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CUSUM statistic and optimality for a minimax criterion in the disorder problems [2]

We observe the process $X = (X_s)_{s\geq 0}$ with

$$dX_s = \begin{cases} \sigma\, dB_s, & s < \theta, \\ \mu\, ds + \sigma\, dB_s, & s \geq \theta, \end{cases}$$

where $\theta \in [0, \infty]$ is a disorder.

Let P^θ be the distribution of X for given θ. Statistic $\gamma_t = \sup_{\theta\leq t} \frac{d\mathsf{P}^\theta}{d\mathsf{P}^\infty}(t, X)$, $t \geq 0$, is called CUSUM statistic.

We give a "martingale" proof of the following lower bound: for any stopping time τ with $\mathsf{E}^\infty\tau < \infty$

$$\sup_{\theta\geq 0} \operatorname*{ess\,sup}_{\omega} \mathsf{E}^\theta\big((\tau-\theta)^+ \mid \mathcal{F}_\tau\big)(\omega) \geq \frac{\mathsf{E}^\infty \int_0^\tau \gamma(t)\, dt}{\mathsf{E}^\infty \gamma_\tau}.$$

Using this inequality we show that the stopping time

$$\tau^*(B) = \inf\{t \geq 0 \colon \gamma_t \geq B\},$$

where B solves the equation $(2\sigma^2/\mu^2)[B - \ln B - 1] = T$, is optimal in the Lorden problem

$$D(T) = \inf_{\tau\in\mathfrak{M}_T} \sup_{\theta\geq 0} \operatorname*{ess\,sup}_{\omega} \mathsf{E}^\theta\big((\tau-\theta)^+ \mid \mathcal{F}_\theta\big)(\omega),$$

where $\mathfrak{M}_T = \{\tau \colon \mathsf{E}^\infty\tau = T\}$.

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