs Model

A general specification for transaction costs functions consistent with the scaling implied by dimensional analysis and leverage neutrality:

$$g(Q_{jt}, P_{jt}, V_{jt}, \sigma_{jt}^2, C) = \left(\frac{\sigma_{jt}^2 \cdot C}{m^2 \cdot P_{jt} \cdot V_{jt}}\right)^{\mathbf{1/3}} \cdot f\left(\left(\frac{\sigma_{jt}^2 \cdot C}{m^2 \cdot P_{jt} \cdot V_{jt}}\right)^{\mathbf{1/3}} \cdot \frac{P_{jt} \cdot Q_{jt}}{C}\right).$$

It is consistent with different assumptions about the shape of the function f.

Market Microstructure Invariance

Extra assumptions are necessary to make our predictions operational.

- Three of the quantities—asset price P_{jt}, trading volume V_{jt}, and return volatility σ_{jt}—can be observed directly or readily estimated from public data feeds.
- Q_{jt} is a characteristic of a bet privately known to a trader.
- **Invariance:** the dollar value of C and the dimensionless scaling parameter m^2 are the same!

These assumptions are related to bet size and transaction costs invariance hypotheses. Preliminary calibration gives $C \approx \$2,000$ and $m^2 \approx 0.25$.

Economic Intuition

Scale m and define C so that

$$E\{|Z_{jt}|\} = 1 \qquad \text{and} \qquad C = E\{G_{jt} \cdot |P_{jt} Q_{jt}|\}.$$

The variables L_{jt} and Z_{jt} have an intuitive interpretation:

- $\frac{1}{L_{jt}} = \frac{C}{E\{P_{jt} \cdot |\tilde{Q}_{jt}|\}}$ is **"illiquidity index"** measuring average cost.
- $Z_{jt} = \frac{P_{jt} \cdot \tilde{Q}_{jt}}{E\{P_{jt} \cdot |\tilde{Q}_{jt}|\}}$ is **"scaled bet size"** relative to the average size.
- $m = \frac{E\{|Q_{jt}|\}}{\left(E\{Q_{jt}^2\}\right)^{1/2}}$ is moment ratio.

Liquidity Index

The liquidity index L_{jt} is consistent in terms of units. It is the correct way to construct empirical measure of Kyle's lambda.

$$L_{jt} := \left(\frac{m^2 \cdot P_{jt} \cdot V_{jt}}{\sigma_{jt}^2 \cdot C} \right)^{1/3} \sim \left(\frac{P_{jt} \cdot V_{jt}}{\sigma_{jt}^2} \right)^{1/3}.$$

Liquidity Index and Other Variables

Liquidity index L_{jt} can be linked to many variables, including composition of order flow. More liquid markets are associated with more bets of larger sizes (2-to-1 ratio):

- Bet size $E\{P_{jt} \cdot |\tilde{Q}_{jt}|\} = C \cdot L_{jt}$.
- Number of bets per day $\gamma_{jt} = \frac{1}{m^2} \cdot \sigma_{jt}^2 \cdot L_{jt}^2$.

Liquidity index L_{jt} appears in market impact formula:

$$G_{jt} = g(L_{jt}, Z_{jt}) = \frac{1}{L_{jt}} \cdot f(Z_{jt}).$$

Transaction Costs Models

Suppose $f(\cdot)$ is a power function of the form $f(Z_{jt}) = \text{const} \cdot |Z_{jt}|^{\omega}$.

- A percentage bid-ask spread cost ($\omega = 0$) implies

$$G_{jt} = \frac{S_{jt}}{P_{jt}} = \text{const} \cdot \frac{1}{L_{jt}}.$$

- A linear market impact cost ($\omega = 1$) implies

$$G_{jt} = \text{const} \cdot \frac{P_{jt} \cdot |Q_{jt}|}{C \cdot L_{jt}^2}.$$

- A square-root market impact cost ($\omega = 1/2$) implies

$$G_{jt} = \text{const} \cdot \sigma_{jt} \cdot \left(\frac{|Q_{jt}|}{V_{jt}} \right)^{1/2}.$$

Tests for Bid-Ask Spread

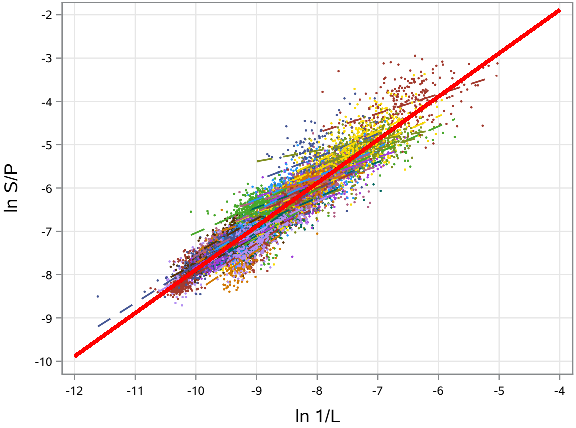
Let S_{jt} denote the percentage bid-ask spread ($\omega = 0$). Since

$$\frac{S_{jt}}{P_{jt}} = const \cdot \frac{1}{L_{jt}},$$

we get

$$\log\left(\frac{S_{jt}}{P_{jt}}\right) = \text{const} + \mathbf{1} \cdot \log\left(\frac{1}{L_{jt}}\right).$$

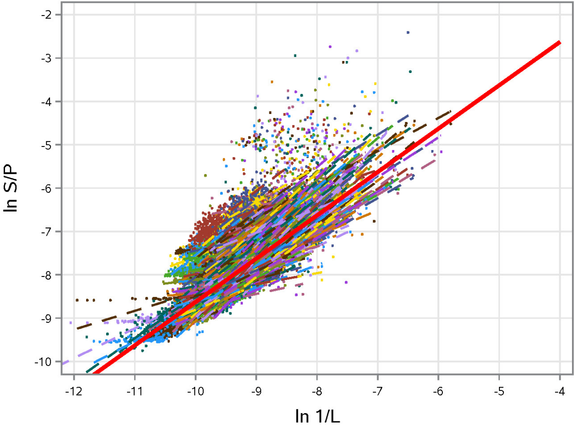
Spread: Results for Russian Data



In aggregate sample, the slope is close to 1! R-square is 0.876.

$$\log(S_{jt}/P_{jt}) = 2.093 + \mathbf{0.998} \cdot \log(1/L_{jt})$$

Spread: Results for U.S. Data



In aggregate sample, the slope is close to 1! R-square is 0.450.

$$\log(S_{jt}/P_{jt}) = 1.011 + \mathbf{0.961} \cdot \log(1/L_{jt})$$

Extensions

The empirical implications of dimensional analysis, leverage invariance, and market microstructure invariance can be generalized to incorporate various trading **frictions**.

Generalized Transaction Costs Formula

Add the **execution horizon** T_{jt} (in units of time), the **tick size** K_{jt}^{MIN} (in dollars per share), and the **lot size** Q_{jt}^{MIN} (in shares).

Re-scale variables to make them dimensionless and leverage neutral using the four variables P_{jt}, V_{jt}, σ_{jt}^2, and C:

- $\frac{|Q_{jt}|}{T_{jt}} \to \frac{|Q_{jt}|}{V_{jt} \cdot T_{jt}}$,
- $K_{jt}^{MIN} \to K_{jt}^{MIN} \cdot \frac{L_{jt}}{P_{jt}}$,
- $Q_{jt}^{MIN} \to Q_{jt}^{MIN} \cdot \frac{\sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}}$.

Generalized Transaction Costs Formula

$$G_{jt} = \frac{1}{L_{jt}} \cdot f\left(\frac{P_{jt} \cdot Q_{jt}}{C \cdot L_{jt}}; \frac{|Q_{jt}|}{V_{jt} \cdot T_{jt}}, \frac{K_{jt}^{MIN} \cdot L_{jt}}{P_{jt}}, \frac{Q_{jt}^{MIN} \cdot \sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}}\right).$$

Optimal Execution Horizon

Suppose the optimal execution horizon T^*_{jt} for an order of Q_{jt} shares depends on P_{jt}, V_{jt}, σ^2_{jt}, C, K^{MIN}_{jt}, and Q^{MIN}_{jt}.

Since $|Q_{jt}|/(V_{jt} \cdot T^*_{jt})$ is dimensionless and leverage neutral, the same logic implies:

$$\frac{|Q_{jt}|}{V_{jt} \cdot T^*_{jt}} = h^* \left(\frac{P_{jt} \cdot Q_{jt}}{C \cdot L_{jt}}, \frac{K^{MIN}_{jt} \cdot L_{jt}}{P_{jt}}, \frac{Q^{MIN}_{jt} \cdot \sigma^2_{jt} \cdot L^2_{jt}}{V_{jt}} \right).$$

If tick size and lot size do not affect execution horizon, $|Q_{jt}|/(V_{jt} \cdot T^*_{jt})$ depends only on $Z_{jt} := P_{jt} \cdot Q_{jt}/(C \cdot L_{jt})$.

Optimal Tick Size and Lot Size

Setting optimal tick size and minimum lot size is of interest for exchange officials and regulators.

Let K_{jt}^{MIN*} and Q_{jt}^{MIN*} denote optimal tick size and optimal minimum lot size, respectively.

Optimal Tick Size and Lot Size

Since the scaled optimal quantities $K_{jt}^{MIN*} \cdot L_{jt}/P_{jt}$ and $Q_{jt}^{MIN*} \cdot L_{jt}^2 \cdot \sigma_{jt}^2/V_{jt}$ are dimensionless and leverage neutral, the scaling laws for these market frictions are

$$K_{jt}^{MIN*} = \text{const} \cdot \frac{P_{jt}}{L_{jt}}, \qquad Q_{jt}^{MIN*} = \text{const} \cdot \frac{V_{jt}}{L_{jt}^2 \cdot \sigma_{jt}^2}.$$

General Formula for Bid-Ask Spread

Here is a formula for bid-ask spread for the market with frictions:

$$\frac{S_{jt}}{P_{jt}} = \frac{1}{L_{jt}} \cdot s\left(\frac{K_{jt}^{MIN} \cdot L_{jt}}{P_{jt}}, \frac{Q_{jt}^{MIN} \cdot \sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}} \right).$$

If tick size and minimum lot size have no influence on quoted bid-ask spreads, then the the relationship simplifies to $S_{jt}/P_{jt} \sim 1/L_{jt}$.

General Formula for Trading Patterns

Here are general formulas for trade sizes $\tilde{X}_{jt}$ and number of trades N_{jt}:

$$\mathrm{Prob}\left\{\frac{P_{jt}\cdot\tilde{X}_{jt}}{C\cdot L_{jt}} < z\right\} = F^{Q}_{jt}\left(z, \frac{K^{MIN}_{jt}\cdot L_{jt}}{P_{jt}}, \frac{Q^{MIN}_{jt}\cdot\sigma^2_{jt}\cdot L^2_{jt}}{V_{jt}}\right).$$

$$N_{jt} = \sigma^2_{jt}\cdot L^2_{jt}\cdot f\left(\frac{K^{MIN}_{jt}\cdot L_{jt}}{P_{jt}}, \frac{Q^{MIN}_{jt}\cdot\sigma^2_{jt}\cdot L^2_{jt}}{V_{jt}}\right).$$

Dimensional Analysis Looks Simple Ex Post

Suppose the variables of interest are functions of share volume V_{jt}, share price P_{jt}, returns volatility σ_{jt} (NOT dollar costs C)

$$\gamma_{jt} = f_{\gamma}(V_{jt}, P_{jt}, \sigma^2_{jt}),$$

Then, we get empirically implausible prediction, also being inconsistent with leverage neutrality principle:

$$\gamma_{jt} \sim \sigma^2_{jt}.$$

Similar analysis for G results in the Barra sqrt model, $G \sim \sigma \cdot \frac{Q}{V}$.

Theory has to provide guidance on which arguments have to be used.

Conclusions

There is a growing empirical evidence that the scaling laws discussed above match patterns in financial data, at least approximately.

Future research:

- Checking the validity of invariance predictions in other samples,
- Improving the accuracy of estimates and the triangulation of proportionality constants.

APPROACH II: META-MODEL

Derivation of Invariance for Econo-physicists

References

These slides are based on the following paper:

- Kyle and Obizhaeva, "Adverse Selection and Liquidity: From Theory to Practice".
- Kyle and Obizhaeva, "The Market Impact Puzzle".

Meta Model

Basic idea: Write down some simple generic equations that are likely to be valid in most theoretical models:

- Orders add up to trading volume;
- Order flow creates returns volatility;
- Each bet moves prices.

Can we derive invariance formulas from these equations? Yes.

Meta-Model

Suppose power function price impact for a bet Q: $\Delta P = \lambda \cdot Q^{\beta}$.
Define $\gamma =$ number of bets per day.
Now assume three-equation "meta-model":

$$V = \gamma \cdot E\left[|Q|\right] \qquad \text{(Definition of volume)}$$

$$\sigma^2 = \gamma \cdot E\left[\left(\frac{\Delta P}{P}\right)^2\right] \qquad \text{(Bets generate all volatility),}$$

$$E\{(\Delta P)^2\} = \lambda^2 \cdot E\left[|Q|^{2\beta}\right] \qquad \text{(Volatility from one bet).}$$

Three easy-to-measure quantities: V, σ, P.
We have five unknown hard-to-measure quantities:

$$\gamma, \quad E\left[|Q|\right], \quad E\left[|Q|^{2\beta}\right], \quad E\left[(\Delta P)^2\right], \quad \lambda.$$

and only three log-linear constraints, so we need more equations.

Empirical Motivation for Invariance

- Emprical Problem: Parameters like bet arrival rate γ and bet size $\mathrm{E}\left[|Q|\right]$ are hard to measure or estimate.
- Can they be replaced with a parameter that is either easier to estimate or does not vary much across assets?
- Empirical strategy: Introduce a parameter C (dollars), which does not vary (much) across assets and time.
- Use "invariant" parameter C to replace parameter which are hard-to-measure and varying across assets, such as γ or $\mathrm{E}\left[|Q|\right]$.
- Assume "transactions cost invariance": Ex ante expected dollar cost of a bet is constant (almost?)

$$C = E\left[|Q \cdot \Delta P|\right] = \lambda \cdot E\left[|Q|^{1+\beta}\right].$$

Augmented Four Equation Meta-Model

Add transactions cost invariance to obtain four equations:

$$V = \gamma \cdot E\left[|Q|\right] \qquad \text{(Definition of volume)}$$

$$\sigma^2 = \gamma \cdot E\left[\left(\frac{\Delta P}{P}\right)^2\right] \qquad \text{(Bets generate all volatility),}$$

$$\mathrm{E}\left[(\Delta P)^2\right] = \lambda^2 \cdot E\left[|Q|^{2\beta}\right] \qquad \text{(Price Impact of one bet),}$$

$$C = \lambda E\left[|Q|^{1+\beta}\right] \qquad \text{(Dollar impact cost of a bet).}$$

Need two invariant moment ratios:

$$m := \frac{E\left[|Q|\right] \cdot \sqrt{E\left[|Q|^{2\beta}\right]}}{E\left[|Q|^{\beta+1}\right]}, \qquad m_\beta := \frac{\left(E\left[|Q|\right]\right)^{\beta+1}}{E\left[|Q|^{\beta+1}\right]}.$$

Assume six parameters are easy to measure or almost constant:

$$P, \quad V, \quad \sigma, \quad C, \quad m, \quad m_\beta.$$

Solve six equations for six hard-to-measure parameters:

$$\gamma, \quad \alpha, \quad E\left[\Delta P^2\right], \quad E\left[|Q|\right], \quad E\left[|Q|^{1+\beta}\right], \quad E\left[|Q|^{2\beta}\right].$$

Solution with Invariance

Define "illiquidity" $1/L$ as volume-weighted expected cost:

$$\frac{1}{L} := \frac{C}{E\left[|P \cdot Q|\right]} = \left(\frac{\sigma^2 \cdot C}{m^2 \cdot P \cdot V}\right)^{1/3}.$$

Then, expected bet size and number of bets are given by

$$E\left[|P \cdot Q|\right] = C \cdot L, \qquad \gamma = \frac{1}{m^2} \cdot \sigma^2 \cdot L^2.$$

Price impact is

$$\frac{\Delta P}{P} = \frac{1}{L} \cdot m_\beta \cdot |Z|^\beta, \qquad \text{where} \qquad Z := \frac{Q}{E\left[|Q|\right]} = \frac{P \cdot Q}{C \cdot L},$$

If C, m, and m_β are invariant across assets, then we have a universal market impact formula and universal formula for size and number of bets. They require estimation of only these three parameters and β! All formulas turned out to be the same as in dimensional analysis!

Calibration of Constants

(Preliminary) Calibration of constants: $C = \$2,000$; if $\beta = 1$, then $m \approx 0.25$ and $m_\beta = m^2$. If $\beta = 1/2$, then $m_\beta = m \approx 0.40$.

Future research:

- Checking the validity of invariance predictions in other samples,
- Improving the accuracy of estimates and the triangulation of proportionality constants.

Conclusion

Meta-model derives an empirical formula for liquidity L without an underlying model of adverse selection. This is consistent with mechanical aspects of trading experienced in markets, where information is invisible.

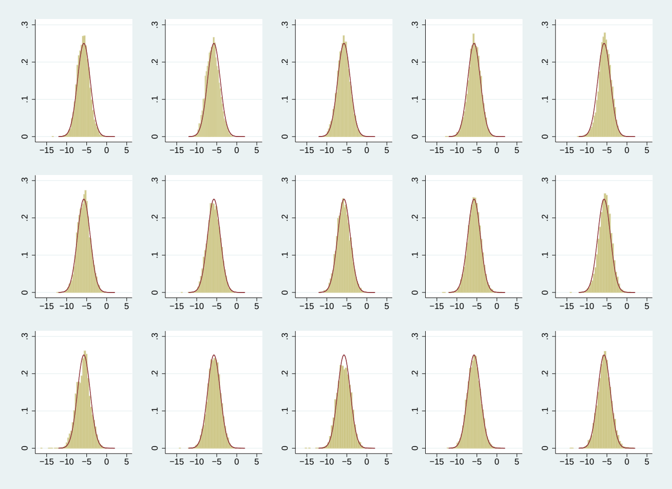
Conclusion

We need theories based in economics to link meta-model to adverse selection. This enables further link to pricing accuracy, probability of informed trading, and precision of signals.

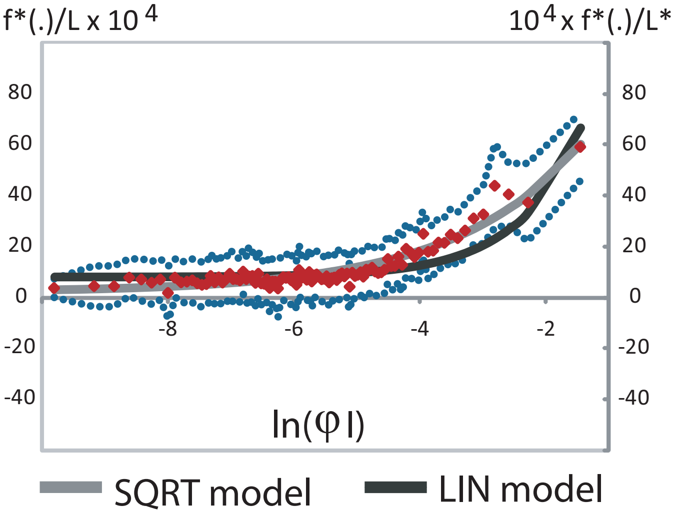
The meta-model and dimensional approach are consistent with theoretical models of both block trading and smooth trading. Theoretical dynamic models link L to resiliency of prices ρ and error variance of prices Σ.

See Kyle and Obizhaeva "Market Microstructure Invariance: A Dynamic Equilibrium Model".

Invariant Log-Normality of Portfolio Transition Order Size

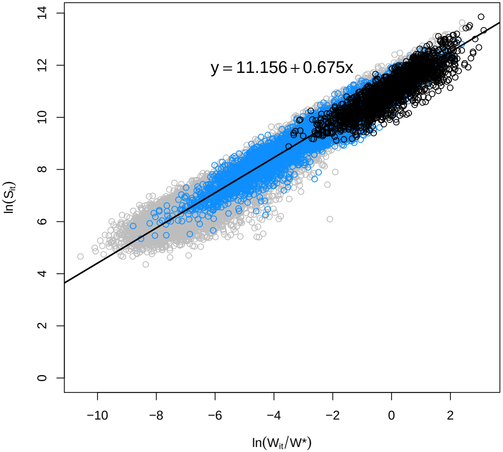


Linear versus Square Root Model

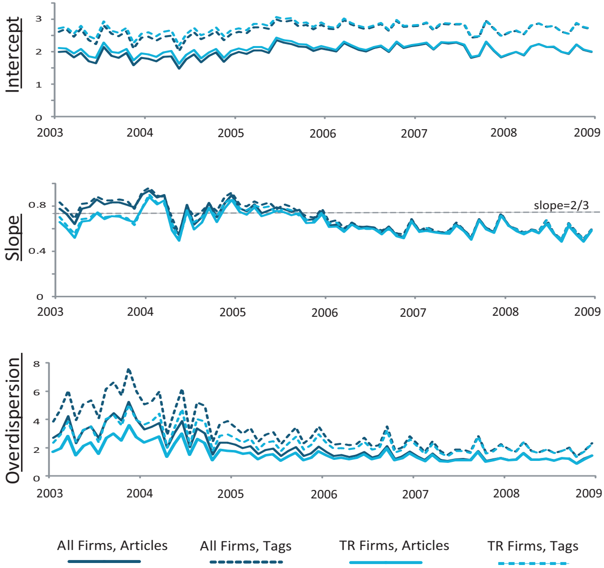


Switching Points: Korean Data

The fitted line for the regression of the number of switching points on trading activity is $\ln(S_{it}) = 11.156 + \mathbf{0.675} \cdot \ln(W_{it}/W^*)$. The invariance-implied slope is **2/3**.



News Articles



NYSE TAQ Prints, 1993